Some Famous Maclaurin Series

This table shows some very famous Maclaurin series (center 0) which you should learn. All of these were obtained by taking derivatives of $f(x)$ over and over, finding a pattern, and using that pattern to get $f^{(n)}(0)$. Recall that the $n$th coefficient is $a_n = \frac{f^{(n)}(0)}{n!}$.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$P(x)$</th>
<th>Pattern of terms</th>
<th>Radius of convergence $R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{1-x}$</td>
<td>$\sum_{n=0}^{\infty} x^n$</td>
<td>$1 + x + x^2 + \cdots$</td>
<td>1</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$\sum_{n=0}^{\infty} \frac{x^n}{n!}$</td>
<td>$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td>$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$</td>
<td>$x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$</td>
<td>$1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

NOTES:
- The geometric series has radius 1. The others, with factorials in the denominator, converge everywhere.
- $\sin$ has odd terms (it’s an odd function), and $\cos$ has even terms (it’s an even function).

SOME OTHER COOL SERIES:
You can find some other series by doing manipulations, such as substitutions and integration. For instance, if you plug $-x$ into the geometric series, followed by an integration, you get

$$\ln(1 + x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

There are similar tricks to get series for $1/(1 + x^2)$ and $\arctan(x)$. You get

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

The series for $\ln(1 + x)$ and $\arctan(x)$ converge on $(-1, 1]$.
- If you put $x = 1$ into the series for $\ln(1 + x)$, you get the alternating harmonic series!
  $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(2)$$
- If you put $x = 1$ into the series for $\arctan(x)$, you get $\arctan(1) = \pi/4$. Multiply 4 to both sides to get the famous series
  $$\text{Gregory series for } \pi: \pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

(If you think that’s cool, look up Machin’s identities on Wikipedia!)