# VORTICES AND A TQFT FOR LEFSCHETZ FIBRATIONS ON 4–MANIFOLDS

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Abstract. Adapting a construction of D. Salamon involving the $U(1)$ vortex equations, we explore the properties of a Floer theory for 3–manifolds that fiber over $S^1$ which exhibits several parallels with monopole Floer homology, and in all likelihood coincides with it. The theory fits into a restricted analogue of a TQFT in which the cobordisms are required to be equipped with Lefschetz fibrations, and has connections to the dynamics of surface symplectomorphisms.

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## 1. Background and summary of results

For some time it has been known that two of the most important invariants of smooth closed 4–manifolds, the Donaldson and Seiberg–Witten invariants, can each be expressed in terms of (3+1)–dimensional topological quantum field theories [6],[28],[19]. In such a "TQFT," to each oriented 3–manifold $Y$ (perhaps equipped with additional data, such as a spin$^c$-structure), one associates canonically a group $V(Y)$ satisfying, among several other conditions, the property that a cobordism $X$ from $Y_1$ to $Y_2$ functorially induces a homomorphism $F_X\colon V(Y_1) \to V(Y_2)$. If $X$ is a smooth closed oriented 4–manifold, divided into two pieces as $X = X_1 \cup_Y X_2$ with $b^+(X_i) > 0$, one views $X_1$ as a cobordism from the empty set $\varnothing$ to $Y$ and $X_2$ as a cobordism from $\varnothing$ to $-Y$ (*i.e.*, $Y$ with its orientation reversed). One has a natural identification $V(-Y) \cong V(Y)^*$, and the 4-dimensional invariant $I_X$ is obtained by a natural calculation in $V(Y)$ involving the images of the maps $F_{X_1}$ and $F_{X_2}$; $I_X$ is independent of the choice of splitting of $X$ into the two pieces $X_1$ and $X_2$.

In the presence of a symplectic structure $\omega$ on the spin$^c$ 4–manifold $(X, s)$, the famous work of C. Taubes collected in [45] shows that the Seiberg–Witten invariant $SW_X(s)$ agrees with a "Gromov invariant" $Gr_{(X,\omega)}(\alpha_s)$ which counts pseudoholomorphic submanifolds of $X$ representing a homology class $\alpha_s$ corresponding to $s$. Kronheimer and Mrowka's work [19] (see [20] for a summary) lays the full foundations for the TQFT underlying $SW_X(s)$, in which the role of the group $V(Y)$ in the above description is played by $HM(Y, s, \eta)$, where $s$ is a spin$^c$ structure and $\eta \in H^2(Y;\mathbb{R})$ is the cohomology class of the perturbation used in the Seiberg–Witten equations. Given the correspondence between $SW$ and $Gr$, it is natural to expect that there might be a TQFT underlying $Gr$ which corresponds to Kronheimer–Mrowka's field theory for $HM$. Progress in this direction has been made by M. Hutchings and his collaborators, who introduce groups $ECH(Y)$ (embedded contact homology) [16] and $HP(Y)$ (periodic Floer homology) [17], [15] in the case that $Y$ is, respectively, a contact manifold or a mapping torus. These groups are conjectured to agree with $HM$ or the (conjecturally equivalent) Heegaard Floer homology $HF^+$ under suitable hypotheses, and do so in each of the several cases that have been computed. However, at this writing a number foundational questions (such as independence of the choice of almost complex structure) remain to be settled for $ECH$ and $HP$, and there do not presently exist full-blown TQFT's incorporating either one of them.

According to results of S. Donaldson [5] and R. Gompf [13], a smooth oriented 4–manifold $X$ admits a symplectic structure if and only if, possibly after blowing $X$ up at finitely many points, there is a *Lefschetz fibration* $f: X \to S^2$ whose fibers are homologically essential. Recall here that a Lefschetz fibration on an oriented 4–manifold is a map to a 2–manifold which is a submersion except at its only finitely many critical points, near each of which there are orientation preserving complex coordinates in terms of which the map has the form $(z, w) \mapsto zw$. As such, the fibers of $f$ are all complex curves of some fixed arithmetic genus, all but finitely many of which are smooth, with the singular fibers having at worst nodal singularities. In the presence of a Lefschetz fibration $f: X \to S^2$ (satisfying certain properties that can always be achieved using the constructions of [5]), Donaldson and I. Smith introduced in [7] an invariant $DS_{(X,f)}(\alpha)$ (for $\alpha \in H_2(X;\mathbb{Z})$) which counts pseudoholomorphic sections of a bundle of symmetric products constructed from $f$. In [46] it was shown that this Donaldson–Smith invariant coincides with Taubes' invariant $Gr$, and hence also with the Seiberg–Witten invariant under the appropriate identification of $H_2(X;\mathbb{Z})$ with the set of spin$^c$ structures on $X$.

The present paper concerns what might be described as a restricted TQFT which underlies the 4-dimensional invariant $DS$. We view this TQFT as a covariant functor *to* the category of modules over a certain ring $A$ *from* a category whose objects are closed oriented 3–manifolds $Y$ equipped with fibrations $f: Y \to S^1$ (along with some additional structure indicated below) with fiber genus at least 2,[1] with $Hom((Y_-, f_-), (Y_+, f_+))$ consisting of Lefschetz fibrations $f: X \to B$ over

[1]Throughout this paper, the genera of the fibers of all surface fibrations will be implicitly assumed to be at least two, unless indicated otherwise.

a base with two boundary components $\partial_- B$ and $\partial_+ B$ such that $f^{-1}(\partial_\pm B) = Y_\pm$ and $f|_{Y_\pm} = f_\pm$. The "additional structure" alluded to earlier on an object $(Y, f)$ consists of a homology class

$$h \in H_1(Y; \mathbb{Z}),$$

a cohomology class

$$c \in H^2(Y; \mathbb{R})$$

which evaluates positively on the fibers of $f$, and

$$\text{a real number } \tau \in (2\pi h \cap [fiber], +\infty).$$

To each such tuple $(Y, f, h, c, \tau)$ and suitable ring $A$ (often, $A$ will be a Novikov ring) we associate an $A$–module $HF(Y, f, h, c, \tau; A)$. These groups have appeared in the literature before: in a paper of D. Salamon [38] they were conjectured to agree with the (at the time not-yet-rigorously-defined) monopole Floer groups of $Y$. As will be explained in more detail below, the fibration $f\colon Y \to S^1$ singles out a canonical spin$^c$ structure, which provides an identification of $Spin^c(Y)$ with $H_1(Y; \mathbb{Z})$. Let $s_h$ be the spin$^c$ structure corresponding to $h \in H_1(Y; \mathbb{Z})$ under this identification. Salamon's conjecture can then be restated as saying that,

$$HF(Y, f, h, c, \tau; A) \cong HM(Y, s_h, \eta(h, c, \tau); A),$$

where, in an appropriate normalization, $\eta(h, c, \tau) = 4\tau c + 2\pi c_1(s_h)$. (The normalization on $\eta$ in this formula is such that $c = 0$ (if it were allowed) would correspond to a "balanced perturbation" [19], *i.e.*, a perturbation as in the hypothesis of Conjecture 1.1 of [23]; our requirement that $c$ pair positively with the fiber thus ensures that reducible solutions to the Seiberg–Witten equations will not enter the picture (so that $\overline{HM}(Y, s_h, \eta(h, c, \tau)) = 0$ and there is just one nontrivial monopole Floer group corresponding to the perturbation $\eta(h, c, \tau)$, making the notation $HM(Y, s_h, \eta(h, c, \tau); A)$ unambiguous) and that, in case $b_1(Y) = 1$, all allowed values of $c$ and $\tau$ will put $\eta(h, c, \tau)$ on the same side of the "wall" familiar from Seiberg–Witten theory.) Note here the general principle that the choice of $h$ in $HF$ corresponds to the choice of a spin$^c$ structure in $HM$, while (given $h$) the choice of $c$ corresponds to the choice of a cohomology class of perturbation 2–forms in $HM$ (up to a scale factor determined by $\tau$). We should caution that the fact that $\eta(h, c, \tau)$ is not a balanced perturbation means that in general $HF$ is not conjectured to coincide with the Heegaard Floer group $HF^+$; rather, they should be related by a change of coefficients as detailed in Chapter VIII of [19].

The following four subsections summarize our results concerning these groups $HF$. Section 2 contains the explicit construction of the groups, after which the results of subsections 1.1, 1.2, 1.3, and 1.4 are proven in Sections 3, 4, 5, and 6, respectively.

1.1. **Coefficient rings.** Typically, the natural choice for the coefficient ring $A$ in $HF(Y, f, h, c, \tau; A)$ will be a Novikov ring $\tilde{\Lambda}_{h,c}$ or $\Lambda_{h,c}$ which (as the notation indicates) depends on the choices of $h \in H_1(Y; \mathbb{Z})$ and $c \in H^2(Y; \mathbb{R})$. Let $R$ be a ring (usually $\mathbb{Z}$, $\mathbb{Z}/2$, or $\mathbb{Q}$), $G$ an abelian group, and $N\colon G \to \mathbb{R}$ a homomorphism. Following the notation of [21], the Novikov ring $Nov(G, N; R)$ is defined to be the set of formal sums $\sum_{g \in G} a_g \cdot g$ ($a_g \in R$) satisfying the property that for every $C \in \mathbb{R}$ we have $\#\{g | a_g \neq 0 \text{ and } N(g) < C\} < \infty$, with addition and multiplication in $Nov(G, N; \mathbb{R})$ defined as the obvious extensions of the corresponding operations on the group ring $R[G]$. For us the most common Novikov rings will be, aside from the universal Novikov ring mentioned below,

$$\tilde{\Lambda}_{h,c} = Nov\left(\ker\langle c_1(s_h), \cdot\rangle, \langle c, \cdot\rangle; R\right) \tag{1}$$

and

$$\Lambda_{h,c} = Nov\left(\frac{\ker\langle c_1(s_h), \cdot\rangle}{\ker\langle c_1(s_h), \cdot\rangle \cap \ker\langle c, \cdot\rangle}, \langle c, \cdot\rangle; R\right) \tag{2}$$

where $\langle c_1(s_h), \cdot\rangle$ and $\langle c, \cdot\rangle$ are the evaluation homomorphisms $H_2(Y;\mathbb{Z}) \to \mathbb{R}$.

Obviously, multiplying $c$ by a positive constant leaves $\Lambda_{h,c}$ unchanged. Recall also that $HM(Y, s, \eta)$ naturally has coefficients in

$$Nov\left(\ker\langle c_1(s), \cdot\rangle, \langle 2\pi(\pi c_1(s) - \eta), \cdot\rangle; R\right),$$

which is the same as $\tilde{\Lambda}_{h,c}$ in the event that $s = s_h$ and $\eta = \eta(h, c, \tau)$, consistently with Salamon's conjecture.

One checks easily that $\Lambda_{h,c}$ as defined above embeds via the ring homomorphism

$$\sum a_g g \mapsto \sum a_g T^{\langle c,g\rangle}$$

as a subring of the universal Novikov ring

$$\Lambda^R_{Nov} = \left\{ \sum_i a_i T^{\lambda_i} | a_i \in R, \, (\forall C > 0)(\#\{i|\lambda_i < C\} < \infty) \right\}.$$

We obtain our groups $HF$ as the homology of a chain complex $CF(Y, f, h, c, \tau)$ which naturally has its coefficients in $\tilde{\Lambda}_{h,c}$. If $A$ is any algebra over $\tilde{\Lambda}_{h,c}$, $HF(Y, f, h, c, \tau; A)$ is by definition the homology of $CF(Y, f, h, c, \tau) \otimes_{\tilde{\Lambda}_{h,c}} A$. Of course, as a particular example of this, the projection

$$\ker\langle c_1(s_h), \cdot\rangle \to \frac{\ker\langle c_1(s_h), \cdot\rangle}{\ker\langle c_1(s_h), \cdot\rangle \cap \ker\langle c, \cdot\rangle}$$

makes $\Lambda_{h,c}$ into an algebra over $\tilde{\Lambda}_{h,c}$.

Observe that when $c = \pm c_1(s_h)$, $\Lambda_{h,c}$ is just the ring $R$ over which we are working while $\tilde{\Lambda}_{h,c} = R[\ker\langle c_1(s_h), \cdot\rangle]$, so no Novikov ring is needed in this case. This choice of $c$ corresponds to the choice made in the construction of periodic Floer homology; see [15]. In this case, for any other $\tilde{c} \in H^2(X;\mathbb{R})$ evaluating positively on the fibers of $f$, $\tilde{\Lambda}_{h,\tilde{c}}$ and $\Lambda_{h,\tilde{c}}$ are obviously algebras over $R[\ker\langle c_1(s_h), \cdot\rangle] = \tilde{\Lambda}_{h,\pm c_1(s_h)}$, so if $\langle c_1(s_h), fiber\rangle \neq 0$ we have a well defined group $HF(Y, F, h, \pm c_1(s_h), \tau; \tilde{\Lambda}_{h,\tilde{c}})$, where the sign at the front of $\pm c_1(s_h)$ is chosen to make its evaluation on the fiber positive.

We mention here that, while the presence of a Novikov ring such as $\tilde{\Lambda}_{h,c}$ as the natural coefficient ring is a standard aspect of Floer theory, the fact that this Novikov ring is described directly in terms of the homology of $Y$ and is (crucially for the invariance theorem below) independent of $\tau$ is more subtle. This fact follows from two basic ingredients: a formula from [34] for a certain cohomology class on the vortex moduli space; and an expression for the evaluation of one of the terms in that formula on certain cycles, derived below as Equation 10, which enables us to choose the forms $\omega_{c,h,\tau}$ at the start of Section 2 in such a way as to arrange that $\tilde{\Lambda}_{h,c}$ be the appropriate Novikov ring.

The following theorem, whose proof uses the $\tau$-independence of the Novikov ring along with the difficult bifurcation analysis carried out by Y.-J. Lee in [21],[22], shows that at least in the majority of cases, the groups $HF$ are independent of $\tau$, and depend on $c$ only to the extent that $c$ determines the appropriate coefficient ring.

**Theorem 1.1.** *Writing $d = \langle PD(h), fiber\rangle$ and letting $g$ be the genus of the fibers of $f\colon Y \to S^1$, assume that either $d \geq g-1$ or $d < (g+1)/2$. Then:*

(i) *$CF(Y, f, h, c, \tau)$ and $CF(Y, f, h, c, \tau')$ are canonically chain homotopy equivalent whenever $\tau, \tau' \in \mathbb{R}$ are such that both chain complexes are defined.*

(ii) *Assume that $\tilde{c}$ and $\pm c_1(s_h)$ both evaluate positively on the fiber. Then for any $\tau > 2\pi d$, $CF(Y, f, h, \tilde{c}, \tau)$ is chain homotopy equivalent to*

$$CF(Y, f, h, \pm c_1(s_h), \tau) \otimes_{R[\ker\langle c_1(s_h), \cdot\rangle]} \tilde{\Lambda}_{h,\tilde{c}}.$$

A similar result holds for the dependence of $HM(Y, s, \eta)$ on $\eta$, as is shown in Section 31 of the current draft version of [19].

1.2. **Grading, module structure, duality, and local coefficients.** As has been observed independently in the thesis of T. Perutz [34], the groups $HF(Y, f, h, c, \tau; A)$ share additional algebraic structure with the monopole Floer groups. Throughout the paper, to a 3–manifold $Y$ associate the graded ring

$$\mathbb{A}(Y) = \mathbb{Z}[U] \otimes \Lambda^*(H_1(Y)/torsion),$$

where $U$ is a formal variable of degree 2 and elements of $H_1(Y)$ have degree 1. Then:

**Proposition 1.2.** *$HF(Y, f, h, c, \tau; A)$ is a naturally $\mathbb{Z}/2$-graded, relatively $\mathbb{Z}/\mathfrak{d}(s_h)$-graded module over $\mathbb{A}(Y)$, where*

$$\mathfrak{d}(s_h) = \gcd_{T \in H_2(Y;\mathbb{Z})} \langle c_1(s_h), T\rangle$$

*and the action of an element of degree $p$ of $\mathbb{A}(Y)$ on $HF(Y, f, h, c, \tau; A)$ decreases the relative grading by $p$.*

Since $HM$ likewise enjoys these properties (as seen, for instance, in sections 3.1 and 3.2 of [19]), it is natural to embellish Salamon's conjecture to state that the (conjectural) isomorphism between $HF$ and $HM$ is an isomorphism of graded modules.

For the "Poincaré duality" property (which, like Proposition 1.2, also appears in [34]), given $(Y, f, h, c, \tau)$, let $(-Y, \bar{f}, h, c, \tau)$ be obtained by reversing the orientation of $Y$ and composing $f$ with complex conjugation.

**Proposition 1.3.** *There is a perfect pairing*

$$\langle\cdot,\cdot\rangle\colon\ CF(Y, f, h, c, \tau; A) \otimes CF(-Y, \bar{f}, -h, c, \tau; A) \to A$$

*which satisfies*

$$\langle \partial_Y a, b\rangle = \langle a, \partial_{-Y} b\rangle$$

*and hence descends to a pairing which identifies $HF(-Y, \bar{f}, -h, c, \tau; A)$ with the dual of $HF(Y, f, h, c, \tau; A)$.*

A handy device in monopole Floer theory is the use of "local coefficients" in $HM$, in which a singular 1–cycle $\gamma$ in $Y$ gives rise to a twisted version $HM(Y, s, \eta; \Gamma_\gamma)$ arising from a twisted coefficient system $\Gamma_\gamma$ on configuration space associated to $\gamma$; homologous 1-cycles yield isomorphic coefficient systems and hence isomorphic

Floer groups, but a homology between the cycles must be specified in order to make these isomorphisms canonical. A parallel situation, which seems most naturally expressed in terms of closed 2–forms on $Y$ rather than their dual 1–cycles, exists for our Floer groups, and we expect the resulting twisted groups to be isomorphic to their monopole Floer counterparts:

**Proposition 1.4.** *To each closed 2–form $\theta \in \Omega^2(Y)$, we may associate a local coefficient system $\Gamma_\theta$ and hence Floer groups $HF(Y, f, h, c, \tau; \Gamma_\theta)$. If $\theta_1, \theta_2$ are closed 2–forms, to each $\zeta \in \Omega^1(Y)$ such that $d\zeta = \theta_2 - \theta_1$ there is associated a canonical isomorphism $\phi_\zeta \colon \Gamma_{\theta_1} \to \Gamma_{\theta_2}$, which then induces an isomorphism of the associated Floer groups.*

1.3. **Cobordisms.** We define a category $\mathsf{FCOB}$ (for "fibered cobordism") as follows. An object $\mathfrak{o}$ of $\mathsf{FCOB}$ is a quintuple $\mathfrak{o} = (Y, f, h, c, \tau)$ where, as before, $Y$ is an oriented 3–manifold (we allow $Y$ to be empty, in which case $f, h, c, \tau$ need not be specified), $f \colon Y \to S^1$ is a fibration, $h \in H_1(Y; \mathbb{Z})$, $c \in H^2(Y; \mathbb{R})$, and $\tau \in (2\pi h \cap [fiber], +\infty)$ (when $h \cap [fiber]$ is outside the interval $[(g + 1)/2, g - 1)$ Theorem 1.1 ensures that the Floer homology $HF(\mathfrak{o})$ associated to the object will be independent of $\tau$ up to canonical isomorphism; these isomorphisms will commute with the homomorphisms decribed below). A morphism $\mathfrak{m} = (X, \tilde{f}, \tau)$ in $Mor(\mathfrak{o}_-, \mathfrak{o}_+)$ consists of a Lefschetz fibration $\tilde{f} \colon X \to B$ defined on a 4–manifold $X$ with oriented boundary $\partial X = (-Y_-) \coprod Y_+$, with two-dimensional image $B$ having boundary components $\partial_- B$, $\partial_+ B$; here $\partial_\pm B = S^1$ if $Y_\pm$ is nonempty, and otherwise $\partial_\pm B = \varnothing$. We require $\tilde{f}^{-1}(\partial_\pm B) = Y_\pm$ and $\tilde{f}|_{Y_\pm} = f_\pm$, and $\tau = \tau_+ = \tau_-$. Furthermore, where we denote by $\partial_\pm \colon H_2(X, \partial X; \mathbb{Z}) \to H_1(Y_\pm; \mathbb{Z})$ the obvious maps induced by restriction to the boundary, we require that the sets

$$C_{c_-, c_+} = \{\tilde{c} \in H^2(X; \mathbb{R}) | \tilde{c}|_{Y_\pm} = c_\pm\}$$

and

$$H_{h_-, h_+} = \{\tilde{h} \in H_2(X, \partial X; \mathbb{Z}) | \partial_\pm \tilde{h} = h_\pm\}$$

both be nonempty.

