Recall: The graphs of $y = \sin(x)$ and $y = \cos(x)$ are pictured below. They are both waves, where $\sin$ starts at a “rising root” and $\cos$ starts at its peak. Here are key features of these graphs:

- The **axis** is the horizontal line through the middle. Here, it’s $y = 0$.
- The **amplitude** is how far the graph rises and falls from the axis.
- The **period** is how long a wave is. You measure distance between rising roots or between peaks (or valleys).

Transforming these: Consider something like $y = a \cdot \sin(bx) + d$ or $y = a \cdot \cos(bx) + d$. It has three transforms.

1. The $b$ in “$bx$” DIVIDES the $x$-values and width by $|b|$. (Usually $b > 0$ though, so $|b|$ is just $b$.)
2. The $a$ in front stretches vertically by $|a|$. (If $a < 0$, flip the waves upside-down.)
3. The addition of $d$ shifts the graph vertically.

These steps produce

$$\begin{align*}
\text{Amplitude} &= |a| \\
\text{Period} &= \frac{2\pi}{b} \\
\text{Axis} &= y = d
\end{align*}$$

**Ex 1:** Find the amplitude, period, and range of each curve. Also make a quick sketch.

(a) $y = 3 \sin(\pi x) + 2$  
(b) $y = -2 \cos(\pi x/2)$

NOTE: For (b), you’ll want to draw the wave upside-down... why?

Some other remarks:
- **Range:** The min is (axis) - (amplitude) and max is (axis) + (amplitude), so the range is $[d - |a|, d + |a|]$.
- (axis) = (max + min) / 2, and (amplitude) = (max - min)/2... why?
- The only difference between sin and cos here is what $y$-value the graph starts with at $x = 0$!

**Ex 2:** Let $f(x) = a \cos(2x) + d$, where $a$ is positive. If $f(x)$ oscillates between $-17$ and $5$, find $a$ and $d$.

NOTE: This only talks about range, so the $2$ on the inside is irrelevant (its a horizontal change). Use the min and max to get axis and amp.

**Ex 3:** Determine the ranges of:  
(a) $10 + 2 \sin(-8x + 9)$  
(b) $10 + 10 \cos^2(x)$

HINT: For (a), ignore horizontal transforms. For (b), $\cos^2(x)$ has range $[0, 1]$, not $[-1, 1]$... why?

**Ex 4:** Determine the coordinates of the first maximum and minimum turning points on the graph of $y = 6 \sin(13x)$ on the interval $[0, 2\pi]$.

NOTE: Normally, the first maximum turning point of $\sin$ is at $(\pi/2, 1)$, and the first minimum is $(3\pi/2, -1)$. In other words, they are $1/4$ and $3/4$ of the way through one full wave. Transform those points!

**Phase Shift**

When you shift a wave horizontally, the amount its “start position” moves is called the **phase shift** (PS). When you have a phase shift, the equation is more like $y = a \sin(bx + c) + d$.

Finding PS: Focus on the angle part $bx + c$... call this $\theta$. Normally, the wave “starts” when $\theta = 0$, so $bx + c = 0$. To find the PS, which is the new “starting $x$” of the wave, solve for $x$!

$$\text{Phase shift} = -\frac{c}{b}$$

**Ex 5:** Find the amplitude, period, phase shift, and axis:

(a) $y = -3 \cos(2x + \pi) - 1$  
(b) $y = \frac{1}{2} \sin(\frac{\pi}{2} x)$
Recognizing the Transforms from a Sine or Cosine

Instead of going from an equation to a graph, let’s go the other way around now! In other words, we’re going to find the constants $a, b, c, d$ in $y = a \cdot \text{trig}(bx + c) + d$ from a picture. For simplicity, we’ll make sure $a, b > 0$ and $c \geq 0$.

