NAME:

**Math 3000 Test 1 9/27/12**

Instructions: Show your work, justifying your answers. If a problem specifies the method of solution, you are expected to use that method. You may not use your book or notes or a calculator.

Suggestion: Work quickly on the problems you can easily do, to leave time for the others.

1. For the following matrix $A$ and vector $\mathbf{b}$, determine the reduced echelon form of the augmented matrix $[A|\mathbf{b}]$ and write the general solution of $A\mathbf{x} = \mathbf{b}$ in standard form.

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 2 \\ 0 & -1 & 1 & 1 & 3 \\ 1 & 3 & 1 & 0 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 2 \\ 0 & -1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 2 \\ 0 & -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 2 & 3 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 4 & 1 & 8 \\ 0 & -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & \frac{3}{2} \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 4 & 1 & 8 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 1 & \frac{3}{2} \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 4 & 0 & \frac{13}{2} \\ 0 & 1 & -1 & 0 & -3/2 \\ 0 & 0 & 0 & 1 & \frac{3}{2} \end{array}\right]$$

$$\Rightarrow \quad x_1 + 4x_3 = \frac{13}{2} \qquad x_1 = \frac{13}{2} - 4x_3$$

$$x_2 - x_3 = -\frac{3}{2} \qquad x_2 = -\frac{3}{2} + x_3$$

$$x_3 = x_3 \qquad x_3 = x_3$$

$$x_4 = \frac{3}{2} \qquad x_4 = \frac{3}{2}$$

Solution:

$$\begin{bmatrix} \frac{13}{2} - 4x_3 \\ -\frac{3}{2} + x_3 \\ x_3 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 13/2 \\ -3/2 \\ 0 \\ 3/2 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

2. (a) Compute the projection of the vector $(3,5)$ onto the vector $(2,7)$.

$$\mathrm{proj}_{(2,7)}(3,5) = \frac{(2,7)\cdot(3,5)}{\|(2,7)\|^2}(2,7) = \frac{2\cdot 3+7\cdot 5}{4+49}(2,7)$$

$$= \frac{41}{53}(2,7)$$

(b) Write down a system of linear equations which you would solve to find the parabola $y = ax^2 + bx + c$ passing through the points $(0,2)$, $(1,3)$ and $(2,1)$. (You don't have to solve this system.)

$$2 = c$$
$$3 = a+b+c$$
$$1 = 4a+2b+c$$

3. In this problem, you should use part a) to help you with the remaining parts.

(a) Find constraint equations that $\mathbf{b} = (b_1, b_2, b_3)$ must satisfy in order for the system $A\mathbf{x} = \mathbf{b}$ to be consistent, where

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -4 \\ 1 & -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 2 & b_1 \\ 2 & -4 & b_2 \\ 1 & -2 & b_3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} -1 & 2 & b_1 \\ 0 & 0 & 2b_1 + b_2 \\ 0 & 0 & b_1 + b_3 \end{array}\right]$$

so $2b_1 + b_2 = 0$

$b_1 + b_3 = 0$

(b) Find constraint equations that $\mathbf{b} = (b_1, b_2, b_3)$ must satisfy in order to be in the span of $(-1, 2, 1)$ and $(2, -4, -2)$.

Same as a)

(c) Find the Cartesian equation of the following plane in $\mathbb{R}^3$:

$$\mathbf{x} = s(-1, 2, 1) + t(2, -4, -2)\ , \quad s, t \in \mathbb{R}.$$

not a plane

(d) Find the Cartesian equation of the following plane in $\mathbb{R}^3$:

$$\mathbf{x} = (3, 2, 1) + s(-1, 2, 1) + t(2, -4, -2)\ , \quad s, t \in \mathbb{R}.$$

not a plane

4. Find the value of $k$ such that the following matrix has rank 2.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ -1 & 1 & k \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ -1 & 1 & k \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & 3 & 3+k \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 3+k+18 \end{bmatrix}$$

The matrix has rank 2 if $3+k+18=0$

$$k=-21$$

5. (a) Suppose $\mathbf{u}, \mathbf{v}$ and $\mathbf{x}$ are unit vectors such that the angle between $\mathbf{u}$ and $\mathbf{x}$ is $45°$, and $\mathbf{v}$ and $\mathbf{x}$ are orthogonal. Find $(\mathbf{u}+\mathbf{v}) \cdot \mathbf{x}$.