We also place the following additional structure on $B$:

**Definition 1.5.** *A "starred surface with boundary" is an oriented surface $B$ with (say) genus $g$ and $n$ boundary components equipped with distinguished points, arcs, and parametrized loops as follows:*

(0) *The distinguished points comprise one "interior base point" $b$, $s \geq 0$ 'interior special points" $p_1, \ldots, p_s$, and one "boundary base point" $q_j$ on each of the $n$ boundary components, and*

(1) *There are $n + s$ distinguished arcs, namely one from $b$ to $p_i$ for each $i$ and one from $b$ to $q_j$ for each $j$, one distinguished loop (namely the boundary component) based at each $q_j$, and $2g$ distinguished interior loops $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$, such that $\alpha_1, \ldots, \alpha_g$ are based at $b$, are linearly independent in $H_1(B; \mathbb{Z})$, and represent homotopy classes having zero geometric intersection number, and $\beta_1, \ldots, \beta_g$ are disjoint, with $\#(\alpha_i \cap \beta_j) = \delta_{ij}$.*

*We require also that the distinguished arcs and loops have no intersections other than the ones implied by the above conditions.*

We equip the base $B$ of the Lefschetz fibration $\tilde{f} \colon X \to B$ with the structure of a starred surface with boundary, with interior special points comprising precisely the critical values of $\tilde{f}$, and with the boundary loop at $\partial_- B$ (resp. $\partial_+ B$) negatively (resp. positively) oriented. Finally, for technical reasons that shall appear in Lemma 5.1 and its proof, in the event that $g(B) > 0$ we take as given in the data of the morphism $\mathfrak{m}$ a set of cohomology classes $b_i \in H^2(\tilde{f}^{-1}(\beta_i); \mathbb{R})$ with the property that there are $\tilde{c} \in C_{c_-,c_+}, \tilde{h} \in H_{h_-,h_+}$ such that $b_i = (\tilde{c} + 2\pi PD(\tilde{h})/\tau)|_{\tilde{f}^{-1}(\beta_i)}$ for $i = 1, \ldots, g(B)$. For the sake of conciseness, we shall nonetheless generally denote morphisms with the notation $\mathfrak{m} = (X, \tilde{f}, \tau)$, suppressing our marking of $B$ and our choice of classes in the $H^2(\tilde{f}^{-1}(\beta_i); \mathbb{R})$.

Note that any two starred surfaces $B_1, B_2$ with interior base points $b_1, b_2$ nonempty boundary may be glued along any of their common boundary components $S_1, S_2$ (if necessary after reversing the parametrization of one of the $S_i$) to obtain a new starred surface $B_1\, {}_{S_1}\sharp_{S_2} B_2$ with (possibly empty) boundary as follows. We join the corresponding boundary components at their corresponding boundary base points; this in particular yields a path $\gamma$ from $b_1$ to $b_2$. To get the new distinguished arcs, delete the loop at the former boundary base point, and extend all the paths (and loops) based at $b_2$ in $B_2$ to paths (and loops) based at $b_1$ in $B_1 \sharp B_2$ by adding on the path $\gamma$ from $b_1$ to $b_2$. This construction applies equally well when $B_1 = B_2 = B$ (as long as $S_1 \neq S_2$); in this case $B_{S_1}\sharp_{S_2} B$ will have genus one larger than $B$, and the path $\gamma$ appears as a new $\alpha$-curve while the loop resulting from the fusing of the old boundary components becomes a new $\beta$-curve. In particular, a starred Riemann surface $B$ with genus $g$ and $n$ boundary components can be cut along its $\beta$-curves to obtain a starred Riemann surface $B^0$ with genus 0 and $n+2g$ boundary components, so that $B$ is recovered from $B^0$ by applying the gluing construction $g$ times.[2]

Given an object $\mathfrak{o} = (Y, f, h, c, \tau)$ of $\mathsf{FCOB}$, we have a well-defined group $HF(\mathfrak{o}; \Lambda^R_{Nov})$, where again $\Lambda^R_{Nov}$ is the universal Novikov ring over the ring $R$; more generally if $\theta \in \Omega^2(Y)$ is closed we may consider the twisted Floer homology group $HF(\mathfrak{o}; \Lambda^R_{Nov} \otimes \Gamma_\theta)$ . A crucial property of $HF$, a similar version of which was independently discovered in [34], is that it is a functor from $\mathsf{FCOB}$ to the category $\mathsf{MOD}_{\Lambda^R_{Nov}}$ of modules over $\Lambda^R_{Nov}$.

Denote by $\varnothing$ the object of $\mathsf{FCOB}$ whose underlying 3–manifold is the empty set. By way of definition, we set $HF(\varnothing; \Lambda^R_{Nov}) = \Lambda^R_{Nov}$.

Given morphisms $\mathfrak{m}_0 \in Mor(\mathfrak{o}_0, \mathfrak{o}_1)$ and $\mathfrak{m}_1 \in Mor(\mathfrak{o}_1, \mathfrak{o}_2)$, we define the composite morphism $\mathfrak{m}_1 \circ \mathfrak{m}_0 \in Mor(\mathfrak{o}_0, \mathfrak{o}_2)$ by the obvious procedure of gluing the total spaces $X_0$ and $X_1$ of the Lefschetz fibrations $\tilde{f}_0, \tilde{f}_1$ underlying $\mathfrak{m}_0$, $\mathfrak{m}_1$ along their common boundary component $Y_1$ to obtain a new Lefschetz fibration $f\colon X \to B$; as noted earlier $B$ inherits the structure of a starred surface with boundary from those of the bases of the $\tilde{f}_i$. If we choose $\tilde{c}_0 \in C_{c_0,c_1}$ and $\tilde{c}_1 \in C_{c_1,c_2}$, so that in particular $\tilde{c}_0$ and $\tilde{c}_1$ have the same restriction to $Y_1$, then the Mayer–Vietoris sequence reveals that the set of $\tilde{c} \in H^2(X; \mathbb{R})$ such that $\tilde{c}|_{X_i} = \tilde{c}_i$ $(i = 0, 1)$ is nonempty and

[2] One might naturally ask whether it is really necessary to incorporate the complication of starred surfaces into the theory; the reason we have done so is that it will help us construct certain 2–forms on symmetric product bundles in an essentially canonical way. If we were only considering a single cobordism, say from $\mathfrak{o}_-$ to $\mathfrak{o}_+$, it would suffice for the construction of the desired 2–form to replace the starred surface structure with an auxiliary choice of a cohomology class from the set $C_{c_-,c_+}$. However, we wish to compose our cobordisms, and the set of possible choices of cohomology class on the composed cobordism having the appropriate restrictions to the pieces is generally a positive-dimensional affine space, so that there is no canonical way to glue cohomology classes.

is an affine space over the image of the boundary map $\delta\colon H^1(Y;\mathbb{R}) \to H^2(X;\mathbb{R})$. Poincaré duality implies a parallel statement for the $h_i$. In particular $\mathfrak{m}_1 \circ \mathfrak{m}_o$ is a morphism (for $C_{c_0,c_2}$ and $H_{h_0,h_2}$ are nonempty, and where relevant the $\beta$-curves on $B$ are just those on the $B_i$, so we can use the same cohomology classes on the preimages of the $\beta$-curves as were used on the $\mathfrak{m}_i$).

**Theorem 1.6.** *To each morphism $\mathfrak{m} = (X, \tilde{f}, \tau)$ from $\mathfrak{o}_-$ to $\mathfrak{o}_+$, where $\mathfrak{o}_\pm = (Y_\pm, f_\pm, h_\pm, c_\pm, \tau)$, and to each closed form $\theta \in \Omega^2(X)$ vanishing near the critical points of $f\colon X \to B$, we may associate a homomorphism*

$$F_{\mathfrak{m},\theta}\colon \mathbb{A}(X) \otimes HF(\mathfrak{o}_-;\Lambda^R_{Nov} \otimes \Gamma_{\theta|_{Y_-}}) \to HF(\mathfrak{o}_+;\Lambda^R_{Nov} \otimes \Gamma_{\theta|_{Y_+}})$$

*where $\mathbb{A}(X) = \mathbb{Z}[U] \otimes \Lambda^*(H_1(X;\mathbb{Z})/torsion)$; in fact, each map $F_{\mathfrak{m},\theta}$ decomposes naturally as a sum*

$$F_{\mathfrak{m},\theta} = \sum_{\tilde{h}\in H_{h_-,h_+}} F_{\mathfrak{m},\theta,\tilde{h}},$$

*and these maps enjoy the following properties:*

(i) *For morphisms $\mathfrak{m}_0 = (X_0, f_0, \tau)$ from $\mathfrak{o}_0 = (Y_0, f_0, h_0, c_0, \tau)$ to $\mathfrak{o}_1 = (Y_1, f_1, h_1, c_1, \tau)$ and $\mathfrak{m}_1 = (X_1, f_1, \tau)$ from $\mathfrak{o}_1$ to $\mathfrak{o}_2 = (Y_2, f_2, h_2, c_2, \tau)$, for $\theta$ a closed 2-form on the total space $X = X_0 \cup_Y X_1$ of the Lefschetz fibration underlying $\mathfrak{m}_1 \circ \mathfrak{m}_0$, for $\tilde{h}_0 \in H_{h_0,h_1}$, $\tilde{h}_1 \in H_{h_1,h_2}$, and for $v \in HF(\mathfrak{o}_0;\Lambda^R_{Nov} \otimes \Gamma_{\theta|_{Y_0}})$ we have*

$$\sum_{\substack{\tilde{h}\in H_{h_0,h_2}:\\ \tilde{h}|_{X_0}=\tilde{h}_0,\tilde{h}|_{X_1}=\tilde{h}_1}} F_{\mathfrak{m}_1\circ\mathfrak{m}_0,\theta,\tilde{h}}(U^{k+l} \otimes 1 \otimes v) =$$
$$F_{\mathfrak{m}_1,\theta|_{X_1},\tilde{h}_0}(U^k \otimes 1 \otimes F_{\mathfrak{m}_0,\theta|_{X_0},\tilde{h}_1}(U^l \otimes 1 \otimes v)).$$

(ii) *Where $i_-\colon \mathbb{A}(Y_-) \to \mathbb{A}(X)$ is the map induced by the action of the inclusion $Y_- \subset X$ on $H_1$, $F_{\mathfrak{m},\theta,\tilde{h}}$ is compatible with the module structure of $HF$ in the sense that, for $\lambda \in \mathbb{A}(Y_-)$ and $v \in HF(\mathfrak{o}_-;\Lambda^R_{Nov} \otimes \Gamma_\theta)$,*

$$F_{\mathfrak{m},\theta,\tilde{h}}(1 \otimes \lambda \cdot v) = F_{\mathfrak{m},\theta,\tilde{h}}(i_-(\lambda) \otimes v).$$

(iii) *Where $-\mathfrak{m} \in Mor(\mathfrak{o}_+, \mathfrak{o}_-)$ is the morphism obtained by reversing the orientation of the boundary components of the base of $\tilde{f}$, with respect to the pairing in Proposition 1.3 we have*

$$\langle F_{\mathfrak{m},\theta,\tilde{h}}(v), w\rangle_{\mathfrak{o}_+} = \langle v, F_{-\mathfrak{m},\theta,\tilde{h}}(w)\rangle_{\mathfrak{o}_-}.$$

(iv) *Suppose that $\tilde{f}\colon X \to \Sigma$ is a Lefschetz fibration on the* closed *manifold $X$ over the closed surface $\Sigma$, so that $\mathfrak{o}_- = \mathfrak{o}_+ = \varnothing$, and $H_{h_-,h_+} = H_2(X;\mathbb{Z})$. Then for a certain homomorphism $A\colon H_2(X;\mathbb{Z}) \to \mathbb{R}$ and for $\mathfrak{m} = (X, f, \tau)$ and $\theta \in \Omega^2(X)$ representing $[\theta] \in H^2(X;\mathbb{R})$ we have,*

$$F_{\mathfrak{m},\theta,\tilde{h}}(U^r \otimes (\eta_1 \wedge \cdots \wedge \eta_k) \otimes 1) = T^{A(\tilde{h})} e^{\langle[\theta],\tilde{h}\rangle} DS_{(X,\tilde{f})}(\tilde{h}; pt^r, \eta_1, \ldots, \eta_k). \tag{3}$$

*Here $DS_{(X,\tilde{f})}(\tilde{h}; pt^r, \eta_1, \ldots, \eta_k)$ is the obvious extension of the Donaldson–Smith invariant [7] to an invariant counting sections of the relative Hilbert scheme of $f$ which correspond to surfaces in $X$ representing $\tilde{h} \in H_2(X;\mathbb{Z})$ and passing through $r$ generic points and through generic cycles representing $\eta_1, \ldots, \eta_k$.*

Our TQFT thus contains the Donaldson–Smith invariant, which, thanks to [45] and [46], is known to agree with the four-dimensional Seiberg–Witten invariant under a natural (given the symplectic or Lefschetz fibration structure on $X$) correspondence between $H_2(X;\mathbb{Z})$ and $Spin^c(X)$. As is well-known (and shown in detail in Chapter VII of [19]), there is a TQFT in Seiberg–Witten theory whose cobordism maps enjoy properties exactly parallel to those of Theorem 1.6, with the Seiberg–Witten invariant appearing in place of the (equivalent, by [45] and [46]) Donaldson–Smith invariant in part (iv). As such, we may further embellish Salamon's conjecture to state that the conjectural isomorphisms between $HF$ and $HM$ commute with the cobordism maps; the agreement of $DS$ with $SW$ would then be a shadow of this relationship.

In [34], Perutz uses and extends constructions similar to this in order to construct a "Lagrangian matching invariant" for the singular Lefschetz fibrations constructed in [1] (which exist on blowups for any 4–manifold with $b^+ > 0$, though it is not known whether Perutz's invariant is independent of the choice of singular Lefschetz fibration on a given 4–manifold), and conjectures that this new invariant, too, agrees with the Seiberg–Witten invariant.

1.4. **Relation to dynamics of surface symplectomorphisms.** Given a symplectomorphism $\phi\colon (\Sigma,\omega)\to(\Sigma,\omega)$ of a symplectic 2–manifold of genus at least 2, form the mapping torus $Y_\phi = \mathbb{R}\times\Sigma/(t+1,x)\sim(t,\phi(x))$; this fibers over $S^1$ and carries a fiberwise symplectic form $\omega_\phi$ obtained by pushing forward the obvious form induced by $\omega$ on $\mathbb{R}\times\Sigma$ via the projection $\mathbb{R}\times\Sigma\to Y_\phi$. Let $j$ be an almost complex structure on the fibers of $f\colon Y_\phi\to S^1$ and $h\in H_1(Y_\phi;\mathbb{Z})$; where $e$ is the Euler class of the vertical tangent bundle of $Y_\phi\to S^1$, under several assumptions on $\phi$ and $j$, including that $[\omega_\phi]\in H^2(Y_\phi;\mathbb{R})$ is proportional to $c_1(s_h)=e+2PD(h)$, the *periodic Floer homology* $HP(\phi,h,j)$ is defined in [17] to be the homology of a chain complex $CP(\phi,h)$ whose generators are "admissible orbit sets" $\alpha=\{(\alpha_i,m_i)\}$ such that $\sum_i m_i[\alpha_i]=h$. Here the $\alpha_i$ are periodic orbits for $\phi$ (which then give rise naturally to loops in $Y$), and the $m_i$ are positive integers such that $m_i=1$ if $\alpha_i$ is hyperbolic. The matrix element $\langle\partial\alpha,\beta\rangle$ for the boundary operator $\partial$ of the chain complex counts certain embedded holomorphic curves $C$ in $\mathbb{R}\times Y$ such that $C\cap(\{t\}\times Y)$ is asymptotic to $\alpha$ (resp. $\beta$) as $t\to-\infty$ (resp. $t\to+\infty$). Note that $CP(\phi,h)$ is independent of $j$ as a graded group; the same is expected to be true of $HP(\phi,h,j)$, but this has not yet been proven.

$HP(\phi,h,j)$ is defined over the coefficient ring $\mathbb{Z}$; more generally, if $[\omega_\phi]=c\in H^2(X;\mathbb{R})$, one could define a periodic Floer homology $HP(\phi,h,c,j)$ over the same Novikov ring $\tilde{\Lambda}_{h,c}$ as in (2).

Let $d=h\cap[fiber]$. $\phi$ then induces a continuous (but usually not differentiable) map $S^d\phi\colon S^d\Sigma\to S^d\Sigma$, where $S^d\Sigma$ is the $d$th symmetric product of $\Sigma$. Letting $Y_{S^d\phi}$ denote the mapping torus of $S^d\phi$, $h\in H_1(Y_\phi;\mathbb{Z})$ naturally determines a homotopy class $p_h$ of sections of $Y_{S^d\phi}$, and any generator $\alpha=\{(\alpha_i,m_i)\}$ of $CP(\phi,h)$ corresponds to a fixed point $D_\alpha$ of $S^d\phi$ such that the "constant section" of $Y_{S^d\phi}$ at $D_\alpha$ represents $p_h$ (in such a situation we say $D_\alpha$ is "in the $p_h$-sector"). The admissibility condition on the hyperbolic orbits in $\alpha$ prevents this from being a one-to-one correspondence. Notably, at least in some simple cases, one can perturb the (usually only Hölder continuous) map $S^d\phi$ to a smooth map $\Phi\colon S^d\Sigma\to S^d\Sigma$ such that the fixed points of $\Phi$ in the $p_h$-sector are precisely the $D_\alpha$ for *admissible* $\alpha$; the non-admissible fixed points disappear on this perturbation to a smooth map. For instance, if $\phi$ is given in local holomorphic coordinates near one of its hyperbolic fixed points by $x+iy \mapsto \lambda x+i\lambda^{-1}y$ where $\lambda > 1$ (*i.e.*, $z \mapsto az+\sqrt{a^2-1}\bar{z}$ where $a = (\lambda+\lambda^{-1})/2 > 1$), then in terms of the natural holomorphic coordinates $\sigma_1 = z_1+z_2$, $\sigma_2 = z_1z_2$ near $\{0, 0\}$ on $S^2\Sigma$, $S^2\phi$ is given by

$$\begin{aligned}(\sigma_1,\sigma_2) &\mapsto \left(a\sigma_1+\sqrt{a^2-1}\overline{\sigma_1}, a^2\sigma_2+(a^2-1)\overline{\sigma_2}+a\sqrt{a^2-1}(z_1\overline{z_2}+\overline{z_1}z_2)\right) \\ &= \left(a\sigma_1+\sqrt{a^2-1}\overline{\sigma_1}, a^2\sigma_2+(a^2-1)\overline{\sigma_2}+\frac{a}{2}\sqrt{a^2-1}(|\sigma_1|^2-|\sigma_1^2-4\sigma_2|)\right).\end{aligned}$$

One easily checks that leaving the first component of this function unchanged and adding an appropriate small imaginary-valued function supported near the origin to the second component results in a smooth function with no fixed points in the coordinate neighborhood under consideration.

