A Difficult Reduction Formula Proven Using Parts

In this handout, we’ll look at a difficult case of a reduction formula. In doing this, we’re going to see:

- A tricky choice of parts
- Use of an identity to change the form of an integral
- “Cycling” back to the original integral

This is also covered in the textbook, but hopefully this covers the steps in more detail. You won’t have to do something this complicated, but you should look at this as an interesting way to put all our skills together.

Our Objective

Deduce the following reduction formula, valid for all integers $n \geq 2$:

\[
\int \sin^n x \, dx = \frac{1}{n} \sin^{n-1} x \cos x \, dx - \frac{n-1}{n} \int \sin^{n-2} x \, dx
\]

(Note that each time you use this formula, the power on the sines drops by 2, not by 1.)

Our Use of Parts

Here, integration by parts is a good idea, so that we can break up $\sin^n x$ into a product and only integrate one part of it. The most important choice is: what should $dv$ be? Remember that we want to take as much as we can for $dv$ as long as we actually can integrate it!

This leads us to pick just a single sine as $dv$, leaving the leftover $n-1$ powers for $u$:

\[
\begin{align*}
  u &= \sin^{n-1} x \\
  du &= (n-1) \sin^{n-2} x \cos x \, dx \\
  dv &= \sin x \, dx \\
  v &= -\cos x
\end{align*}
\]

Note the use of the Chain Rule with $du$.

This produces

\[
\int \sin^n x \, dx = \sin^{n-1} x \cos x - \int -(n-1) \sin^{n-2} x \cos^2 x \, dx
\]

Unfortunately, this doesn’t seem reduced yet; we had $n$ powers to start, and we still have $n$ powers in total at the end. In fact, it looks like doing parts a second time would be really ugly; no matter what you’d do, the choice of $u$ would feature the derivative product rule, further lessening the chances to get something manageable at the end.

The Trick to Get the Original Integral
The original integral used only sines, not sines and cosines. We will turn the $\cos^2 x$ back into sines via the famous trigonometric identity $\sin^2 x + \cos^2 x = 1$. This gives us $\cos^2 x = 1 - \sin^2 x$, turning our integration problem into

$$\sin^{n-1} x \cos x + (n-1) \left( \int \sin^{n-2} x (1 - \sin^2 x) \, dx \right)$$

Let’s expand this out and distribute the $(n-1)$ carefully. We get

$$\int \sin^n x \, dx = \sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

At this point, though, we see the same type of integral on the left and right sides of the equation! (Thus, we have successfully “cycled” back to the original problem.) When we add $(n-1)$ copies of the original problem to both sides, we end up with $n$ copies in total:

$$n \int \sin^n x \, dx = \sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx$$

Lastly, we divide by $n$.

$$\int \sin^n x \, dx = \frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

**DISCLAIMER**

This is **not** the way we will deal with powers of sines most of the time. In Section 8.2, we will see much more efficient methods.

However, there is a lesson to be learned from this problem: sometimes a little careful algebra can turn an integral into a much more recognizable form, relating back to a previous problem! Here, we “converted two cosines into two sines” to be able to get back $\sin^n x$. It turns out that several other reduction formulas use a similar gimmick to get the cycling they need. If you’re curious about this, talk to me about how the reduction formula for $\int (x^2 + a^2)^n \, dx$ works; it has a similar trick where $x^2$ is rewritten as $(x^2 + a^2) - a^2$ in order to get the cycling to occur.