1. Find $d = \text{axis} = (\text{max} + \text{min})/2$ and $a = \text{amplitude} = (\text{max} - \text{min})/2$.
2. Measure the period. Usually, I look for two adjacent peaks/crests/maxes or two valleys/troughs/mins.
   - Since $(\text{period}) = 2\pi/b$, solve for $b$!
3. There are technically infinitely many possible phase shifts! (The wave repeats infinitely often.) By convention, our choice will be the greatest negative “starting point” of the wave (though $x = 0$ is possible).
   - This means you go left of the origin but as close as possible, and that $x$-coordinate is your phase shift.
   - Remember: sin starts at the axis and rises. cos starts at a peak/crest.
4. Think of this as a “two-layer problem” where we have $\cos(\theta)$ or $\cos(\theta) + d$.
5. Solve the basic equation $\cos(\theta) = -1/2$; there’s only one answer in $(0, \pi)$. Use that $\theta$ value to find $b$!
   - Double-check that $b$ satisfies $0 < b < 9.$

HINT: On the far right, you can see the parts labeled (the solid dot is PS).

NOTE: If this were a COSINE function, how would that change the phase shift and the equation?

Tricky Problem Type: Getting the Transformed Wave from Two Points

Ex 6: The graph of a sine function with a positive coefficient is shown (it’s the graph on the left).

(a) Find its amplitude, period, and phase shift.
(b) Write the equation in the form $y = a \sin(bx + c)$ with $a, b > 0$.

HINT: On the far right, you can see the parts labeled (the solid dot is PS).

NOTE: If this were a COSINE function, how would that change the phase shift and the equation?

Ex 7: A function $f(x)$ is of the form $f(x) = a + \cos(bx)$, where $a$ and $b$ are constants and $0 < b < 9$. If $f(0) = 2$ and $f(\pi/9) = 0.5$, find $a$ and $b$.

MAIN APPROACH:
1. You have a form given, and you have two points. Plug the points in!
2. When you plug in 0, you get $a + \cos(0) = 2$. Since $\cos(0) = 1$ (whereas $\sin(0) = 0$), you get $a = 1$.
3. When you plug in $\pi/9$, you get $1 + \cos(b\pi/9) = 0.5$. Hence, $\cos(b\pi/9) = -0.5 = -1/2$.
4. Think of this as a “two-layer problem” where we have $\cos(\theta) = -1/2$ and $\theta = b\pi/9$. Let’s focus first on $\cos(\theta) = -1/2$.
   - To get an interval for $\theta$: We know $0 < b < 9$. Multiply by $\pi/9$ to find $0 < b\pi/9 < \pi$. Thus, $\theta$ must be in $(0, \pi)$. This tells you that $\theta$ must be in Quad I or II; Quad I would handle positive values, so our $\theta$ must be Quad II.
5. Solve the basic equation $\cos(\theta) = -1/2$; there’s only one answer in $(0, \pi)$. Use that $\theta$ value to find $b$!
   - Double-check that $b$ satisfies $0 < b < 9.$

Ex 8: Suppose $g(x) = a + \tan(bx)$, where $-3.5 < b < 3.5$. Determine $a$ and $b$ if $g(0) = 3$ and $g(\pi/7) = 2$.

NOTE: If $\theta = b\pi/7$, you find $-\pi/2 < \theta < \pi/2$. This time, the angle is NEGATIVE for Quadrant IV! I recommend: find $\theta_R$, then just use $\theta = -\theta_R$ for Quad IV.

Transforming the Other Four Trigs

We can do the same types of transforms to the other trig functions, and we make equations like $y = a \tan(bx + c) + d$, $y = a \sec(bx + c) + d$, etc. There are minor changes:

- $\text{PS} = -c/b$, but think about where the graphs “start”! At $x = 0$, tan has $(0, 0)$ and sec has $(0, 1)$, whereas the “co” functions cot and csc have vertical asymptotes.
- “Amplitude” and “axis” don’t make as much sense, since there’s no min or max. Instead, use $a$ and $d$ to transform vertically! The ranges of tan and cot will always be $(-\infty, \infty)$. For sec and csc, though, think of the range as having a “gap” from $-1$ and 1, and your transformations adjust that gap.
- tan and cot have half the usual period! It’s $\pi/b$ instead of $2\pi/b$.

Ex 9: Find the ranges and periods of the following curves (check with a graphing calculator!):

(a) $-3 \tan(5x)$
(b) $4 \sec(x/2)$
(c) $9 \csc(2x + 3) + 1$