With this in mind, we state a basic property of our groups $HF$ which suggests a connection to $HP$.

**Theorem 1.7.** *For any symplectomorphism $\phi$, $HF(Y_\phi, f, h, [\omega_\phi], \tau)$ arises as the homology of a chain complex whose generators are the fixed points in the $p_h$-sector of a smooth map $\Phi_\tau\colon S^d\Sigma \to S^d\Sigma$, where $\Phi_\tau \to S^d\phi$ in $C^0$-norm as $\tau \to \infty$.*

Note that the fixed points of $\Phi_\tau$ will, for large $\tau$, all be close to fixed points of $S^d\phi$; one would like to conclude that they will all be close to the fixed points coming from the admissible orbit sets that generate $HP$, but it is not clear that this is the case. We hope that further analysis of the maps $\Phi_\tau$ might make it possible to establish a correspondence between the generators and flowlines for $HP$ and those for $HF$ when $\tau$ is large enough and hence to equate the two groups, but this seems out of reach at present.

While we cannot go so far as to prove that $HF$ and $HP$ are equivalent, our results do suffice to imply the existence of periodic points with certain periods for a certain class of surface symplectomorphisms. Recall from [41] that a symplectomorphism $\phi\colon \Sigma \to \Sigma$ is called *monotone* provided that, where $Y_\phi$ is the mapping torus of $\phi$ and $\omega_\phi$ is the form on $Y_\phi$ induced by the symplectic form on $\Sigma$, the cohomology classes $e(T^{vt}Y_\phi)$ and $[\omega_\phi]$ are proportional. There are monotone symplectomorphisms in every mapping class, and if $\phi$ is monotone then so is any $\psi\circ\phi$ where $\psi$ is the flow of any possibly-time-dependent vector field symplectically dual to a (possibly-time-dependent) closed one-form representing an element of the image of $id-\phi^*\colon H^1(\Sigma;\mathbb{R}) \to H^1(\Sigma;\mathbb{R})$, so in particular the intersection of the space of monotone symplectomorphisms of $\Sigma$ with any component of the space of orientation-preserving diffeomorphisms of $\Sigma$ is infinite-dimensional. The following (which is a special case of known results previously established by rather different means) is the most general statement we can make about the dynamics of such symplectomorphisms.

**Corollary 1.8.** *Let $\phi\colon \Sigma \to \Sigma$ be a monotone symplectomorphism of a surface of genus $g \geq 2$. Then the induced map $S^{2g-2}\phi\colon S^{2g-2}\Sigma \to S^{2g-2}\Sigma$ on the $(2g-2)$th symmetric product of $\Sigma$ has a fixed point.*

As will be clear from our proof, for $\phi$ belonging to particular mapping classes the number $2g-2$ can be lowered depending on the properties of the Seiberg–Witten basic classes of the total spaces of Lefschetz fibrations having $\phi$ as monodromy (in particular, if $\phi$ is the monodromy around a loop of a Lefschetz fibration obtained via Donaldson's construction by blowing up a Lefschetz pencil, the basic class corresponding to a section of square $-1$ forces $\phi$ itself to have a fixed point). This connection seems to deserve further study.

Note that a fixed point of $S^{2g-2}\phi$ is equivalent to, for some partition $2g-2=\sum_{i=1}^{m} n_i d_i$, periodic orbits $o_1,\ldots,o_m$ of $\phi$ with minimal periods $d_1,\ldots,d_m$ respectively. For $g>2$, Corollary 1.8 can also be deduced via elementary methods: by considering the relationship of (what are now called) the Lefschetz numbers of the iterates of $\phi$ to the characteristic polynomial of the action of $\phi$ on $H^1(\Sigma;\mathbb{Z})$, Nielsen showed in [31] that an orientation-preserving homeomorphism $\phi$ must have a periodic point of period at most $2g-2$ and that this estimate is best possible; by examining his argument more carefully one can show that it implies that one of the Lefschetz numbers $L(\phi), L(S^2\phi), L(S^{2g-2}\phi)$ is nonzero, so that in any event $S^{2g-2}\phi$ has a fixed point. Since Nielsen's argument does not work for the case $g=2$, he asked in [31] whether orientation-preserving homeomorphisms of surfaces of genus 2 always have points of period at most 2; this question remained open for decades before eventually being answered affirmatively by Dicks and Llibre [4], using methods quite different from those we use in the special case considered here.

1.5. **Remarks and Acknowledgements.** During the development of this work, I learned of the thesis [34] of T. Perutz, which (among several other things) develops ideas parallel to some of those discussed in this paper. By and large, due to slight technical differences in our formulations, the proofs given here are independent of similar results in [34] (a notable exception is that I appeal to Perutz's calculation of a certain cohomology class, which is needed in the proof of Theorem 1.1 (which has no analogue in [34]) and simplifies some other arguments); this leads to some redundancy between some results in [34] and parts of Sections 4 and 5 of this paper, which seems justified because it ensures that the relevant results are in the particular form that we need, and hopefully makes this paper more readable than it otherwise would be. I am very grateful to T. Perutz for sending me his thesis and for some interesting conversations. Thanks are also due to C. Taubes for advice regarding vortices which played an essential role in the proof of Theorem 1.7; to M. Hutchings, Y.-J. Lee, and T. Mrowka for answering some questions regarding their work; and to the anonymous referee for his or her detailed comments and useful suggestions. This work was partially supported by an NSF postdoctoral fellowship.

## 2. Defining $HF$

Let $Y, f, h, c, \tau$ be as in the previous section. Our groups $HF(Y, f, h, c, \tau)$ are defined, adapting [38], as the Floer homologies of certain symplectomorphisms that the monodromy of $f$ induces on the symmetric products of the fibers of $f$. Note that whereas [38] begins with an explicit presentation of $Y$ as a mapping torus, we do not begin with such data; since the fibration $f\colon Y \to S^1$ only specifies the monodromy of $f$ up to smooth isotopy, we will need additional data to describe its symplectic behavior. The triple $(h, c, \tau)$ (principally, just $c$) provides these data: recall that $h \in H_1(X;\mathbb{Z})$ was arbitrary; $c \in H^2(X;\mathbb{R})$ was a class having positive pairing with the fiber of $f$, and $\tau > 2\pi h \cap [fiber]$.

Set $d = h \cap [fiber]$. First, if $d<0$, set $HF(Y, f, h, c, \tau; A) = 0$; this is consistent with adjunction relations in monopole Floer theory. If $d = 0$, set $HF(Y, f, h, c, \tau; A) = A$ if $h = 0$ and $HF(Y, f, h, c, \tau; A) = 0$ otherwise. Restrict attention now to the case $d > 0$.

We then have $\langle c + \frac{2\pi}{\tau}PD(h), [fiber]\rangle > 0$, which enables us to use the Thurston trick to find closed forms $\omega = \omega_{c,h,\tau} \in \Omega^2(Y)$ representing $c + \frac{2\pi}{\tau}PD(h)$ which restrict symplectically to every fiber of $f$. The $\omega$–orthogonal complement of $\ker f_* \subset TY$ then defines a horizontal subspace of $TY$, and so picking a basepoint in $S^1$ and flowing along the horizontal lift of the vector field $\partial_\theta$ on $S^1$ defines a symplectomorphism $\phi_\omega \colon \Sigma \to \Sigma$, where $\Sigma$ denotes the fiber of $f \colon Y \to S^1$ over the chosen basepoint in $S^1$. We now show that the Hamiltonian isotopy class of the symplectomorphism $\phi_\omega$ depends only on the class $[\omega] = c + \frac{2\pi}{\tau}PD(h) \in H^2(Y;\mathbb{R})$. (This fact is assuredly well-known and we indicate the proof only for completeness.)

**Proposition 2.1.** *Let $f \colon Y \to S^1$ be a fibration, and $\omega_0, \omega_1$ two closed forms on $Y$ which restrict to each fiber as volume forms and represent the same class in $H^2(Y;\mathbb{R})$. Then $\{\phi_{s\omega_1+(1-s)\omega_0}\}_{s\in[0,1]}$ is a Hamiltonian isotopy from $\phi_{\omega_0}$ to $\phi_{\omega_1}$.*

*Proof.* Note first that, for any $t \in S^1$, $\omega_0$ and $\omega_1$ are nonvanishing 2–forms on the 2–manifold $f^{-1}(t)$ which induce the same orientation, so the same statement applies to $\omega_s := s\omega_1 + (1-s)\omega_0$ for each $s$; in particular the symplectomorphism $\{\phi_{s\omega_1+(1-s)\omega_0}\}$ is well-defined.

Since $\omega_0$ and $\omega_1$ are cohomologous, write $\omega_1 = \omega_0 + d\alpha$. Now consider the fibration

$$\pi \colon [0,1] \times Y \to [0,1] \times S^1$$
$$(s, y) \mapsto (s, f(y));$$

the form $\Omega = \omega_0 + d(s\alpha)$ is then closed and restricts to each $Y = \pi^{-1}(\{s\} \times S^1)$ as $\omega_s$. If we let $\gamma_s$ be the loop in $[0,1] \times S^1$ obtained by juxtaposing the paths (each with domain $[0,1]$ for the parameter $t$) $t \mapsto (0, e^{2\pi i t})$, $t \mapsto (st, 1)$, $t \mapsto (s, e^{-2\pi i t})$, and $t \mapsto (s(1-t), 1)$, then the monodromy around $\gamma_s$ (using the horizontal lift of $\dot{\gamma_s}$ given by the $\Omega$–orthogonal complement of $\ker d\pi$) is $\phi_{\omega_s} \circ \phi_{\omega_0}^{-1}$ (modulo the identification of $\pi^{-1}(0,1)$ with $\pi^{-1}(s,1)$ via horizontal translation). But according to Proposition 6.31 of [27], since $\Omega \in \Omega^2([0,s] \times Y)$ is closed and $\gamma_s$ is contractible, the $\Omega$–monodromy around $\gamma_s$ is Hamiltonian. Thus each $\phi_s$ differs from $\phi_0$ by a Hamiltonian isotopy, and the proposition is proven. $\square$

Thus, specifying the cohomology class $c + \frac{2\pi}{\tau}PD(h) \in H^2(Y;\mathbb{R})$ specifies the monodromy $\phi_\omega$ of the fibration $f \colon Y \to S^1$ up to Hamiltonian symplectomorphism; this makes it reasonable to expect basic symplectic properties (*e.g.*, Floer homology) of $\phi$ to depend only on $c$, $h$, and $\tau$. The reader might wonder at this point why we are using the formula $c + \frac{2\pi}{\tau}PD(h)$ to refer to the cohomology class of the form on $Y$ rather than just, say, $c$ (as is done in [34]); the reason is that with this choice we ensure that the coefficient ring $\tilde{\Lambda}_{h,c}$ over which the still-to-be defined group $HF(Y, f, h, c, \tau)$ is naturally a module will depend only on $h$ and $c$ and not on $\tau$ (indeed, as will be seen in the proof of Theorem 1.1, under a technical assumption on $h \cap [fiber]$ this choice ensures that the groups themselves are independent of $\tau$).

We now explain the definition of the groups $HF$ when $d > 0$. Given the data $(Y, f, h, c, \tau)$, we choose a closed, fiberwise symplectic form $\omega$ representing the class $c + \frac{2\pi}{\tau}PD(h)$. As above, choosing a basepoint $1 \in S^1$ and using the horizontal distribution induced by $\omega$, we obtain a fiberwise symplectomorphism

$$(Y, \omega) \cong \left( \frac{\mathbb{R} \times \Sigma}{(t+1, x) \sim (t, \phi_\omega(x))}, \omega \right),$$

where $\Sigma = f^{-1}(1)$ and the symplectomorphism $\phi_\omega\colon (\Sigma, \omega|_\Sigma) \to (\Sigma, \omega|_\Sigma)$ is the monodromy of $f$. We shall follow [38] to define a family of symplectic forms $\Omega_{d,\omega,\tau}$ on the symmetric products $S^d\Sigma$, and a family of maps $\Phi_{d,\omega,\tau}\colon S^d\Sigma \to S^d\Sigma$ which are $\Omega_{d,\omega,\tau}$–symplectomorphisms.

These symplectomorphisms are obtained from a construction involving the $U(1)$ *vortex equations* on the fiber $\Sigma$. This construction is carried out in Sections 4 and 5 of [38]; we recall it here. Let $L$ be a Hermitian line bundle on $\Sigma$ of degree $d$. The configuration space $\mathcal{C}_L$ is defined as the space of pairs $(A, \theta)$ where $A$ is a Hermitian connection on $L$ and $\theta$ is a section of $L$ (in practice these should be viewed as elements of Sobolev spaces $L^2_k$ for $k > 2$, but for the most part we shall suppress these standard details; incidentally, our $\theta$ will be $(2\tau)^{-1/2}$ times the section $\Theta_0$ used in [38]). $J$ will denote an $\omega$–compatible complex structure on $\Sigma$. The $U(1)$ vortex equations for a pair $(A, \theta) \in \mathcal{C}_L$ are then

$$\begin{aligned} \bar\partial_{J,A}\theta &= 0 \\ iF_A &= \tau(1 - |\theta|^2)\omega. \end{aligned} \tag{4}$$

Now the tangent space to $\mathcal{C}_L$ is given by $T_{(A,\theta)}\mathcal{C}_L = \Omega^1(\Sigma; i\mathbb{R}) \times \Omega^0(E)$. There exists a *universal symplectic form* $\tilde\Theta \in \Omega^2(\mathcal{C}_L)$ given by

$$\tilde\Theta((\alpha_1, \theta_1), (\alpha_2, \theta_2)) = -\int_\Sigma \alpha_1 \wedge \alpha_2 + 2\tau \int_\Sigma \Im\langle\theta_1, \theta_2\rangle\omega.$$

Where again $J$ is an arbitrary complex structure on $\Sigma$ compatible with $\omega$,

$$\mathcal{X}_J = \{(A, \theta) \in \mathcal{C}_L | \bar\partial_{J,A}\theta = 0\}$$

is a symplectic submanifold of $(\mathcal{C}_L, \tilde\Theta)$. A Kähler structure is induced on both $\mathcal{X}_J$ and $\mathcal{C}_L$ by $\tilde\Theta$ together with the complex structure $(\alpha, \theta) \mapsto (*\alpha, i\theta)$.

One has an action of the gauge group $\mathcal{G} = L^2_{k+1}(\Sigma, S^1)$ by

$$u \cdot (A, \theta) = (A - u^{-1}du, u\theta);$$

in fact this is a Hamiltonian action on $\mathcal{C}_L$, with moment map

$$\mu\colon (A, \theta) \mapsto *iF_A + \tau|\theta|^2$$

(where $*\omega = 1$). The set $\tilde{\mathcal{M}}_{\Sigma,d}(J, \tau)$ of solutions to the vortex equations is thus $\mathcal{X}_J \cap \mu^{-1}(\tau)$, and the set

$$\mathcal{M}_{\Sigma,d}(J, \tau) = \tilde{\mathcal{M}}_{\Sigma,d}(J, \tau)/\mathcal{G}$$

of gauge equivalence classes of solutions to the vortex equations is the symplectic reduction of $\mathcal{X}_J$ by the action of $\mathcal{G}$.

An extension by O. Garcia-Prada [11] of a theorem of C. Taubes implies that the map

$$\mathcal{M}_{\Sigma,d}(J, \tau) \to S^d(\Sigma, J)$$

which sends an equivalence class $[A, \theta]$ to the zero set of $\theta$ is an isomorphism of complex manifolds. (We write $S^d(\Sigma, J)$ here to emphasize that the complex structure, and indeed even the $C^\infty$ charts, on $S^d\Sigma$ depend on the complex structure on $\Sigma$.) The form $\tilde\Theta$ on $\mathcal{C}_L$ descends to a symplectic form $\Theta_{J,\tau}$ on each $\mathcal{M}_{\Sigma,d}(J, \tau)$.

Let $\mathcal{J}(\Sigma)$ be the space of (almost) complex structures on $\Sigma$, and consider the space

$$\tilde{\mathcal{X}} = \{(J, A, \theta) | J \in \mathcal{J}(\Sigma), (A, \theta) \in \mathcal{X}_J\}.$$

$\tilde{\mathcal{X}}$ obviously fibers over $\mathcal{J}(\Sigma)$ with fiber $\mathcal{X}_J$, and $\tilde{\Theta}$ defines a closed, fiberwise symplectic form on this fibration. Carrying out the above symplectic reduction process fiberwise, we obtain a closed, fiberwise symplectic form $\hat{\Theta}_\tau$ on the fibration

$$\hat{\mathcal{X}} = \{(J, D)|D \in S^d(\Sigma, J)\} \to \mathcal{J}(\Sigma).$$

If

$$\begin{aligned} \gamma\colon [0,1] &\to \mathcal{J}(\Sigma) \\ t &\mapsto J_t \end{aligned}$$

is a smooth path of almost complex structures on $\Sigma$, we then obtain a symplectic fibration on $\gamma^*\hat{\mathcal{X}}$ with closed fiberwise symplectic 2–form $\gamma^*\hat{\Theta}_\tau$. Using the parallel translation given by the $(\gamma^*\hat{\Theta}_\tau)$–orthogonal complement of $T^{vt}\gamma^*\hat{\mathcal{X}}$ then gives a symplectomorphism

$$F_{\{J_t\}}\colon (S^d(\Sigma, J_0), \Theta_{J_0,\tau}) \to (S^d(\Sigma, J_1), \Theta_{J_1,\tau}). \tag{5}$$

With this preparation in hand, let us return to the data consisting of a fibration $f\colon Y \to S^1$ along with $h \in H_1(X;\mathbb{Z})$, $c \in H^2(X;\mathbb{R})$, and $\tau \in \mathbb{R}$; as earlier, we set $d = h \cap (fiber)$ and choose a fiberwise symplectic 2–form $\omega = \omega_{c,h,\tau}$ representing $c+\frac{2\pi}{\tau}PD(h)$; letting $\phi_\omega$ be the resulting monodromy map $\Sigma \to \Sigma$, the pair $(f\colon Y \to S^1, \omega)$ is identified as a symplectic fibration with the mapping torus of $\phi_\omega$. Now let $\bar{J}$ be an $\omega$–compatible almost complex structure on $T^{vt}Y$; the datum of $\bar{J}$ amounts to a path $J_t$ of almost complex structures such that $J_1 = \phi_\omega^* J_0$. Now if $\phi$ is any diffeomorphism of $\Sigma$ and $J$ any complex structure on $\Sigma$ there is a tautological Kähler isomorphism

$$S^d\phi\colon S^d(\Sigma, \phi^*J) \to S^d(\Sigma, J)$$

(we should caution here that if we were to instead view $S^d\phi$ as a map $S^d(\Sigma, J) \to S^d(\Sigma, J)$ it typically would not even be differentiable). We then set

$$\Phi_{d,\omega,\tau} = S^d\phi_\omega \circ F_{\{J_t\}}\colon (S^d(\Sigma, J_0), \Theta_{J_0,\tau}) \to (S^d(\Sigma, J_0), \Theta_{J_0,\tau}).$$

$\Phi_{d,\omega,\tau}$ is the composition of two symplectomorphisms and hence is a symplectomorphism.

