

NAME:

Math 3000 Test 2 11/01/12

Instructions: Show your work, justifying your answers. If a problem specifies the method of solution, you are expected to use that method. You may not use your book or notes or a calculator.

Suggestion: Work quickly on the problems you can easily do, to leave time for the others.

1. Give a 2×2 matrix A so that for any $\mathbf{x} \in \mathbb{R}^2$, $A\mathbf{x}$ is the vector obtained by first projecting \mathbf{x} onto the vector $(1, 2)$ and then rotating the resulting vector $\pi/2$ clockwise.

$$\text{proj}_{(1,2)}(1,0) = \frac{(1,0) \cdot (1,2)}{\|(1,2)\|^2} (1,2) = \frac{1}{5} (1,2) = \left(\frac{1}{5}, \frac{2}{5}\right) = \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \end{bmatrix}$$

so $\text{proj}_{(1,2)} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ 0 & 0 \end{bmatrix}$

$$\text{proj}_{(1,2)}(0,1) = \frac{(0,1) \cdot (1,2)}{\|(1,2)\|^2} (1,2) = \frac{2}{5} (1,2) = \left(\frac{2}{5}, \frac{4}{5}\right) = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \end{bmatrix}$$

Rotation by $\frac{\pi}{2}$ clockwise: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ so rotation by $\frac{\pi}{2}$ clockwise = $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

So

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{4}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

2. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 & -3 & -3 \\ 0 & 1 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

3. (a) Find the LU decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & 0 \end{bmatrix} \xrightarrow{E_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -3 \end{bmatrix} = U$$

$$(E_1 A = U, A = E_1^{-1} U)$$

$$L = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

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$$E_1^{-1}$$

(b) Let A be an $n \times n$ matrix and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be nonzero vectors satisfying $A\mathbf{x} = \mathbf{x}$ and $A\mathbf{y} = 2\mathbf{y}$. Show that $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent.

Suppose $a\vec{x} + b\vec{y} = \vec{0}$. Want: $a = b = 0$.

$$A(a\vec{x} + b\vec{y}) = \vec{0}$$

$$a\vec{x} + 2b\vec{y} = \vec{0}$$

Subtracting equations gives $b\vec{y} = \vec{0}$ so $b = 0$. Then $a\vec{x} = \vec{0}$ so $a = 0$.

4. Show that if A and B are symmetric $n \times n$ matrices then AB is symmetric if and only if $AB = BA$. (Recall that a matrix A is symmetric if $A = A^T$.)

Proof : AB is symmetric $\Leftrightarrow (AB)^T = AB$
 $\Leftrightarrow B^T A^T = AB$
 $\Leftrightarrow BA = AB$ (since $A = A^T$, $B = B^T$). \square

5. Determine the intersection of

$$\text{Span}((1, 1, 1), (0, 1, 2)) \text{ and } \text{Span}((1, 1, 0), (2, 1, 1))$$

$$a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$a = c + 2d$$

$$a - c - 2d = 0$$

$$a + b = c + d$$

$$a + b - c - d = 0$$

$$a + 2b = d$$

$$a + 2b - d = 0$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 1 & 1 & -1 & -1 \\ 1 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 1 & 1 & -1 & -1 \\ 1 & 2 & 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

a, b, c are pivot variables, d = free variable

$$a - 3d = 0$$

$$a = 3d$$

$$b + d = 0$$

$$b = -d$$

$$c - d = 0$$

$$c = d$$

$$\text{so } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3d \\ -d \\ d \\ d \end{bmatrix} = d \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

The intersection is

$$\text{Span} \left(3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right)$$

which equals $\text{Span} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right)$

$$\text{Span} \left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right)$$

$$\text{Span} \left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right).$$

6. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ -1 & 0 & 3 & 4 \\ 2 & 2 & -2 & -3 \end{bmatrix}.$$

(a) Give constraint equations for $\mathbf{C}(A)$.

(b) Give a basis for $\mathbf{N}(A)$.

$$\text{(a)} \quad \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & b_1 \\ -1 & 0 & 3 & 4 & b_2 \\ 2 & 2 & -2 & -3 & b_3 \end{array} \right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & b_1 \\ 0 & 2 & 4 & 5 & b_1 + b_2 \\ 0 & -2 & -4 & -5 & -2b_1 + b_3 \end{array} \right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & b_1 \\ 0 & 2 & 4 & 5 & b_1 + b_2 \\ 0 & 0 & 0 & 0 & -2b_1 + b_3 + b_1 + b_2 \end{array} \right]$$

Constraint equation:

$$-2b_1 + b_3 + b_1 + b_2 = 0$$

$$-b_1 + b_2 + b_3 = 0$$

(b) Continuing from (a): Put A into reduced echelon form. Continuing from the last step of a):

$$\left[\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{cccc} 1 & 0 & -3 & -4 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{cccc} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 = 3x_3 + 4x_4$$

$$x_2 = -2x_3 - \frac{5}{2}x_4$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_3 + 4x_4 \\ -2x_3 - \frac{5}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}$$

Then $\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbf{N}(A)$.

7. Let A be an $m \times n$ matrix and let B be the matrix obtained from A by the row operation of adding 3 times row 1 to row 2. Show that $\mathbf{R}(A) = \mathbf{R}(B)$.

Let $\vec{A}_1, \dots, \vec{A}_m$ and $\vec{B}_1, \dots, \vec{B}_m$ denote the rows of A and B , respectively.

Then $\vec{B}_1 = \vec{A}_1$, $\vec{B}_2 = \vec{A}_2 + 3\vec{A}_1$, $\vec{B}_3 = \vec{A}_3, \dots, \vec{B}_m = \vec{A}_m$. Note: $\vec{A}_2 = \vec{B}_2 - 3\vec{B}_1$.

We will show $\mathbf{R}(B) \subset \mathbf{R}(A)$ and $\mathbf{R}(A) \subset \mathbf{R}(B)$.

$\mathbf{R}(B) \subset \mathbf{R}(A)$: If $\vec{x} \in \mathbf{R}(B)$ then $\vec{x} = c_1 \vec{B}_1 + \dots + c_m \vec{B}_m$ for some $c_1, \dots, c_m \in \mathbb{R}$. Then $\vec{x} = c_1 \vec{A}_1 + c_2 (\vec{A}_2 + 3\vec{A}_1) + \dots + c_m \vec{A}_m = (c_1 + 3c_2) \vec{A}_1 + c_2 \vec{A}_2 + \dots + c_m \vec{A}_m$

so $\vec{x} \in \mathbf{R}(A)$. Hence $\mathbf{R}(B) \subset \mathbf{R}(A)$.

$\mathbf{R}(A) \subset \mathbf{R}(B)$: If $\vec{x} \in \mathbf{R}(A)$ then $\vec{x} = c_1 \vec{A}_1 + \dots + c_m \vec{A}_m$ for some $c_1, \dots, c_m \in \mathbb{R}$. Then $\vec{x} = c_1 \vec{B}_1 + c_2 (\vec{B}_2 - 3\vec{B}_1) + \dots + c_m \vec{B}_m$
 $= (c_1 - 3c_2) \vec{B}_1 + c_2 \vec{B}_2 + \dots + c_m \vec{B}_m$

so $\vec{x} \in \mathbf{R}(B)$. Hence $\mathbf{R}(A) \subset \mathbf{R}(B)$.