$$\cos 45^\circ = \frac{\vec{u}\cdot\vec{x}}{\|\vec{u}\|\|\vec{x}\|} = \vec{u}\cdot\vec{x} \qquad \text{So } \vec{u}\cdot\vec{x} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\vec{v}\cdot\vec{x} = 0$$

$$\text{So } (\vec{u}+\vec{v})\cdot\vec{x} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

(b) Find the products $AB$ and $BA$ if defined, or explain why not if they are not defined, for

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 2-1 & 0-1 & 2+3 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ 1 & 0 & 1 \end{bmatrix}$$

BA is not defined because $B \in M_{2\times 3}$, $A \in M_{2\times 2}$

not equal

6. Let $\overset{\vec{x}}{\cancel{\mathbf{v}}}$ and $\overset{\vec{y}}{\cancel{\mathbf{w}}}$ be unit vectors in $\mathbb{R}^2$. Show that the angle between $\mathbf{x}$ and $\mathbf{x}+\mathbf{y}$ equals the angle between $\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$. Deduce that $\mathbf{x}+\mathbf{y}$ bisects the angle between $\mathbf{x}$ and $\mathbf{y}$.

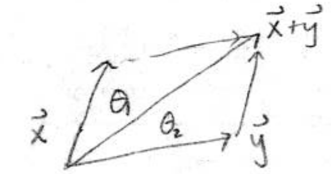
Let $\theta_1$ be the angle between $\vec{x}$ and $\vec{x}+\vec{y}$ and $\theta_2$ be the angle between $\vec{y}$ and $\vec{x}+\vec{y}$. We want to show $\theta_1=\theta_2$. It is enough to show $\cos\theta_1=\cos\theta_2$ because the function $f(\theta)=\cos\theta$ is 1-1 if the domain is $[0,\pi]$, so $\cos\theta_1=\cos\theta_2$ implies $\theta_1=\theta_2$.

$$\cos\theta_1 = \frac{\vec{x}\cdot(\vec{x}+\vec{y})}{\|\vec{x}\|\,\|\vec{x}+\vec{y}\|} = \frac{\vec{x}\cdot\vec{x}+\vec{x}\cdot\vec{y}}{1\cdot\|\vec{x}+\vec{y}\|} = \frac{1+\vec{x}\cdot\vec{y}}{\|\vec{x}+\vec{y}\|} \quad \text{as } \vec{x}\cdot\vec{x}=\|\vec{x}\|^2=1^2=1$$

$$\cos\theta_2 = \frac{\vec{y}\cdot(\vec{x}+\vec{y})}{\|\vec{y}\|\,\|\vec{x}+\vec{y}\|} = \frac{\vec{y}\cdot\vec{x}+\vec{y}\cdot\vec{y}}{1\cdot\|\vec{x}+\vec{y}\|} = \frac{1+\vec{x}\cdot\vec{y}}{\|\vec{x}+\vec{y}\|} \quad \text{as } \vec{x}\cdot\vec{y}=\vec{y}\cdot\vec{x} \text{ and } \vec{y}\cdot\vec{y}=\|\vec{y}\|^2=1^2=1.$$

Hence $\cos\theta_1=\cos\theta_2$, which is what we wanted.

To see that $\vec{x}+\vec{y}$ bisects the angle between $\vec{x}$ and $\vec{y}$,




Since $\theta_1=\theta_2$, $\vec{x}+\vec{y}$ bisects the angle $\theta_1+\theta_2$ between $\vec{x}$ and $\vec{y}$.

7. Let $A$ be an $m \times n$ matrix and let $r$ be the rank of $A$.

(a) What is the condition on the rank which implies that the system $A\mathbf{x} = \mathbf{0}$ has only one solution?

(b) What is the condition on the rank which implies that the system $A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b} \in \mathbb{R}^n$?

(c) Give a proof or counterexample to the following assertion: If the system $A\mathbf{x} = \mathbf{0}$ has only one solution, then the system $A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b} \in \mathbb{R}^n$.

a) $r = n$ (no free variables)

b) $r = m$ (no constraint equations, so no nonzero rows in an echelon form of A)

c) Counterexample: We look for a counterexample: find a matrix A where $r = n$, $r < m$. Take $n = 1$, $m = 2$, $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{x} = [c]$.

Then $A\vec{x} = \vec{0}$ means $[c] = \vec{0}$ so $c = 0$, so $\vec{x} = \vec{0}$. Therefore $A\vec{x} = \vec{0}$ has only one solution (namely $\vec{x} = \vec{0}$). But $A\vec{x} = \begin{bmatrix} c \\ 0 \end{bmatrix}$ so the system $A\vec{x} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is not consistent if $b_2 \neq 0$.