**Proposition 2.2.** *If $\omega_0$ and $\omega_1$ are two representatives of $c + \frac{2\pi}{\tau}PD(h)$ as above, and $J_{0,t}$ (resp $J_{1,t}$) are $\omega_0$– (resp. $\omega_1$–) compatible almost complex structures, with $J_{s,t}$ a family of $(s\omega_1 + (1-s)\omega_0)$–compatible almost complex structures interpolating between them, then $\Phi_{d,\omega_0,\tau}$ is Hamiltonian equivalent to $F^{-1}_{\{J_{s,0}\}} \circ \Phi_{d,\omega_1,\tau} \circ F_{\{J_{s,0}\}}$.*

*Proof.* First note that, by the proof of Proposition 2.1, the forms $\omega_s = s\omega_1 + (1-s)\omega_0$ are all fiberwise symplectic on $Y$; in light of this the family of almost complex structures $J_{s,t}$ exists by the contractibility of $\mathcal{J}(\Sigma)$. Defining $\Gamma\colon [0,1]^2 \to \mathcal{J}(\Sigma)$, we have a closed fiberwise symplectic 2–form $\Gamma^*\hat{\Theta}$ on the fibration

$$\Gamma^*\hat{\mathcal{X}} = \{(s,t,D)|D \in S^d(\Sigma, J_{s,t})\} \to [0,1]^2,$$

and since each

$$S^d\phi_{\omega_s}\colon (S^d(\Sigma, J_{s,1}), \Theta_{J_{s,1},\tau}) \to (S^d(\Sigma, J_{s,0}), \Theta_{J_{s,0},\tau})$$

is a symplectomorphism, $\Gamma^*\hat{\Theta}$ descends to the mapping torus

$$\Gamma^*\hat{\mathcal{X}}/(s,1,D) \sim (s,0,S^d\phi(D)),$$

making this fibration over the cylinder a locally Hamiltonian fibration. Now the monodromy of this fibration around a loop similar to that in the proof of Proposition 2.1 (using, as usual, the form induced by $\Gamma^*\hat{\Theta}$ to determine the horizontal distribution) is $F^{-1}_{\{J_{s,0}\}} \circ \Phi_{d,\omega_1,\tau} \circ F_{\{J_{s,0}\}} \circ \Phi^{-1}_{d,\omega_0,\tau}$. But since the loop is contractible this monodromy is Hamiltonian by Proposition 6.31 of [27]. □

We can now finally define $HF(Y, f, h, c, \tau)$. If $\Phi\colon (X, \omega) \to (X, \omega)$ is a symplectomorphism of a symplectic manifold $X$ with nondegenerate fixed points, let $Y_\Phi \to S^1$ denote the mapping torus of $\Phi$ and $\omega_\Phi$ denote the 2–form on $Y_\Phi$ induced by the pullback of $\omega$ to $\mathbb{R} \times Y$. Recall then that, for $\mathcal{P} \in \pi_1(\Gamma(Y_\Phi))$, the Floer homology $HF^{symp}(\Phi, \mathcal{P})$ is obtained, naïvely, by Floer–Morse theory on the subset of $\Gamma(Y_\Phi)$ consisting of sections representing the homotopy class $\mathcal{P}$ using the action 1–form

$$\mathcal{Y}_\gamma(\xi) = -\int_0^1 \omega_\Phi(\dot{\gamma}(t), \xi(t))\, dt, \tag{6}$$

for $\gamma \in \mathcal{P}$ and $\xi \in T_\gamma\mathcal{P} = \gamma^* T^{vt} Y_\phi$ (if $\Phi$ has degenerate fixed points, a perturbation is used; see Section 3 of [21], which is the most thorough reference for this subject). The complex $CF^{symp}(\Phi, \mathcal{P})$ is then generated by fixed points $x$ of $\Phi$ with the property that the "constant section" of $Y_\Phi$ at $x$ belongs to the homotopy class $\mathcal{P}$; the boundary operator counts holomorphic cylinders in $\mathbb{R} \times Y_\Phi$. $CF^{symp}$ naturally has its coefficients in, depending on convention, either the Novikov ring

$$Nov\left(\frac{\ker\langle c_1(T^{vt}Y_\phi), \cdot\rangle}{\ker\langle c_1(T^{vt}Y_\phi), \cdot\rangle \cap \ker\langle\omega, \cdot\rangle}, \langle\omega, \cdot\rangle; R\right)$$

or the larger Novikov ring

$$Nov\left(\ker\langle c_1(T^{vt}Y_\phi), \cdot\rangle, \langle\omega, \cdot\rangle; R\right)$$

for a ring $R$ (in our context, for $g/2 + 1 < d < g - 1$, the virtual moduli methods of [24] are required, and so $R$ will need to be a field of characteristic zero).

We consider the case $X = S^d(\Sigma, J)$ and $\Phi = \Phi_{d,\omega,\tau}$. As is shown in Section 7 of [38], there is a one-to-one correspondence $\mathcal{P}_h \leftrightarrow h$ between

$$\pi_1(\Gamma(Y_{\Phi_{d,\omega,\tau}})) \text{ and } \{h \in H_1(Y_{\phi_\omega}; \mathbb{Z}) | h \cap (fiber) = d\}$$

(roughly speaking, a homotopy class $\mathcal{P}$ of sections of the bundle $Y_{\Phi_{d,\omega,\tau}} \to S^1$ of symmetric products corresponds to the homology class represented by the union of points appearing in the divisors represented by some section in $\mathcal{P}$; we'll be clearer about this later).

As such, we can set

$$HF(Y, f, h, c, \tau) = HF^{symp}(\Phi_{d,\omega,\tau}, \mathcal{P}_h),$$

where, once again, $\omega$ is a fiberwise symplectic representative of $c + \frac{2\pi}{\tau} PD(h)$. By Proposition 2.2 and the standard fact that $HF^{symp}$ is invariant under conjugation by symplectomorphisms and under Hamiltonian isotopy, we see immediately that $HF(Y, f, h, c, \tau)$ does not depend on the choice of $\omega$. We shall soon verify that the Novikov ring over which it is defined is as promised in the introduction, and that it is independent of $\tau$ at least for $d$ outside a certain range. First, however, a digression regarding the topology of $Y_\Phi$ is in order.

## 3. Basic properties of $HF$

3.1. **Topology of (relative) symmetric products.** We review here some basic facts regarding the cohomology of symmetric products. A standard reference for some of this material is [26]; for the relative versions see also the appendices and Section 2.1 of [34].

First note that if $\Sigma$ is a Riemann surface we obtain a natural map

$$\uparrow\colon H^*(\Sigma;\mathbb{Z}) \to H^*(S^d\Sigma;\mathbb{Z})$$

as follows. Inside the product $\Sigma\times S^d\Sigma$ we have a divisor $\mathcal{D} = \{(p,D)\in\Sigma\times S^d\Sigma | p\in D\}$, where we view $D\in S^d\Sigma$ as a set of points in $\Sigma$ (in other references, such as [34], $\mathcal{D}$ is called $\Delta$, but we prefer to use $\Delta$ to denote the diagonal stratum in the symmetric product). Letting $\pi_1$ and $\pi_2$ denote the projections of $\Sigma\times S^d\Sigma$ onto either factor, the map $\uparrow$ is defined by

$$\uparrow c = (\pi_2)_!(\mathfrak{d}\cup\pi_1^* c),$$

where $\mathfrak{d} = PD[\mathcal{D}]\in H^2(\Sigma\times S^d\Sigma;\mathbb{Z})$. Note the simple geometric interpretation of this map: if $c\in H^*(\Sigma;\mathbb{Z})$, let $A$ be a cycle Poincaré dual to $c$; then the Poincaré dual of $\uparrow c$ is represented by a cycle whose image is the set $\{D\in S^d\Sigma | D\cap A\neq\varnothing\}$ of degree $d$ effective divisors on $\Sigma$ which contain a point of $A$.

Dually, there is a map

$$\begin{aligned}\downarrow\colon H_*(S^d\Sigma;\mathbb{Z}) &\to H_*(\Sigma;\mathbb{Z})\\ A &\mapsto (\pi_1)_*(\mathfrak{d}\cap\pi_2^! A);\end{aligned}$$

again we may intuitively visualize this as sending $A\in H_*(S^d\Sigma;\mathbb{Z})$ to the homology class in $\Sigma$ represented by union of all the points in $\Sigma$ which appear in the set of divisors which is the image of some chain representing $A$. Note that for $c\in H^*(\Sigma;\mathbb{Z})$ and $A\in H_*(S^d\Sigma;\mathbb{Z})$, we have

$$\langle\uparrow c, A\rangle = \langle(\pi_2)_!(\mathfrak{d}\cup\pi_1^*c), A\rangle = \langle c, (\pi_1)_*(\mathfrak{d}\cap\pi_2^! A)\rangle = \langle c, \downarrow A\rangle.$$

We can now describe the cohomology of $S^d\Sigma$; for proofs (in a somewhat different language) see [26].

**Proposition 3.1.** (i) *The map $\uparrow\colon H^1(\Sigma;\mathbb{Z})\to H^1(S^d\Sigma;\mathbb{Z})$ is an isomorphism.*

(ii) *Identifying $H^1(\Sigma;\mathbb{Z})$ with $H^1(S^d\Sigma;\mathbb{Z})$ by $\uparrow$, where $\omega$ is a positive generator of $H^2(\Sigma;\mathbb{Z})$ and $U=\uparrow\omega$, one has*

$$H^*(S^d\Sigma;\mathbb{Z}) = \frac{\mathbb{Z}[U]\otimes_{\mathbb{Z}}\Lambda^*H^1(\Sigma;\mathbb{Z})}{\langle U^i\otimes(\gamma_1\wedge\cdots\wedge\gamma_j) | i,j\geq 0,\ i+j>d, \gamma_i\in H^1(\Sigma;\mathbb{Z}))\rangle}$$

*as graded rings. In particular $H^*(S^d\Sigma;\mathbb{Z})$ is naturally a module over $\mathbb{Z}[U]\otimes_{\mathbb{Z}}\Lambda^*H^1(\Sigma;\mathbb{Z})$.*

We will also be interested in *relative* symmetric products associated to surface fibrations. To wit, let $\pi\colon X\to B$ be a fibration with fiber $\Sigma$ a closed surface of genus $g\geq 2$ ($B$ is a compact manifold, possibly with boundary, in which case $\partial X = \pi^{-1}(\partial B)$). By choosing an almost complex structure $J$ on $T^{vt}X$ and appealing to the parametrized Riemann mapping theorem to obtain "restricted charts" which are smooth horizontally and holomorphic vertically (see, *e.g.*, [7]), one can construct a fibration

$$\Pi\colon X_d(\pi)\to B$$

carrying a vertical almost complex structure $\tilde{J}$ such that each fiber $\Pi^{-1}(b)$ is identified as a complex manifold with $(S^d\pi^{-1}(b), J|_{\pi^{-1}(b)})$. The maps $\uparrow$ and $\downarrow$ extend to the relative context: inside the fiber product

$$X_\pi \times_\Pi X_d(\pi)$$

we have a codimension–2 submanifold

$$\mathcal{D} = \{(b,p,D) \in B \times X \times X_d(\pi) | p \in \pi^{-1}(b), D \in \Pi^{-1}(b), p \in D\}$$

determining a class

$$[\mathcal{D}] \in H_{2d+2}(X_\pi \times_\Pi X_d(\pi), \partial(X_\pi \times_\Pi X_d(\pi))).$$

Again let $\mathfrak{d} = PD[\mathcal{D}]$ and define

$$\begin{aligned} \uparrow\colon\ H^*(X;\mathbb{Z}) &\to H^*(X_d(\pi);\mathbb{Z}) \\ c &\mapsto (\pi_2)_!(\mathfrak{d} \cup \pi_1^* c), \end{aligned}$$

and similarly for $\downarrow$, where $\pi_1$ and $\pi_2$ are the projections from $X_\pi \times_\Pi X_d(\pi)$ to $X$ and $X_d(\pi)$, respectively.

The following formula, proven using the family Atiyah–Singer theorem, expresses the first Chern class of the vertical tangent bundle $T^{vt}X_d(\pi)$ (with its induced almost complex structure) in terms of the Euler class of $T^{vt}X$.

**Lemma 3.2.** ([34], *Lemma 2.1.1*) *Assume that $B$ is closed. Then*

$$c_1(T^{vt}X_d(\pi)) = \frac{1}{2}\left(\uparrow(e(T^{vt}X)) + (\pi_2)_!(\mathfrak{d}^2)\right).$$

We now give a geometric interpretation of the class $(\pi_2)_!(\mathfrak{d}^2)$ appearing in Lemma 3.2. $\Delta \subset X_d(f)$ will denote the real–codimension 2 subvariety of $X_d(f)$ consisting of divisors $D$ having one or more points repeated. $\Delta$ is easily seen to represent an element $[\Delta] \in H_{2r-2+\dim B}(X_d(f), \partial X_d(f))$.

**Proposition 3.3.** *If $B$ is closed, then $(\pi_2)_!(\mathfrak{d}^2) = PD[\Delta] + \uparrow(e(T^{vt}X))$*

*Proof.* Let $v \in \Gamma(T^{vt}X)$ be a transversally-vanishing vertically-valued vector field on $X$. $v$ then induces a vertically-valued vector field $V$ on $X_\pi \times_\Pi X_d(\pi)$ by, with respect to the splitting $T^{vt}_{(b,p,D)}X_\pi \times_\Pi X_d(\pi) = T^{vt}_{(b,p,D)}X \oplus T^{vt}_{(b,p,D)}X_d(\pi)$, setting $V(b,p,D) = (v(b,p),0)$. So $\mathfrak{d}^2$ is represented by the Poincaré dual of the vanishing locus $\mathcal{V}$ of the projection of $V|_{\mathcal{D}}$ to the normal bundle $N\mathcal{D}$. This latter is easily seen to be

$$\begin{aligned} &\{(b,p,D) \in X_\pi \times_\Pi X_d(f) | D = 2p + D' \text{ for some } D' \in X_{d-2}(\pi)\} \\ &\qquad\qquad \cup \{(b,p,D) \in X_\pi \times_\Pi X_d(f) | v(p) = 0\}. \end{aligned}$$

Hence $(\pi_2)_!(\mathfrak{d}^2) = PD(\pi_2)_*[\mathcal{V}]$ is represented by the Poincaré dual of the homology class represented by the union

$$\Delta \cup \{D \in X_d(f) | D \cap v^{-1}(0) \neq \varnothing.\}$$

But the second set in this union represents $PD(\uparrow(e(T^{vt}X)))$ since $v$ is a transversally vanishing section of $T^{vt}X$. The conclusion is then immediate. ∎

3.2. **Novikov rings.** We shall now recall how exactly Novikov rings enter into the general picture of symplectic Floer theory; [21] is a good reference for those seeking further details. After this, we shall be prepared to verify that the Novikov ring over which $HF(Y, f, h, c, \tau)$ is defined is indeed the ring $\tilde{\Lambda}_{h,c}$ of the introduction. As many references exist for technicalities relating to the relevant Fredholm theory and compactness, our treatment shall be essentially formal, but our conclusion will remain valid in full generality, with the exception that the ring $R$ below needs to be a field of characteristic zero when the virtual moduli methods of [24] are required. Returning to the notation of Section 2, assume that $\Phi : (X, \omega) \to (X, \omega)$ is a symplectomorphism. The configuration space for $HF^{symp}(\Phi, \mathcal{P})$ is then the space of sections of the mapping torus $Y_\Phi$ of $\Phi$ belonging to the homotopy class $\mathcal{P}$, and generators for the Floer complex will be those sections of $Y_\Phi$ corresponding to fixed points of $\Phi$. We have a natural evaluation map

$$ev_{\mathcal{P}} \colon H_1(\mathcal{P}; \mathbb{Z}) \to H_2(Y_\Phi; \mathbb{Z}).$$

Let $A_0 \in H_1(\mathcal{P}; \mathbb{Z})$ be such that $\langle c_1(T^{vt}Y_\Phi), ev(A_0)\rangle$ is a positive generator for $Im(\langle c_1(T^{vt}Y_\Phi), ev(\cdot)\rangle \colon H_1(\mathcal{P}; \mathbb{Z}) \to \mathbb{Z})$. Let $p \colon \tilde{\mathcal{P}} \to \mathcal{P}$ be the universal abelian cover of the configuration space $\mathcal{P}$, with covering group $H_1(\mathcal{P}; \mathbb{Z})$. One then has a natural relative $\mathbb{Z}$–grading $\tilde{gr}(\tilde{x}, \tilde{y})$ for any nondegenerate $\tilde{x}, \tilde{y} \in \tilde{\mathcal{P}}$ given by the Maslov index, such that

$$\tilde{gr}(\tilde{x}, \tilde{y}) + \tilde{gr}(\tilde{y}, \tilde{z}) = \tilde{gr}(\tilde{x}, \tilde{z}) \text{ and } \tilde{gr}(\tilde{x}, A \cdot \tilde{x}) = \langle c_1(T^{vt}Y_\Phi), ev(A)\rangle. \tag{7}$$

Where $\mathcal{F} \subset \mathcal{P}$ denotes the set of critical points for the action 1–form $\mathcal{Y}$ of (6), for all $x \in \mathcal{F}$ choose and fix lifts $\tilde{x}$ in $\tilde{\mathcal{P}}$ with the property that

$$|\tilde{gr}(\tilde{x}, \tilde{y})| < \langle c_1(T^{vt}Y_\Phi), ev(A_0)\rangle$$

for each $x, y$, as is possible using the second property in (7).

Where $J$ is a generic $\mathbb{R}$–invariant almost complex structure on $\mathbb{R} \times T^{vt}(Y_\Phi)$, the boundary operator for the Floer complex counts finite energy solutions

$$\begin{aligned} U \colon \mathbb{R} \times S^1 &\to \mathbb{R} \times Y_\Phi \\ (s, t) &\mapsto (s, u(s, t)) \end{aligned}$$

to a perturbed Cauchy–Riemann equation $\bar{\partial}_J U = X_H(U)$; each of these is asymptotic as $s \to \pm\infty$ to generators $x^\pm \in \mathcal{F}$, and $s \mapsto u(s, \cdot)$ determines a path in $\mathcal{P}$ from $x^-$ to $x^+$. This path then lifts to a unique path $\tilde{u}$ in $\tilde{\mathcal{P}}$ from $\tilde{x}^-$ to $A \cdot \tilde{x}^+$ for some $A \in H_1(\mathcal{P}; \mathbb{Z})$. Accordingly, given $A$, let $\mathcal{M}_{J,H}(x^-, x^+; A)$ be the set of those $U$ which, as above, satisfy $\bar{\partial}_J = X_H(U)$ and determine a path $s \mapsto \tilde{u}(s, \cdot)$ in $\tilde{\mathcal{P}}$ from $\tilde{x}^-$ to $A \cdot \tilde{x}^+$. For generic Hamiltonian perturbations $X_H$, this will be a manifold of dimension $\tilde{gr}(\tilde{x}^-, A \cdot \tilde{x}^+)$ with a free $\mathbb{R}$–action. Note that the choice of $A_0$ determines a splitting $H_1(\mathcal{P}; \mathbb{Z}) = \mathbb{Z}A_0 \oplus \ker(\langle c_1(T^{vt}Y), ev(\cdot)\rangle)$; let $p_2$ denote the projection onto the second summand in this splitting.

The Floer complex $CF^{symp}(\Phi, \mathcal{P})$ is then the free module generated by the elements of $\mathcal{P}$ over the Novikov ring

$$Nov\left(\ker\langle c_1(T^{vt}Y_\Phi), ev(\cdot)\rangle, \langle [\omega_\Phi], ev(\cdot)\rangle; R\right) \tag{8}$$

where $R$ is an arbitrary ring (the notation is as in the introduction); the differential is given by the formula

$$\partial\langle x^-\rangle = \sum_{\tilde{gr}(\tilde{x}^-, A \cdot \tilde{x}^+) = 1} \#(\mathcal{M}(x, y; A)/\mathbb{R})[p_2(A)]\langle x^+\rangle.$$

Here $\#(\mathcal{M}(x,y;A)/\mathbb{R})$ refers to a signed count of points in the indicated compact 0–manifold, using coherent orientations as in [10].

Recall now that $HF(Y,f,h,c,\tau) = HF^{symp}(\Phi_{d,\omega,\tau},\mathcal{P}_h)$ where $\omega$ is a fiberwise symplectic 2–form on $Y$ representing $c + \frac{2\pi}{\tau}PD(h)$ and $d = h \cap (fiber)$. We can be somewhat clearer than before about the definition of the homotopy class $\mathcal{P}_h$ of sections of $Y_{\Phi_{d,\omega,\tau}}$ corresponding to $h$; note that $Y_{\Phi_{d,\omega,\tau}}$ is the relative symmetric product built from the fibration $f\colon Y \to S^1$; $\mathcal{P}_h$ is characterized by the property that, where

$$ev\colon \pi_0(\Gamma(Y_{\Phi_{d,\omega,\tau}})) \to H_1(Y_{\Phi_{d,\omega,\tau}};\mathbb{Z})$$

is the obvious evaluation map, we have $\downarrow(ev(\mathcal{P}_h)) = h$, the map $\downarrow$ having been defined in the last section.

**Lemma 3.4.** *Let $\gamma \in H_1(\mathcal{P}_h)$, with $ev(\gamma) = T \in H_2(Y_{\Phi_{d,\omega,\tau}};\mathbb{Z})$. Then*

$$\langle PD(\Delta), T\rangle_{Y_{\Phi_{d,\omega,\tau}}} = \langle 2PD(h) - e(T^{vt}Y), \downarrow T\rangle_Y$$

*where $\Delta$ is the diagonal in the relative symmetric product $Y_{\Phi_{d,\omega,\tau}}$.*

*Proof.* Assume $\gamma$ is represented by a loop $\alpha\colon S^1 \to \mathcal{P}_h$. Define

$$\tilde{\alpha}\colon S^1 \times S^1 \to S^1 \times Y_{\Phi_{d,\omega,\tau}}$$
$$(\theta,t) \mapsto (\theta, \alpha(\theta,t))$$

and let $\tilde{T} \in H_2(S^1 \times Y_{\Phi_{d,\omega,\tau}};\mathbb{Z})$ be the class represented by $\tilde{\alpha}$. By perturbing $\alpha$, we may assume that $\tilde{\alpha}$ is transverse to the diagonal in $S^1 \times Y_{\Phi_{d,\omega,\tau}}$ (which is a relative symmetric product over the torus, of course). Where $i\colon Y \cong \{1\} \times Y \to S^1 \times Y$ is the inclusion, we have

$$\downarrow\tilde{T} = i_*(\downarrow T) + [S^1] \times h. \tag{9}$$

Let $C = \{p \in S^1 \times Y | \exists D \in Im(\tilde{\alpha})\, p \in D\}$ be the image of the cycle in $S^1 \times Y$ representing $\downarrow T$, so that $C$ is "swept out" by $\tilde{\alpha}$. Where $v$ is a transversally vanishing section of $T^{vt}(S^1 \times Y)$ all of whose zeroes occur over points in $S^1 \times S^1$ which are not contained in the finite set $\{(s,t) \in S^1 \times S^1 | \tilde{\alpha}(s,t) \in \Delta\}$ and $\phi^t\colon Y \to Y$ is its time–$t$ flow, set

$$C^t = \{p \in S^1 \times Y | \phi^t(p) \in C\}.$$

For $t$ small, one sees that there is one intersection of $C$ with $C^t$ for each point of $v^{-1}(0) \cap C$, and another intersection of $C$ with $C^t$ for each intersection of the image of $\tilde{\alpha}$ with $\Delta$, and that moreover these all occur with correctly–corresponding signs. This shows that

$$\langle PD(\downarrow\tilde{T}), \downarrow\tilde{T}\rangle_{S^1\times Y} = \langle PD(\Delta), \tilde{T}\rangle_{S^1 \times Y_{\Phi_{d,\omega,\tau}}} + \langle e(T^{vt}(S^1 \times Y)), \downarrow\tilde{T}\rangle_{S^1\times Y}.$$

But from (9) we see

$$\langle PD(\downarrow\tilde{T}), \downarrow\tilde{T}\rangle_{S^1\times Y} = 2\langle PD(h), \downarrow T\rangle_Y,$$

while it is straightforward to see that

$$\langle PD(\Delta), \tilde{T}\rangle_{S^1 \times Y_{\Phi_{d,\omega,\tau}}} = \langle PD(\Delta), T\rangle_{Y_{\Phi_{d,\omega,\tau}}}$$

and

$$e(T^{vt}(S^1 \times Y), \downarrow\tilde{T}\rangle_{S^1\times Y} = \langle e(T^{vt}Y), \downarrow T\rangle_Y,$$

proving the lemma. □

We now invoke the following very useful calculation of Perutz.

**Theorem 3.5.** ([34]*, p. 70*)

$$[(\Omega_{d,\omega,\tau})_{\Phi_{d,\omega,\tau}}] = 2\pi\left(\tau \uparrow [\omega] - \pi(\pi_2)_!(\mathfrak{d}^2)\right) \in H^2(Y_{\Phi_{d,\omega,\tau}};\mathbb{R}).$$

With this in hand we can identify the Novikov ring over which $HF(Y,f,c,h,\tau)$ is defined.

Combining Proposition 3.3 and Lemma 3.4 and using several times the duality between $\uparrow$ and $\downarrow$, we see that, for $\gamma \in H_1(\mathcal{P}_h;\mathbb{Z})$, we have

$$\langle(\pi_2)_!(\mathfrak{d}^2), ev(\gamma)\rangle = \langle PD[\Delta] + \uparrow e(T^{vt}Y), ev(\gamma)\rangle = \langle 2PD(h), \downarrow (ev(\gamma))\rangle, \tag{10}$$

so by Lemma 3.2

$$\langle c_1(T^{vt}Y_{\Phi_{d,\omega,\tau}}), ev(\gamma)\rangle = \frac{1}{2}\langle e(T^{vt}Y) + 2PD(h), \downarrow ev(\gamma)\rangle. \tag{11}$$

Meanwhile since we are choosing $\omega$ as a representative of $c + \frac{2\pi}{\tau}PD(h)$, we have

$$\begin{aligned}\langle[(\Omega_{d,\omega,\tau})_{\Phi_{d,\omega,\tau}}], ev(\gamma)\rangle &= 2\pi\langle\tau(c + 2\pi PD(h)/\tau) - \pi(2PD(h)), \downarrow ev(\gamma)\rangle \\ &= 2\pi\tau\langle c, \downarrow ev(\gamma)\rangle.\end{aligned} \tag{12}$$

Therefore, in light of (8), $HF(Y,f,h,c,\tau)$ is defined over the Novikov ring

$$Nov\left(\ker\langle e(T^{vt}Y) + 2PD(h), \cdot\rangle, \langle c, \cdot\rangle; R\right).$$

In the introduction, allusions were made to "a spin$^c$ structure $s_h$ corresponding to $h \in H_1(Y;\mathbb{Z})$;" we clarify that slightly here: in Section 8 of [38], Salamon defines a canonical spin$^c$ structure on the total space of $f\colon Y \to S^1$ which has (rank 2, Hermitian) spinor bundle $S = \underline{\mathbb{C}} \oplus T^{vt}Y$ (the reader who prefers to think of spin$^c$ structures as given by nonvanishing vector fields can identify this as the structure specified by a vector field which is transverse to the fibers of $f$). $s_h$ is then defined by tensoring $S$ with a line bundle $L$ such that $c_1(L) = PD(h)$ and extending Clifford multiplication trivially. Evidently, then,

$$c_1(s_h) = c_1(L \oplus (T^{vt}Y \otimes L)) = e(T^{vt}Y) + 2PD(h),$$

proving that the Novikov ring we are considering is precisely the ring $\tilde{\Lambda}_{h,c}$ specified in Section 1.1. (If one instead uses the convention that the natural coefficient ring for $CF^{symp}(\Phi,\mathcal{P})$ is

$$Nov\left(\frac{\ker\langle c_1(T^{vt}Y_\Phi), ev(\cdot)\rangle}{\ker\langle c_1(T^{vt}Y_\Phi), ev(\cdot)\rangle \cap \langle[\omega_\Phi], ev(\cdot)\rangle}, \langle[\omega_\Phi], ev(\cdot)\rangle; R\right)$$

one evidently instead obtains the smaller ring $\Lambda_{h,c}$ here.)

3.3. **Invariance.** In this subsection, we shall prove Theorem 1.1. We remark first of all that for invariance results such as Theorem 1.1 which equate the homologies of two Floer chain complexes $CF_-$ and $CF_+$ which depend on different auxiliary data, the usual technique of proof has for some time been the "method of continuation," wherein one defines a chain map $CF_- \to CF_+$ by counting finite-energy solutions to some modified version of the Cauchy–Riemann equations on the cylinder $\mathbb{R}_t \times S^1$ which has the property that such solutions are asymptotic to generators of $CF_\pm$ as $t \to \pm\infty$. In our setting, in which we consider the effect on $HF(Y,f,h,c,\tau)$ of varying the parameter $\tau$, naïve attempts to use this method do not appear to work. Indeed, the method of continuation would suggest that, to equate $HF(Y,f,h,c,\tau_-)$ with $HF(Y,f,h,c,\tau_+)$ for (say) $\tau_- < \tau_+$, we should consider maps

$$u\colon \mathbb{R} \times S^1 \to \mathbb{R} \times Y_d(f)$$

which satisfy a perturbed Cauchy–Riemann equation for an almost complex structure on $\mathbb{R} \times Y_d(f)$, which is compatible with a form $\Omega$ on $\mathbb{R} \times Y_d(f)$ which agrees with the form induced by $\Omega_{d,\omega,\tau_\pm}$ as the $\mathbb{R}$ parameter approaches $\pm\infty$. But since $\Omega_{d,\omega,\tau_+}$ and $\Omega_{d,\omega,\tau_-}$ are not cohomologous, such a form $\Omega$ could not be closed, and this would prevent us from obtaining the energy bounds on these maps $u$ which are needed to show the continuation method validly defines a map between the two chain complexes.

Instead of the continuation method, we make use of recent work of Lee ([21], [22]) which will enable us to understand in fairly explicit terms how the chain complexes $CF(Y, f, h, c, \tau)$ vary as $\tau$ increases from $\tau_-$ to $\tau_+$. It is interesting to note that A. Floer himself used a similar method in his original paper [9] on Lagrangian Floer homology, though some details needed to justify this approach did not appear until Lee's work. Lee was concerned with torsion invariants rather than homology in her work, and so did not explicitly prove an invariance theorem for homology, even though such a theorem readily follows from her analysis (Lee also did not consider the effect of smoothly varying the symplectic form; however this does not complicate the analysis as long as the "$H^1$–codirectionality" hypothesis discussed below is maintained). Since this result may be useful in other contexts, we state it in general form here and give an outline of the algebraic arguments needed in the proof, referring readers to [21] and [22] for the subtle geometric and analytic arguments necessary to show that the Floer complexes behave as we claim. Recall from Definition 2.3.1 of [21] that a $2n$-dimensional symplectic manifold $(X, \omega)$ is $w^+$–monotone provided that every sphere with Chern number strictly between 0 and $n - 1$ has positive symplectic area.

**Theorem 3.6** (Lee). *Let $\{\omega_t\}_{t\in[0,1]}$ be a smooth family of symplectic forms on a manifold $X$ such that the symplectic manifolds $(X, \omega_t)$ are $w^+$–monotone, let $\phi_t \colon X \to X$ be a smooth family of diffeomorphisms such that $\phi_t^*\omega_t = \omega_t$, and let $\mathcal{P} \in \pi_0(\Gamma(Y_{\phi_t}))$ be such that the action functionals $\mathcal{Y}^t$ for the Floer complexes $CF^{symp}(\phi_t, \mathcal{P})$ are $H^1$–codirectional, in the sense that, where $K \leq \pi_1(\Gamma(Y_{\phi_t}))$ denotes the kernel of the spectral flow homomorphism we have $[\mathcal{Y}_t]|_K = f(t)[\mathcal{Y}_0]|_K$ where $f$ is a nonnegative continuous function; thus where $\Lambda_t$ is the Novikov ring over which $CF^{symp}(\phi_t, \mathcal{P})$ is naturally defined $\Lambda_0$ is a module over $\Lambda_t$, with $\Lambda_0 = \Lambda_t$ when $f(t) \neq 0$. Then*

$$CF^{symp}(\phi_0, \mathcal{P}) \text{ is chain homotopy equivalent to } CF^{symp}(\phi_1, \mathcal{P}) \otimes_{\Lambda_1} \Lambda_0.$$

We first explain how Theorem 1.1 follows from this result. In Theorem 1.1, we consider two cases where we allow one entry from the standard data $(Y, f, h, c, \tau)$ to vary in a one-parameter family; namely, we either:

(i) Let $\tau$ vary from $\tau_-$ to $\tau_+$ (say $\tau_t = (1-t)\tau_- + t\tau_+$), fixing $(Y, f, h, c)$, or
(ii) Let $c$ vary from $\tilde{c}$ to $\alpha c_1(s_h)$ (say $c_t = t\alpha c_1(s_h) + (1-t)\tilde{c}$; here $\alpha \in \mathbb{R} \setminus \{0\}$), fixing $(Y, f, h, \tau)$.

Recall that in the statement of Theorem 1.1 we have assumed that either $d \geq g-1$ or $d < (g+1)/2$; since $\pi_2(S^d\Sigma)$ is an infinite cyclic group generated by a sphere on which the symplectic form is positive and $c_1$ evaluates as $d - g + 1$ [26] this is the assumption needed to ensure that $S^d\Sigma$ is $w^+$–monotone and so for Theorem 3.6 to apply. For the range $(g+1)/2 \leq d < g-1$ it seems likely that the methods of [24] could be used to prove similar results to the ones we cite here, but this does not appear to be at all straightforward.

In either case, let $\Omega_t$ be the symplectic form obtained by Salamon's construction using the data at time $t$. The relevant action one–form, defined on the space of sections $\gamma$ of $Y_d(f)$ representing the homotopy class $\mathcal{P}_h$, is then

$$\mathcal{Y}^t_\gamma(\xi) = -\int_0^1 \Omega_t(\dot{\gamma}(s), \xi(s))\, ds;$$

the action of this 1–form on loops in $\mathcal{P}_h$ is given by Equation 12. Meanwhile the action of $c_1(T^{vt}Y_d(f))$ (which plays the role here of Lee's spectral flow homomorphism $\psi\colon \pi_1(\mathcal{P}_h) \to \mathbb{Z}$) is given by $\langle c_1(s_h), \downarrow ev(\cdot)\rangle$. So in case (i) above, the classes of the forms $\mathcal{Y}^t$ in $H^1(\mathcal{P}_h; \mathbb{R})$ satisfy

$$[\mathcal{Y}^t]|_{\ker\psi} = \frac{\tau_t}{\tau_0}[\mathcal{Y}^0]|_{\ker\psi}$$

while in case (ii)

$$[\mathcal{Y}^t]|_{\ker\psi} = (1-t)[\mathcal{Y}^1]|_{\ker\psi}.$$

Thus in both cases, our path of 1–forms $\mathcal{Y}^t$ on the infinite-dimensional space $\mathcal{P}_h$ is $H^1$–codirectional. This fact makes Theorem 3.6 relevant to our situation.

We now briefly outline the facts from [21] which enter into the proof of Theorem 3.6. Lee's work implies the existence of a "regular homotopy of Floer systems" (RHFS) between the (partially-defined) flows on the space $\mathcal{P} \subset Y_{\phi_t}$ which underlie complexes $CF^{symp}(\phi_0, \mathcal{P})$ and $CF(\phi_1, \mathcal{P})$ Namely, there is a path $(J_t, H_t)_{t\in[0,1]}$ of $\omega_t$–compatible almost complex structures and Hamiltonian perturbations such that as $t$ varies the chain complexes $CF^{symp}(\phi_t, \mathcal{P})$ change only at certain values of $t$ corresponding to "handleslides" and (just finitely many) "death-births," all in the complement of a set $S_{reg}$ of second category in $[0,1]$; the Floer complex corresponding to $(J_t, H_t)$ is well-defined for each $t \in S_{reg}$.

If $[t_0, t_1] \subset [0,1]$ is an interval in which the complex changes only by handleslides (that is, Floer flow lines between generators of equal index; there may be infinitely many of these, but only finitely many with energy below any given bound) and $t_0, t_1 \in S_{reg}$, the generators for the chain complex remain unchanged throughout the interval, while the differentials $\partial_0, \partial_1$ of the chain complexes at times $t_0, t_1$ are related by

$$\begin{aligned}\langle \partial_1 x, y[p_2(A)]\rangle =& \langle \partial_0 x, y[p_2(A)]\rangle + \\ & \sum_{z,B} \sum_{s\in\mathcal{M}^{[t_0,t_1]}_{hs}(\mathbf{x},\mathbf{z};B)} \epsilon(s)\langle \partial_0 z, y[p_2(B-A)]\rangle + \\ & \sum_{z,C} \sum_{s\in\mathcal{M}^{[t_0,t_1]}_{hs}(\mathbf{z},\mathbf{y};C)} -\epsilon(s)\langle \partial_1 x, z[p_2(C-A)]\rangle\end{aligned} \tag{13}$$

where in general $\mathcal{M}^{[t_0,t_1]}_{hs}(\mathbf{u},\mathbf{v};B)$ denotes the moduli space of handleslides between the generators $\mathbf{u}$ and $\mathbf{v}$ having Novikov ring weight $B$, and for each handleslide $s$ $\epsilon(s) = \pm 1$ is a sign determined by coherent orientations.

If we then set

$$Tx = x + \sum_{z,B} \sum_{s\in\mathcal{M}^{I}_{hs}(\mathbf{x},\mathbf{z};B)} \epsilon(s) z[p_2(B)],$$

then the matrix element $\langle \partial_0 Tx, y[p_2(A)]\rangle$ is the sum of the first two terms on the right hand side of (13), while $\langle T\partial_1 x, y[p_2(A)]\rangle$ is the difference of the term on the left hand side and the last term on the right. Hence $T$ defines a chain map $CF^{symp}(\phi_1, \mathcal{P}) \otimes \Lambda_0 \to CF^{symp}(\phi_0, \mathcal{P})$ (note that this is well-defined over $\Lambda_0$ as a result of the finiteness condition on handleslides mentioned earlier). But $T$ is an invertible map (if we write $T = I + U$ where $U$ is the identity, $\sum_{n=0}^{\infty}(-U)^n$ will be a well-defined endomorphism over the Novikov ring and will obviously be inverse to $T$), so it actually defines an isomorphism of chain complexes.

This reduces the invariance problem to showing that the Floer homology is unchanged on intervals containing death-births, of which there are only finitely many. Let $I \subset [0,1]$ be an open interval containing a single death or birth, say at $\bar{t}$ (we'll assume it's a birth; the death case may be obtained by reversing various arrows and inequality signs in the discussion below). For $t < t' < \bar{t}$ such that $t, t' \in S_{reg}$ we have isomorphisms of chain complexes

$$T_{t,t'} \colon CF^{symp}(\phi_t, \mathcal{P}) \otimes \Lambda_0 \to CF^{symp}(\phi_{t'}, \mathcal{P}) \otimes \Lambda_0$$

as above; these form a directed system indexed by a dense subset of $\{t \in I | t < t'\}$, so we may let $(CF^-_{\bar{t}}, \partial^-)$ be the direct limit of the $CF^{symp}(\phi_t, \mathcal{P})$ under this directed system. Likewise let $(CF^+_{\bar{t}}, \partial^+)$ be the inverse limit of the directed system indexed by the subset $\{t \in I \cap S_{reg} | t > \bar{t}\}$ of $I$, given by the isomorphisms $T_{t,t'}$. We pass to these limits in order to allow ourselves to ignore handleslides in the following discussion, since $\bar{t}$ might not be contained in any open interval over which there are no handleslides.

Lee's axioms for an RHFS in Section 4 of [21] (see also [14] for a more explicit description in the finite dimensional context) imply the following description of the relationship of $(CF^+_{\bar{t}}, \partial^+)$ to $(CF^-_{\bar{t}}, \partial^-)$. We have

$$CF^+_{\bar{t}} = CF^-_{\bar{t}} \oplus \Lambda_0 \langle x^+, x^- \rangle$$

for some two generators $x^\pm$ which differ in relative grading by 1; these new generators are "born" from a degenerate critical point $x^0$ of the action form $\mathcal{Y}^{\bar{t}}$ which appears at $t = \bar{t}$.

Using Lee's axioms RHFS2 and RHFS2c, one can deduce the following description for the relationship between the Floer boundary operators $\partial^-$ and $\partial^+$. There are maps $v \colon \Lambda_0 \to CF^-_{\bar{t}}$, $\mu \colon CF^-_{\bar{t}} \to \Lambda_0$ and an invertible element $\alpha \in \Lambda_0$ such that, with respect to an ordered basis for $CF^+_{\bar{t}}$ consisting of an ordered basis for $CF^-_{\bar{t}}$ followed by $(x^+, x^-)$, the differential $\partial^+$ may be written in block form as

$$\partial^+ = \begin{pmatrix} \partial^- + \alpha^{-1} v \circ \mu & v & 0 \\ 0 & 0 & 0 \\ \mu & \alpha & 0 \end{pmatrix}.$$

Note that the fact that $(\partial^+)^2 = 0$ then implies that $\mu \circ v = \mu \circ \partial^- = \partial^- \circ v = 0$. Define a map $i \colon CF^-_{\bar{t}} \to CF^+_{\bar{t}}$ by the block matrix

$$i = \begin{pmatrix} Id \\ -\alpha^{-1}\mu \\ 0 \end{pmatrix}$$

and a map $p \colon CF^+_{\bar{t}} \to CF^-_{\bar{t}}$ by

$$p = \begin{pmatrix} Id & 0 & -\alpha^{-1} v \end{pmatrix}.$$

The fact that $\mu \circ \partial^- = 0$ implies that $i$ is a chain map, while the fact that $\partial^- \circ v = 0$ implies that $p$ is a chain map. Now obviously $p \circ i$ is the identity, while defining

$K\colon CF^+_{\bar{t}} \to CF^+_{\bar{t}}$ by

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha^{-1} \\ 0 & 0 & 0 \end{pmatrix},$$

one easily computes (using the fact that $\mu \circ v = 0$) that

$$\partial^+ K + K\partial^+ = 1 - i \circ p = \begin{pmatrix} 0 & 0 & \alpha^{-1}v \\ \alpha^{-1}\mu & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$i$ and $p$ thus put $CF^-_{\bar{t}}$ and $CF^+_t$ into chain homotopy equivalence.

Theorem 3.6 is immediate from this, for the interval $[0, 1]$ contains just finitely many values of $t$ (say $t_1, \ldots, t_n$) at which death-births occur; write $t_0 = 0, t_{n+1} = 1$). Our treatment of handleslides shows that the chain complexes $CF^{symp}(\phi_t, \mathcal{P}) \otimes \Lambda_0$ are mutually isomorphic for all $t \in S_{reg} \cap [t_i, t_{i+1}]$, and our treatment of births shows that the direct limit $\lim\limits_{\rightarrow}^{t<t_i} CF^{symp}(\phi_t, \mathcal{P}) \otimes \Lambda_0$ under these isomorphisms is chain homotopy equivalent to $\lim\limits_{\leftarrow}^{t>t_i} CF^{symp}(\phi_t, \mathcal{P}) \otimes \Lambda_0$, implying that the various $CF^{symp}(\phi_t, \mathcal{P}) \otimes \Lambda_0$ for all regular $t$ in the entire interval are chain homotopy equivalent. This completes our account of the proof of Theorem 3.6; as explained earlier, the assumption that $d \notin [(g + 1)/2, g - 1)$ makes Theorem 1.1 a special case of this more general result.

## 4. Further algebraic properties

With $HF(Y, f, h, c, \tau; A)$ defined as the Floer homology of the symplectomorphism $\Phi_{d,\omega,\tau}$, the algebraic properties alluded to in Section 1.2 now follow quickly from standard properties of Floer homology. First, as always in Floer theory, we have a absolute $\mathbb{Z}/2$ grading provided here by the Lefschetz index: a generator of $CF(Y, f, h, c, \tau; A)$ is a fixed point $p$ of (possibly a Hamiltonian perturbation of) $\Phi_{d,\omega,\tau}$, and the absolute grading of $p$ is just

$$sign \det(Id - (d\Phi_{d,\omega,\tau})_p).$$

The relative grading also follows from the standard setup and equation (11); quite generally the ambiguity in the relative grading of a generator of the Floer homology of $HF^{symp}(\psi; \mathcal{P})$ of a symplectomorphism $\psi$ in the fixed point class $\mathcal{P}$ is given by

$$2 \gcd_{\gamma \in H_1(\mathcal{P})} \langle c_1(T^{vt}Y_\psi), ev(\gamma)\rangle,$$

and in our case with $\psi = \Phi_{d,\omega,\tau}$ and $\mathcal{P} = \mathcal{P}_h$, this is precisely the divisibility of $c_1(s_h) = e(T^{vt}Y_\phi) + 2PD(h)$, as stated in Proposition 1.2.

Poincaré duality for $HF$ is still another simple consequence of the setup: replacing the tuple $\mathfrak{o} = (Y, f, h, c, \tau)$ with $\bar{\mathfrak{o}} = (-Y, \bar{f}, -h, c, \tau)$ of course preserves the orientations of the fibers, while we have $PD_{-Y}(-h) = PD_Y(h)$, so the same fiberwise symplectic form $\omega$ representing $c + \frac{2\pi}{\tau}PD_{\pm Y}(\pm h)$ can be used for both $\mathfrak{o}$ and $\bar{\mathfrak{o}}$ and the same fiberwise complex structures $J_t$ can be used on $Y \to S^1$ and $-Y \to S^1$. Using these same auxiliary data, the horizontal vector field whose flow we use to define the monodromy $\Phi^{\bar{\mathfrak{o}}}_{d,\omega,\tau}$ in Salamon's construction for $\bar{\mathfrak{o}}$ will then be precisely the opposite of the vector field which generates the monodromy $\Phi^{\mathfrak{o}}_{d,\omega,\tau}$. Hence $\Phi^{\bar{\mathfrak{o}}}_{d,\omega,\tau} = (\Phi^{\mathfrak{o}}_{d,\omega,\tau})^{-1}$. Now it is quite generally the case that, for Floer chain complexes $CF^{symp}(\psi, \mathcal{P})$ of symplectomorphisms $\psi$, $CF^{symp}(\psi^{-1}, \bar{\mathcal{P}})$ is naturally the dual complex to $CF^{symp}(\psi, \mathcal{P})$ under an appropriate identification

$\mathcal{P} \leftrightarrow \bar{\mathcal{P}}$ of fixed point classes: a fixed point of $\psi$ is of course also a fixed point of $\psi^{-1}$ and, tautologically, two such are Nielsen–equivalent for $\psi$ iff they are for $\psi^{-1}$, so as groups $CF^{symp}(\psi^{-1}, \bar{\mathcal{P}}) = CF^{symp}(\psi, \mathcal{P})$. The differentials are related by the observation that since the mapping torus fibration $Y_{\psi^{-1}} \to S^1$ is (up to equivalence of fibrations) the conjugate of $Y_\psi \to S^1$, a cylinder in $\mathbb{R} \times Y_\psi$ which serves as a flowline from the generator $a$ to the generator $b$ in $CF^{symp}(\psi)$ is the same thing as a cylinder in $\mathbb{R}\times Y_{\psi^{-1}}$ which serves as a flowline from $b$ to $a$ in $CF^{symp}(\psi^{-1})$. In our context, the fixed point class $\mathcal{P}_h$ for $\Phi^{\mathfrak{o}}_{d,\omega,\tau}$ corresponds tautologically to the fixed point class $\mathcal{P}_{-h}$ for $\Phi^{\bar{\mathfrak{o}}}_{d,\omega,\tau}$, and so $CF(\mathfrak{o})$ and $CF(\bar{\mathfrak{o}})$ are the same as groups and have dual differentials, which proves Proposition 1.3 (the pairing between $CF(\mathfrak{o})$ and $CF(\bar{\mathfrak{o}})$ promised therein is of course obtained by, for generators $\mathbf{x}$ and $\mathbf{y}$ of the identical groups, setting $\langle \mathbf{x}, \mathbf{y}\rangle$ to be 1 if $\mathbf{x} = \mathbf{y}$ and 0 otherwise and then extending linearly).

The only remaining item from Proposition 1.2 is the structure of $HF(Y, h, c, \tau; A)$ as a module over

$$\mathbb{A}(Y) = \mathbb{Z}[U] \otimes \Lambda^*(H_1(Y;\mathbb{Z})/torsion).$$

We obtain this by considering the quantum cap product structure in Floer theory, which we describe here in the case that virtual cycle methods are not needed. Quite generally, the boundary operator $\partial\colon CF \to CF$ in a Floer theory with configuration space $\mathcal{C}$ counts paths $\gamma\colon [-\infty, \infty] \to \mathcal{C}$ with prescribed endpoints which are (formally) gradient lines for some Morse function on $\mathcal{C}$. Letting $\mathcal{M}(\mathbf{x}, \mathbf{y}; A)$ denote the moduli space of flowlines from a generator $\mathbf{x}$ of $CF$ to a generator $\mathbf{y}$ having relative homotopy class $A$, if $k$ denotes the common index of these flowlines then evaluation of the flowline at time zero determines a $k$–dimensional chain $ev_*\mathcal{M}(\mathbf{x}, \mathbf{y}; A)$ in $\mathcal{C}$. So if $a \in C^k(\mathcal{C};\mathbb{Z})$ we get a degree–$(-k)$ map

$$a\cdot\colon CF \to CF$$

by setting

$$a \cdot \mathbf{x} = \sum_{\mathbf{y},A} \langle a, ev_*\mathcal{M}(\mathbf{x}, \mathbf{y}; A)\rangle [p_2(A)]\mathbf{y};$$

considering the boundary components of the $\mathcal{M}_k(\mathbf{x}, \mathbf{y}; A)$ reveals that $a\cdot$ is a chain map. Further, the map that it induces on the Floer homology $HF$ depends only on the cohomology class of $a$, and the resulting map $H^*(\mathcal{C};\mathbb{Z}) \times HF \to HF$ makes $HF$ into a module over $H^*(\mathcal{C};\mathbb{Z})$. For more details on this see [47] and [25], in the latter of which it is shown that the quantum cap product can be made compatible with virtual cycle machinery.

In our context, the configuration space $\mathcal{C}$ is a homotopy class $\mathcal{P}_h$ of sections of the degree–$d$ relative symmetric product $Y_d(f) \to S^1$ of a fibered 3–manifold $f\colon Y \to S^1$. Thus the quantum cap product construction makes $HF(Y, f, h, c, \tau; A)$ into a module over $H^*(\mathcal{P}_h;\mathbb{Z})$. To obtain the asserted module structure over $\mathbb{A}(Y)$ (and so exhibit still another parallel with the monopole and Heegaard Floer theories), we thus just need to exhibit a natural graded ring homomorphism

$$\mathbb{Z}[U] \otimes \Lambda^*(H_1(Y;\mathbb{Z})/torsion) \to H^*(\mathcal{P}_h;\mathbb{Z}),$$

where $U$ has degree 2. In this direction, note that if $g\colon K \to \mathcal{P}_h$ is a chain of dimension $k$, we get a dimension–$(k + 1)$ chain $e(g)\colon S^1 \times K \to Y_d(f)$ by setting $e(g)(t, k) = (g(k))(t)$; $e$ evidently defines a degree–1 chain map

$$e\colon C_*(\mathcal{P}_h;\mathbb{Z}) \to C_{*+1}(Y_d(f);\mathbb{Z});$$

dualizing this and passing to cohomology yields a homomorphism

$$e^*\colon\ H^*(Y_d(f);\mathbb{Z}) \to H^{*-1}(\mathcal{P}_h;\mathbb{Z}).$$

Recalling the map $\uparrow\colon\ H^*(Y;\mathbb{Z}) \to H^*(Y_d(f);\mathbb{Z})$, we now set

$$U = e^*\left(\uparrow (PD[pt])\right) \in H^2(\mathcal{P}_h;\mathbb{Z})$$

and use the homomorphism

$$\begin{aligned} H_1(Y;\mathbb{Z}) &\to H^1(\mathcal{P}_h;\mathbb{Z}) \\ \gamma &\mapsto e^* \uparrow PD(\gamma). \end{aligned}$$

Note that the image of any torsion elements of $H_1(Y;\mathbb{Z})$ will be trivial, and so the above map factors through a map

$$H_1(Y;\mathbb{Z})/torsion \to H^1(\mathcal{P}_h;\mathbb{Z}).$$

Now that we have chosen an element $U \in H^2(\mathcal{P}_h;\mathbb{Z})$ and a homomorphism $H_1(Y;\mathbb{Z})/torsion \to H^1(\mathcal{P}_h;\mathbb{Z})$, a unique ring homomorphism

$$\mathbb{A}(Y) = \mathbb{Z}[U] \otimes \Lambda^*(H_1(Y;\mathbb{Z})/torsion) \to H^*(\mathcal{P}_h;\mathbb{Z})$$

is forced on us by the graded ring structure of $H^*(\mathcal{P}_h;\mathbb{Z})$. This completes the proof of the existence of the module structure over the promised ring.

More geometrically, the map on $HF(Y, f, h, c, \tau; A)$ induced by the element

$$U^r \otimes \gamma_1 \wedge \cdots \wedge \gamma_k \in \mathbb{Z}[U] \otimes \Lambda^*(H_1(Y;\mathbb{Z})/torsion)$$

counts holomorphic sections

$$\begin{aligned} \mathbb{R} \times S^1 &\to \mathbb{R} \times Y_d(f) \\ (s,t) &\mapsto u(s,t) \end{aligned}$$

with the property that the cycle obtained by taking the union of the points in the divisors $u(0,t)$ for the various $t \in S^1$ contains a generic set of $r$ points and meets generic representatives of the classes $\gamma_1, \ldots, \gamma_k$.

Finally, we explain the construction of the local coefficient systems $\Gamma_\theta$ for closed forms $\theta \in \Omega^2(Y)$. First, we choose once and for all a de Rham representative $\delta$ of the class $PD(\mathcal{D}) \in H^2(Y \times_{S^1} Y_d(f);\mathbb{R})$. Now from our initial data set $(Y, f, h, c, \tau)$ we have constructed a closed form $\omega \in \Omega^2(Y)$ restricting to each fiber of $f\colon\ Y \to S^1$ as a volume form; pulling back this form by $\pi_1\colon\ Y \times_{S^1} Y_d(f) \to Y$ gives a closed 2–form on $Y \times_{S^1} Y_d(f)$ which restricts as a volume form to each fiber of $\pi_2\colon\ Y \times_{S^1} Y_d(f) \to Y_d(f)$; this gives rise via integration down the fibers of $\pi_2$ to a form–level extension

$$(\pi_2)_!\colon\ \Omega^*(Y \times_{S^1} Y_d(f)) \to \Omega^{*-2}(Y_d(f))$$

of the Gysin map, and so to a form–level extension

$$\begin{aligned} \uparrow_\delta\colon\ \Omega^*(Y) &\to \Omega^*(Y_d(f)) \\ \theta &\mapsto (\pi_2)_!(\delta \wedge \pi_1^*\theta) \end{aligned} \tag{14}$$

of $\uparrow$ which is a cochain map and so takes closed forms to closed forms. This yields a local system $\Gamma_\theta$ on the configuration space $\mathcal{P}_h$ as follows: to each $\mathbf{x} \in \mathcal{P}_h$ take $(\Gamma_\theta)_{\mathbf{x}} = \mathbb{R}$ as the fiber over $\mathbf{x}$; since an element of $\mathcal{P}_h$ gives rise via evaluation to a loop in $Y_d(f)$, a path $\gamma$ from $\mathbf{x}$ to $\mathbf{y}$ in $\mathcal{P}_h$ gives rise via evaluation to a cylinder

$C_\gamma \subset Y_d(f)$ with boundary components prescribed by $\mathbf{x}$ and $\mathbf{y}$, and we define the isomorphism

$$\phi_\theta([\gamma])\colon\ (\Gamma_\theta)_{\mathbf{x}} \to (\Gamma_\theta)_{\mathbf{y}}$$

to be multiplication by

$$\exp\left(\int_{C_\gamma} \uparrow_\delta \theta\right);$$

that $\uparrow_\delta \theta$ is closed of course ensures that this depends only on the homotopy class of $\gamma$ relative to its boundary. The resulting twisted Floer groups are then obtained by letting the twisted Floer chain complex be the direct sum over the fixed points $x$ of $\Phi_{d,\omega,\tau}$ of the groups $\Gamma_\theta(\mathbf{x}) \otimes \tilde{\Lambda}_{h,c}$ (where $\mathbf{x} \in \mathcal{P}_h$ is the section of $Y$ corresponding to $x$) and weighting each term in the standard Floer differential (which corresponds to a path $\gamma$ in $\mathcal{P}_h$) by $\phi_\theta([\gamma])$; see Section 2.7 of [19] for the analogous construction in Morse theory. If $\zeta \in \Omega_1(Y)$ satisfies $d\zeta = \theta_2 - \theta_1$, then if $\gamma$ is a path in $\mathcal{P}_h$ from $\mathbf{x}$ to $\mathbf{y}$, we have

$$\frac{\phi_{\theta_2}([\gamma])}{\phi_{\theta_1}([\gamma])} = \exp\left(\int_{C_\gamma} \uparrow_\delta d\zeta\right) = \exp\left(\int_{C_\gamma} d \uparrow_\delta \zeta\right) = \frac{\exp(\int_{\mathbf{y}} \uparrow_\delta \zeta)}{\exp(\int_{\mathbf{x}} \uparrow_\delta \zeta)},$$

and so

$$\begin{gathered}\Gamma_{\theta_1}(\mathbf{x}) \to \Gamma_{\theta_2}(\mathbf{x}) \\ z \mapsto z \exp\left(\int_{\mathbf{x}} \uparrow_\delta \zeta\right)\end{gathered}$$

defines an isomorphism of local systems $\Gamma_{\theta_1} \cong \Gamma_{\theta_2}$, as claimed in the introduction.

The construction of the $\Gamma_\theta$ depends on a choice of de Rham representatives of the classes $PD(\Delta) \in H^2(Y \times_{S^1} Y_d(f);\mathbb{R})$ and $c+\frac{2\pi}{\tau}PD(h) \in H^2(Y;\mathbb{R})$; it is easy to see that different choices of these representatives give rise to local systems which are isomorphic, with the isomorphism depending on cohomologies between the different choices.

## 5. Maps from cobordisms

The basic ingredient in the construction of the maps on $HF$ obtained from "fibered cobordisms" between objects $(Y, f, h, c, \tau)$ in the category $\mathsf{FCOB}$ is the fact that, given a Lefschetz fibration $f\colon\ X \to B$ with closed fibers, a fiberwise positive class $c \in H^2(X;\mathbb{R})$, and a class $h \in H_2(X, \partial X;\mathbb{Z})$ with $h \cap [fiber] = d > 0$, one can construct a *relative Hilbert scheme* $F\colon\ X_d(f) \to B$ such that for regular values $t$ of $f$ one has $F^{-1}(t) = Sym^d f^{-1}(t)$, and which carries a symplectic form $\Omega$ such that for each component $S$ of $\partial B$, $(F^{-1}(S), \Omega|_{F^{-1}(S)})$ is isomorphic as a locally Hamiltonian fibration to the mapping torus $Y_{\Phi_{d,\omega,\tau,S}}$, $\Phi_{d,\omega,\tau,S}$ being the Salamon monodromy map defining the Floer group $HF(Y, f|_{f^{-1}(S)}, \partial_S h, c|_{f^{-1}(S)}, \tau)$. In the case that $\partial B$ has two components and $X$ is a morphism in $\mathsf{FCOB}$ from $\mathfrak{o}_-$ to $\mathfrak{o}_+$, $(X_d(f), \Omega)$ then defines a symplectic cobordism from the mapping torus used to define $HF(\mathfrak{o}_-)$ to that used to define $HF(\mathfrak{o}_+)$, and the map between the Floer groups is obtained by adding half-infinite cylindrical ends to this symplectic cobordism and then counting pseudoholomorphic cylinders with prescribed asymptotics.

The construction of the relative Hilbert scheme $X_d(f)$, including the proof of the crucial fact that it is smooth in spite of the presence of singular fibers in the Lefschetz fibration $f\colon\ X \to B$, is carried out in detail in [42]. The existence of a natural deformation equivalence class of symplectic structures on $X_d(f)$ is also proven in [42], but in order to obtain the proper behavior of the symplectic form on $\partial X_d(f)$ we shall need somewhat more refined results, which we now set about proving.

Recall that in Section 2, given an object $(Y, f, h, c, \tau)$ in $\mathsf{FCOB}$ and a closed form $\omega \in \Omega^2(Y)$ representing $c + \frac{2\pi}{\tau} PD(h)$, when $d > 0$ we have obtained a symplectic form $\Omega_{d,\omega,\tau}$ on $S^d\Sigma$ and a map $\Phi_{d,\omega,\tau}\colon S^d\Sigma \to S^d\Sigma$ which preserves $\Omega_{d,\omega,\tau}$. (When $d \leq 0$ we have set $HF = 0$ except when $h = 0$, in which case $HF$ is the coefficient ring $A$; in all such cases we hereby define the cobordism maps of this section to be the identity, and so restrict attention to the case $d > 0$ hereinafter.) The mapping torus $Y_{\Phi_{d,\omega,\tau}}$ then carries a closed fiberwise symplectic form $(\Omega_{d,\omega,\tau})_{\Phi_{d,\omega,\tau}}$; note that $Y_{\Phi_{d,\omega,\tau}} \cong Y_d(f)$ as fibrations over $S^1$. Recall also the definition of a starred surface $B$ in the introduction, which in particular specifies a set $\{\beta_1, \ldots, \beta_{g(B)}\}$ of disjoint curves on $B$.

**Lemma 5.1.** *Fix $\tau > 2\pi d$, and let $f\colon X \to B$ be a fibration by closed surfaces of genus $g \geq 2$ over a compact connected starred surface $B$ whose boundary decomposes into connected components as $\partial B = \partial_1 B \cup \cdots \cup \partial_n B$. Write $Y_i = f^{-1}(\partial_i B)$, and let $c_i \in H^2(f^{-1}(\partial_i B); \mathbb{R})$, $h_i \in H^1(f^{-1}(\partial_i B); \mathbb{Z})$, and $b_j \in H^2(f^{-1}(\beta_j); \mathbb{R})$ be such that there exist $\tilde{c} \in H^2(X; \mathbb{R})$, $PD(\tilde{h}) \in H^2(X; \mathbb{Z})$ both evaluating positively on the fibers (say $\langle PD(\tilde{h}), [fiber]\rangle = d$) with $c_i = \tilde{c}|_{Y_i}$, $PD(h_i) = PD(\tilde{h})|_{Y_i}$, and $b_j = (\tilde{c} + 2\pi PD(\tilde{h})/\tau)|_{f^{-1}(\beta_j)}$. Let $\omega_i \in \Omega^2(Y_i)$ be closed fiberwise symplectic forms representing $c_i + 2\pi PD(h_i)/\tau = (c + 2\pi PD(h)/\tau)|_{Y_i} \in H^2(Y_i; \mathbb{R})$. Then there is a fiberwise symplectic form $\tilde{\Omega} = \Omega_{\vec{c},\vec{h},\vec{b}} \in \Omega^2(X_d(f))$, determined canonically up to isotopy by the $c_i, h_i$, and $b_j$ and independent of the choices of $\tilde{c}$ and $\tilde{h}$, such that*

$$[\tilde{\Omega}|_{(Y_i)_d(f)}] = [(\Omega_{d,\omega_i,\tau})_{\Phi_{d,\omega_i,\tau}}]$$

*for each $i$.*

*Proof.* First note that if $g\colon Z \to S^1$ is a fibration and if $\Omega_-, \Omega_+$ are cohomologous closed fiberwise symplectic forms on $Z$ (say $\Omega_+ = \Omega_- + d\alpha$) which tame a common fiberwise almost complex structure, then on $[-1,1]_t \times Z$, if we set $\tilde{\Omega} = \Omega_- + d(\rho(t)\alpha)$ where $\rho\colon [-1,1] \to [0,1]$ is a smooth function which is identically zero near $-1$ and identically 1 near 1, then $\tilde{\Omega}$ is a closed fiberwise symplectic form on $Z \times [-1,1] \to S^1 \times [-1,1]$ which, for small $\epsilon$, restricts to $Z \times [-1, -1+\epsilon]$ as the pullback of $\Omega_-$ and to $Z \times [1-\epsilon, 1]$ as the pullback of $\Omega_+$. Using this device, in the context of the lemma it becomes straightforward to construct a closed, fiberwise Kähler form $\tilde{\Omega}$ over all of $B$ satisfying the desired properties as soon as we have constructed it over the surface obtained by cutting $B$ along its $\beta$-curves in such a way that $\tilde{\Omega}$ restricts over $\beta_j$ to $\Omega_{d,\omega_j,\tau}$, $\omega_j$ being a fiberwise symplectic representative of $b_j \in H^2(f^{-1}(\beta_j); \mathbb{R})$. This observation reduces us to the case that $B$ has genus zero.

Having made this reduction, consider first the case that $B = S^2$, which is simpler in that here there are no boundary conditions for the form to satisfy; the only datum given to us is the fibration map $f\colon X \to B$, which is a bundle whose structure group is the identity component $Diff_0(\Sigma)$ of the diffeomorphism group of $\Sigma$. Now it follows from Moser stability [29] that $Diff_0(\Sigma)$ retracts to $Symp(\Sigma, \omega)$, while since $g \geq 2$ the Earle–Eells theorem [8] states that $Diff_0(\Sigma)$ is contractible; hence the structure group of the fibration reduces to $Symp(\Sigma, \omega)$, which is contractible. The fibration thus admits a symplectic trivialization which is unique up to isotopy, giving us a canonical symplectic identification of $X$ with $S^2 \times \Sigma$ with $f$ being the projection to the first factor, and this induces an identification $X_d(f)$ with $S^2 \times S^d\Sigma$. So we take for $\tilde{\Omega}$ the pullback of the Salamon form $\Omega_{d,\omega,\tau}$ on any of the individual fibers. Having dispensed with this trivial case, assume that $B$ has at least one boundary component. Let $\tilde{\omega}$ be a fiberwise symplectic form on the total space of $f\colon X \to B$ which represents the class $\tilde{c} + 2\pi PD(\tilde{h})/\tau \in H^2(X;\mathbb{R})$ for an arbitrary choice of $\tilde{c}, \tilde{h}$ as in the statement of the lemma (such $\tilde{\omega}$ exists by the Thurston trick). After trivializing $f$ over a tubular neighborhood $S \subset B$ of the contractible "star" consisting of the arcs from the interior base point $b$ to the boundary basepoints that are part of the data of the starred surface $B$ and modifying $\tilde{\omega}$ by an exact form, we can assume that $\tilde{\omega}|_{f^{-1}(S)}$ is just the pullback to $f^{-1}(S) = S \times \Sigma$ of a volume form $\omega$ on $\Sigma$. Let $U^{(1)}$ be the union of $S$ with tubular neighborhoods of the boundary components (so $U^{(1)}$ is a regular neighborhood of a 1-skeleton for $B$ since we've reduced to $g(B) = 0$).

Using Salamon's construction we can obtain a closed fiberwise Kähler form $\tilde{\Omega}^{(1)}$ over $U^{(1)}$ as follows. Start with the trivial extension of Salamon's form $\Omega = \Omega_{d,\omega,\tau}$ on $S^d f^{-1}(0)$ to $S \times S^d f^{-1}(0)$, and then, over the strips $(-\epsilon, \epsilon) \times \partial_i B \subset U^{(1)}$ corresponding to the boundary components attach strips $(-\epsilon, \epsilon) \times (Y_i)_d(f)$ to $D^2 \times S^d f^{-1}(0)$. To describe $\tilde{\Omega}^{(1)}$ on these latter strips, let $\Phi_i\colon S^d f^{-1}(0) \to S^d f^{-1}(0)$ be the Salamon monodromy map associated to the mapping torus of the monodromy of $\tilde{\omega}$ around these loops; the form on the strip $(-\epsilon, \epsilon) \times (Y_i)_d(f)$ can then just be taken to be the pullback of the form $(\Omega)_{\Phi_i}$ on $(Y_i)_d(f)$.

Now $U^{(1)}$ has $n+1$ boundary components $C, \partial_1 B, \ldots, \partial_n B$ with $[C] = \sum[\partial_i B]$ in homology, and $B$ is obtained by attaching a disc $D'$ to $C$. Since $C$ is contractible in $B$, the $\tilde{\omega}$–monodromy around it is Hamiltonian, from which it follows as in Proposition 2.2 that the $\tilde{\Omega}$–monodromy around $C$ (which is just the composition of the monodromies around the $\partial_i B$) is Hamiltonian as well. Write $F\colon X_d(f) \to B$ for the map defining the relative symmetric product. Then picking $p \in C = \partial D$, $\tilde{\Omega}|_{F^{-1}(C)}$ is cohomologous to the pullback of $\Omega|_{F^{-1}(p)}$ to $S^1 \times F^{-1}(p)$ as a result of the fact that they have Hamiltonian–equivalent monodromies, and so just as in the first paragraph of this proof we can interpolate between these two forms and so glue $\tilde{\Omega}|_{F^{-1}(C)}$ to the pullback of $\tilde{\Omega}|_{F^{-1}(p)}$ to $D' \times F^{-1}(p')$, thus producing the desired form $\tilde{\Omega}$ on all of $F^{-1}(B)$. (The conclusion about $[\tilde{\Omega}|_{(Y_i)_d(f)}]$ follows from the fact that the monodromy of $\tilde{\Omega}$ around $\partial_i B$ is Hamiltonian–equivalent to the Salamon monodromy $\Phi_{d,\omega_i,\tau}$.) $\square$

In order to extend the previous lemma from genuine fibrations to Lefschetz fibrations, consider now an elementary Lefschetz fibration $p\colon E \to D^2$, that is, a Lefschetz fibration over the disc with just one singular fiber $E_0$, which lies over the origin and contains just one node. If $Y = \partial E$, the image of the restriction map $H^2(E;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ then has rank one, generated by $(2-2g)^{-1} e(T^{vt}Y)$. In particular, letting $c \in H^2(E;\mathbb{R})$ and $h \in H_2(E, \partial E;\mathbb{Z})$ meet the fibers positively, the class $[\omega_Y] = c|_Y + \frac{2\pi}{\tau} PD(\partial h) \in H^2(Y;\mathbb{R})$ of the form $\omega_Y$ on $Y$ used in the construction of the Salamon monodromy map for the object $(Y, p|_Y, \partial h, c, \tau)$ will be some negative (since $g \geq 2$) multiple of $e(T^{vt}Y)$.

According to Lemma 3.15 of [35] (which for our present purposes replaces an erroneous lemma in [7] which we had referred to in earlier versions of this paper; we thank the referee for pointing out this issue and suggesting a way of resolving it), where $\eta \in H^2(E;\mathbb{Z})$ restricts to the singular fiber $E_0$ as the orientation class (so $\eta$ is a positive multiple of $c + \frac{2\pi}{\tau}PD(h)$), any cohomology class in $H^2(E_d(p);\mathbb{Z})$ of form

$$s \uparrow \eta + t(d \uparrow \eta - (\pi_2)_!(\mathfrak{d}^2)/2)$$

with $s, t > 0$ is represented by Kähler forms. In particular, for $\tau > 2\pi d$, there are Kähler forms on $E_d(p)$ whose restrictions to the boundary $Y_d(p)$ represent the class $[(\Omega_{d,\omega_Y,\tau})_{\Phi_{d,\omega_Y,\tau}}] = 2\pi(\tau \uparrow [\omega_Y] - \pi(\pi_2)_!(\mathfrak{d}^2))$ of Theorem 3.5.

This positions us to state the result underlying the construction of the maps induced by fibered cobordisms.

**Proposition 5.2.** *Let $f\colon X \to B$ be a Lefschetz fibration on a 4–manifold $X$ over a starred surface $B$ with boundary $\partial B = \partial_1 B \cup \cdots \cup \partial_n B$ such that the critical values of $f$ are precisely the interior special points of $B$. Write $Y_i = f^{-1}(\partial_i B)$, and let $c_i \in H^2(f^{-1}(\partial_i B);\mathbb{R})$, $h_i \in H^1(f^{-1}(\partial_i B);\mathbb{Z})$, and $b_j \in H^2(f^{-1}(\beta_j);\mathbb{R})$ be such that there exist $\tilde{c} \in H^2(X;\mathbb{R})$, $PD(\tilde{h}) \in H^2(X;\mathbb{Z})$ both evaluating positively on the fibers (say $\langle PD(\tilde{h}), [fiber]\rangle = d$) with $c_i = \tilde{c}|_{Y_i}$, $PD(h_i) = PD(\tilde{h})|_{Y_i}$, and $b_j = (\tilde{c} + 2\pi PD(\tilde{h})/\tau)|_{f^{-1}(\beta_j)}$. Let $\omega_i \in \Omega^2(Y_i)$ be closed fiberwise symplectic forms representing $c_i + 2\pi PD(h_i)/\tau = (c + 2\pi PD(h)/\tau)|_{Y_i} \in H^2(Y_i;\mathbb{R})$. Then for $\tau > 2\pi d$ there is a fiberwise symplectic form $\tilde{\Omega} = \Omega_{\vec{c},\vec{h},\vec{b}} \in \Omega^2(X_d(f))$, determined canonically up to isotopy by the $c_i, h_i, \tau$, and $b_j$ and independent of the choices of $\tilde{c}$ and $\tilde{h}$, such that*

$$[\tilde{\Omega}|_{(Y_i)_d(f)}] = [(\Omega_{d,\omega_i,\tau})_{\Phi_{d,\omega_i,\tau}}]$$

*for each $i$.*

*Proof.* Let $p_1, \ldots, p_k$ be the critical values of $f$, so that since the $p_i$ are the interior special points of $B$ our data include arcs from the interior base point $b$ of $B$ to the $p_i$. Let $U_1, \ldots, U_k$ small disjoint neighborhoods of $p_i$, and take an almost complex structure on $X$ with respect to which $f\colon X \to B$ is pseudoholomorphic and whose restriction to each $f^{-1}(U_i)$ is integrable. Write $B^0 = B \setminus \cup U_i$ and $F\colon X_d(f) \to B$ for the map defining the relative Hilbert scheme. $B^0$ then inherits from $B$ the structure of a starred surface with boundary; $B^0$ has no interior special points, and has new boundary components $\partial U_i$ corresponding to the special points of $B$, with the arcs from $b$ to the new boundary components just given by the portions of the arcs from $b$ to the special points of $B$ that lie in $B^0$.

We now apply Lemma 5.1 to $B^0$ (taking for the boundary data $c, PD(h)$ on the new boundary components $f^{-1}(\partial U_i)$ appropriate multiples of $e(T^{vt}f^{-1}(\partial U_i)))$. This gives us a closed fiberwise symplectic form $\tilde{\Omega}^0 = \tilde{\Omega}^0_{\vec{c},\vec{h},\vec{b}}$ on $F^{-1}(B^0)$, while the remarks before the proposition give us Kähler forms $\tilde{\Omega}^i$ on each $F^{-1}(U_i)$; furthermore the restrictions of $\tilde{\Omega}^0$ and $\tilde{\Omega}^i$ to $F^{-1}(\partial U_i)$ are cohomologous and have restrictions which are compatible with the complex structures on each fiber of $F^{-1}(\partial U_i) \to S^1$. But then it is straightforward to glue these together to obtain the desired form on the total space: letting $t$ be the first coordinate on $[-1,1] \times F^{-1}(U_i)$, $\rho\colon [-1,1] \to [0,1]$ a monotone smooth function which is identically 0 near $-1$ and identically 1 near 1, and $\alpha_i \in \Omega^1(F^{-1}(\partial U_i))$ such that $\tilde{\Omega}^i = \tilde{\Omega}^0 + d\alpha_i$, the form $\tilde{\Omega}^0 + d(\rho(t)\alpha_i)$ will be a closed form equal to $\tilde{\Omega}^0$ for $t$ near $-1$ and to $\tilde{\Omega}^i$ for $t$ near 1, and its restriction to each fiber will be a convex combination of Kähler forms and so will be Kähler. $\square$

With this preparation, the existence of the maps promised in Theorem 1.6 follows from an application of the fairly standard idea that a symplectic cobordism gives rise to maps on Floer homology groups. Let $\mathfrak{m} = (X, \tilde{f}, \tau)$ be a morphism between objects $(Y_\pm, f_\pm, h_\pm, c_\pm, \tau)$ in $\mathsf{FCOB}$, so that in particular $\tilde{f}\colon X \to B$ is a Lefschetz fibration with boundary components $Y_\pm$ (either of which is allowed to be empty), and the sets

$$H_{h_-,h_+} = \{\tilde{h} \in H_2(X, \partial X; \mathbb{Z}) | \partial_\pm \tilde{h} = h_\pm \in H_1(Y_\pm; \mathbb{Z})\}$$

and

$$C_{c_-,c_+} = \{\tilde{c} \in H^2(X; \mathbb{Z}) | \tilde{c}|_{Y_\pm} = c_\pm\}$$

are nonempty. Further, we have, as part of the data of the morphism, classes $b_j \in H^2(f^{-1}(\beta_j); \mathbb{R})$ as in Proposition 5.2. From that Proposition, we then obtain a fiberwise symplectic form $\tilde{\Omega}$ on $X_d(\tilde{f})$ restricting to a fiberwise Kähler form cohomologous to the Salamon form $\Omega_{d,\omega_\pm,\tau}$ on the boundary components, where $\omega_\pm \in \Omega^2(Y_\pm)$ is a fiberwise symplectic representative of $c_\pm + \frac{2\pi}{\tau} PD(h_\pm)$. Let $\bar{f}\colon \bar{X} \to \bar{B}$ be the Lefschetz fibration obtained by adding trivial cylindrical ends $(-\infty, -1] \times Y_- \to (-\infty, -1] \times S^1$ and $[1, \infty) \times Y_+ \to [1, \infty) \times S^1$ to $X \to B$ (of course if one or both of $Y_\pm$ is the empty object we don't add an end corresponding to that object). Then our usual device involving a cutoff function gives us a closed, fiberwise symplectic form $\bar{\Omega}$ on $\bar{X}_d(\bar{f})$ extending $\tilde{\Omega} \in \Omega^2(X_d(\tilde{f}))$ and equal to the pullback of $\Omega_{d,\omega_-,\tau}$ on $(-\infty, -2] \times (Y_-)_d(f)$ and to the pullback of $\Omega_{d,\omega_+,\tau}$ on $[2, \infty) \times (Y_+)_d(f)$.

Note that our construction of $\tilde{\Omega}$ is compatible with the composition of morphisms, in the sense that if $\mathfrak{m}_0 \in Mor(\mathfrak{o}_0, \mathfrak{o}_1)$ and $\mathfrak{m}_1 \in Mor(\mathfrak{o}_1, \mathfrak{o}_2)$, then after isotoping the forms $\tilde{\Omega}_i \in \Omega^2((X_i)_d(\tilde{f}_i))$ obtained from Proposition 5.2 to coincide near their common boundary component $(Y_i)_d(f_i)$, the form $\tilde{\Omega}$ on the relative Hilbert scheme associated to $\mathfrak{m}_1 \circ \mathfrak{m}_0$ is (up to isotopy) obtained by gluing together the forms $\tilde{\Omega}_i$ coming from the two pieces. This compatibility property is the main motivation for the additional technical data that we have included in our definition of a morphism.

Now if $\omega_{\bar{B}}$ is a volume form on the base and $\bar{F}\colon \bar{X}_d(\bar{f}) \to \bar{B}$ is the map defining the relative Hilbert scheme, then $\bar{\Omega} + K\bar{F}^*\omega_{\bar{B}}$ will be a symplectic form for large enough $K \in \mathbb{R}$; let $J$ be an almost complex structure on $\bar{X}_d(\bar{f})$ which is compatible with this symplectic structure, which makes $\bar{F}$ a pseudoholomorphic map, and which agrees with the standard complex structure on the relative Hilbert scheme on the preimages of small neighborhoods of each of the critical points of $\bar{f}$. We shall define our maps on the Floer homology groups by counting certain $J$–holomorphic sections of $\bar{F}$ with prescribed asymptotics in $(Y_\pm)_d(f)$.

To be more specific about which sections are counted and how, consider any class $\tilde{h} \in H_{h_-,h_+} \subset H_2(X, \partial X; \mathbb{Z})$. Now the fiber product $X_d(f) \times_B X$ contains a universal divisor $\mathcal{D}$ (for regular values $t \in B$ of $X \to B$, $\mathcal{D}$ meets the fiber $S^d f^{-1}(t) \times f^{-1}(t)$ over $t$ in $\{(D, p) | p \in D\}$; the extension of $\mathcal{D}$ over the critical values follows from the algebro-geometric description of the relative Hilbert scheme of an elementary Lefschetz fibration, the details of which will not be relevant to us here). Hence we get a map $\downarrow\colon H_2(X_d(f), \partial X_d(f); \mathbb{Z}) \to H_2(X, \partial X; \mathbb{Z})$ analogous to the map on relative symmetric products considered earlier. Where $\Gamma(X_d(f))$ denotes the space of sections of $X_d(f)$, there is a natural evaluation map $\pi_0(\Gamma(X_d(f))) \to$

$H_2(X_d(f), \partial X_d(f); \mathbb{Z})$. Composing this with $\downarrow$ gives a map

$$\pi_0(\Gamma(X_d(f)) \to H_2(X, \partial X; \mathbb{Z})$$
$$\gamma \mapsto h_\gamma;$$

Lemma 4.1 of [42] shows that this map is injective.

Our maps $F_{\mathfrak{m},\theta,\tilde{h}}$ in Theorem 1.6 will be the maps induced on homology of certain chain maps $\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}$ on the Floer chain complexes. As always taking $d = \tilde{h} \cap [fiber]$, if our class $\tilde{h} \in H_2(X, \partial X; \mathbb{Z})$ is not in the image of $(\gamma \mapsto h_\gamma)$ we then define $\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}$ to be zero; otherwise let $\gamma_{\tilde{h}}$ denote the unique preimage of $\tilde{h}$ under $(\gamma \mapsto h_\gamma)$. $\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}$ will then be constructed using pseudoholomorphic sections of $\bar{F}\colon \bar{X}_d(f) \to \bar{B}$ in the homotopy class $\gamma_{\tilde{h}}$, with weights depending on $\theta$ and $h$.

We now define the promised maps

$$\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}(U^r \otimes \eta_1 \wedge \cdots \wedge \eta_k \otimes \cdot)\colon\ CF(\mathfrak{o}_-; \Lambda^R_{Nov} \otimes \Gamma_\theta) \to CF(\mathfrak{o}_+; \Lambda^R_{Nov} \otimes \Gamma_\theta)$$

where $r \geq 0$ and $\eta_i \in H_1(X; \mathbb{Z})$. Assume first that neither of $\mathfrak{o}_\pm$ is the empty object. Then $CF(\mathfrak{o}_\pm; \Lambda^R_{Nov})$ is freely generated over $\Lambda^R_{Nov}$ by the "constant sections" $\mathbf{x}$ of $(Y_\pm)_d(f)$ corresponding to the fixed points of Salamon's symplectomorphism $\Phi_{d,\omega_\pm,\tau}$. If $\mathbf{x}_-$ is a generator of $CF(\mathfrak{o}_-; \Lambda^R_{Nov})$ and $\mathbf{x}_+$ is a generator of $CF(\mathfrak{o}_+; \Lambda^R_{Nov})$, and if $J$ is an almost complex structure on $\bar{X}_d(\bar{f})$ which is compatible with the symplectic structure, which makes the projection $\bar{X}_d(\bar{f}) \to \bar{B}$ pseudoholomorphic for some chosen complex structure on $\bar{B}$, and which agrees with the Kähler structure near the singular fibers, then let

$$\mathcal{M}_{J,\mathfrak{m},\tilde{h}}(\mathbf{x}_-, \mathbf{x}_+)$$

denote the moduli space of $J$–holomorphic sections of $\bar{X}_d(\bar{f})$ which

(i) are asymptotic to a cylinder on $\mathbf{x}_-$ on the $(Y_-)_d(f_-)$ end of $\bar{X}_d(\bar{f})$,
(ii) are asymptotic to a cylinder on $\mathbf{x}_+$ on the $(Y_+)_d(f_+)$ end of $\bar{X}_d(\bar{f})$, and
(iii) represent the homotopy class $\gamma_{\tilde{h}}$.

Provided that $d \notin [(g+1)/2, g-1)$, so that the $w^+$–monotonicity condition of [21] holds (implying that moduli spaces will generically not contain bubble trees involving multiply covered spheres of negative Chern number[3]), $\mathcal{M}_{J,\mathfrak{m},\tilde{h}}(\mathbf{x}_-, \mathbf{x}_+)$ will, for generic $J$, be a smooth manifold of a certain dimension $\delta(\tilde{h})$ which admits a compactification $\bar{\mathcal{M}}_{J,\mathfrak{m},\tilde{h}}(\mathbf{x}_-, \mathbf{x}_+)$ by

(i) "broken sections" consisting of a chain of Floer flowlines in $(Y_-)_d(f_-)$, followed by a section of $\bar{X}_d(\bar{f})$, followed by a chain of Floer flowlines in $(Y_+)_d(f_+)$. Each section in the entire chain begins where the previous one ends, and the homotopy class of the whole sequence is $\gamma_{\tilde{h}}$;
(ii) "cusp sections" consisting of a section of $\bar{X}_d(\bar{f})$, together with spherical bubbles in various fibers of $\bar{X}_d(\bar{f})$; and
(iii) combinations of (i) and (ii).

These additional strata will have codimension at least 1, and the only codimension one strata will be those made up of broken sections with just one Floer flowline component. When $d \in [(g-1)/2, g-1)$, so that virtual moduli methods are required,

[3]Superficially, one would also need to rule out bubbles in the singular fibers of the relative Hilbert scheme; however, Lemma 4.8 of [7] shows that any such bubble would be homologous to a sphere in a smooth fiber, so such bubbles do not complicate the analysis.

we can still find rational chains in an ambient space containing $\mathcal{M}_{J,\mathfrak{m},\tilde{h}}(\mathbf{x}_-,\mathbf{x}_+)$ which satisfy properties parallel to these; see [24].

The dimension $\delta(\tilde{h})$ is obtained as follows (see Theorem 3.3.11 of [40]): take an arbitrary section $s$ of $X_d(f)$ representing the class $\gamma_{\tilde{h}}$. $s^*T^{vt}X_d(f)$ is then a symplectic bundle over the open 2–manifold $B$ and so is trivial; choosing any trivialization $\tau$ we can then compute the Conley–Zehnder indices $\mu^\tau_{CZ}(\mathbf{x}_\pm)$ of $T^{vt}(Y_\pm)_d(f_\pm)|_{\mathbf{x}_\pm}$ with respect to the trivialization; then

$$\delta(\tilde{h}) = \mu^\tau_{CZ}(\mathbf{x}_+) - \mu^\tau_{CZ}(\mathbf{x}_-) + 2d\chi(B)$$

is independent of the various choices and is the desired virtual dimension.

Now if $\alpha \in \Omega^2(\bar{X}_d(\bar{f}))$ is any closed form, then the integral of $\alpha$ over any cylinder in $\bar{X}_d(\bar{f})$ representing an element of $\mathcal{M}_{J,\mathfrak{m},\tilde{h}}(\mathbf{x}_-,\mathbf{x}_+)$ depends only on $\tilde{h},\mathbf{x}_-$, and $\mathbf{x}_+$; denote this common value by $\alpha(\tilde{h},\mathbf{x}_-,\mathbf{x}_+)$. This remark in particular applies to both the forms $\uparrow\theta$ (which we can easily make sense of thanks to the fact that we assumed in the statement of Theorem 1.6 that $\theta$ vanishes near the critical points of $f$) and to the form $\bar{\Omega}$ introduced after Proposition 5.2. For $\mathbf{x}_- \in \mathcal{A}_-$, we set

$$\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}(U^k\otimes\eta_1\wedge\cdots\wedge\eta_l\otimes\mathbf{x}_-) = \sum_{\mathbf{x}_+\in\mathcal{A}_+} T^{\bar{\Omega}(\tilde{h},\mathbf{x}_-,\mathbf{x}_+)}e^{\uparrow\theta(\tilde{h},\mathbf{x}_-,\mathbf{x}_+)}\langle\mathbf{x}_-|U^r\otimes\eta_1\wedge\cdots\wedge\eta_k|\mathbf{x}_+\rangle,$$

where

$$\langle\mathbf{x}_-|U^r\otimes\eta_1\wedge\cdots\wedge\eta_k|\mathbf{x}_+\rangle$$

is defined by:

- if $\delta(\tilde{h}) \neq 2k+l$, then $\langle\mathbf{x}_-|U^r\otimes\eta_1\wedge\cdots\wedge\eta_k|\mathbf{x}_+\rangle = 0$.
- if $\delta(\tilde{h}) = 2k+l$, let $A_1,\ldots,A_r$ be generic representatives of the class $PD(\uparrow PD[pt]) \in H^{2d-2}(\bar{X}_d(\bar{f}))$, and let $D_{\eta_1},\ldots,D_{\eta_k}$ be generic representatives of the classes $PD(\uparrow PD(\eta_i)) \in H^{2d-1}(\bar{X}_d(\bar{f}))$. The set

  $$\begin{aligned}\mathcal{M}_{J,\mathfrak{m},\tilde{h}}(A_1,\ldots,A_r,D_1,\ldots,D_k;\mathbf{x}_-,\mathbf{x}_+) = \\ \{(u,x_1,\ldots,x_r,y_1,\ldots,y_k) \in \mathcal{M}_{J,\mathfrak{m},\tilde{h}}(\mathbf{x}_-,\mathbf{x}_+)\times B^r\times B^k| \\ u(x_i)\in A_i, u(y_j)\in D_j\ \forall i,j\}\end{aligned}$$

  will have virtual dimension $\delta(\tilde{h}) + 2(2k+l) - 4k - 3l = 0$, and we let $\langle\mathbf{x}_-|U^r\otimes\eta_1\wedge\cdots\wedge\eta_k|\mathbf{x}_+\rangle$ be the signed number of elements in this set for generic $J$ (counted virtually as in [24],[25] if necessary).

Since the chains $A_i$ have even codimension and the $D_i$ have odd codimension, this construction induces a map

$$\tilde{F}_{\mathfrak{m},\theta,h}\colon\ \mathbb{A}(X)\otimes CF(\mathfrak{o}_-;\Lambda^R_{Nov}\otimes\Gamma_\theta)\to CF(\mathfrak{o}_+;\Lambda^R_{Nov}\otimes\Gamma_\theta),$$

which might depend on the additional choices (the finiteness condition for the Novikov ring is trivial, since there are only finitely many pairs $(\mathbf{x}_-,\mathbf{x}_+)$ and so $\bar{\Omega}(\tilde{h},\mathbf{x}_-,\mathbf{x}_+)$ takes just finitely many values for a given choice of $\tilde{h}$).

**Lemma 5.3.** *Given a morphism $\mathfrak{m} = (X,\tilde{f},\tau)$ between the objects $\mathfrak{o}_\pm = (Y_\pm,f_\pm,h_\pm,c_\pm,\tau)$ and a closed form $\theta\in\Omega^2(X)$ which vanishes near the critical points of $X$, for each $\tilde{h}\in H_{h_-,h_+}$, the map*

$$\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}\colon\ \mathbb{A}(X)\otimes CF(\mathfrak{o}_-;\Lambda^R_{Nov}\otimes\Gamma_{\theta|_{Y_-}})\to CF(\mathfrak{o}_+;\Lambda^R_{Nov}\otimes\Gamma_{\theta|_{Y_+}})$$

*is a chain map, and the induced map on homology*

$$F_{\mathfrak{m},\theta,\tilde{h}}\colon\ \mathbb{A}(X)\otimes HF(\mathfrak{o}_-;\Lambda^R_{Nov}\otimes\Gamma_{\theta|_{Y_-}})\to HF(\mathfrak{o}_+;\Lambda^R_{Nov}\otimes\Gamma_{\theta|_{Y_+}})$$

*is independent of the choices of chains $A_i, D_j$ and of the almost complex structure $J$. The sum $\tilde{F}_{\mathfrak{m},\theta}=\sum_{\tilde{h}\in H_{h_-,h_+}}\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}$ is a well-defined chain map $CF(\mathfrak{o}_-;\Lambda^R_{Nov})\to CF(\mathfrak{o}_+;\Lambda^R_{Nov})$. Furthermore if $X=[0,1]\times Y_-$ is the trivial cobordism, then $F_{\mathfrak{m},\theta,\tilde{h}}$ coincides with the map defining the $\mathbb{A}(Y)$–module structure of $HF(\mathfrak{o}_-;\Lambda^R_{Nov}\otimes\Gamma_{\theta_{Y_-}})$.*

*Proof.* To see that $\sum_{\tilde{h}\in H_{h_-,h_+}}\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}$ is well-defined we just need to check the finiteness condition for the Novikov ring. However this follows directly from Gromov compactness, which ensures that for any $c$ the set

$$\{u\in\cup_{\tilde{h},\mathbf{x}_-,\mathbf{x}_+}\bar{\mathcal{M}}_{J,\mathfrak{m},\tilde{h}}(\mathbf{x}_-,\mathbf{x}_+)|\int_B u^*\bar{\Omega}<c\}$$

is compact, together with the definition that $\Omega(\tilde{h},\mathbf{x}_-,\mathbf{x}_+)=\int_B u^*\bar{\Omega}$ for any section $u$ in the homotopy class $\gamma_{\tilde{h}}$ which is asymptotic to $\mathbf{x}_\pm$.

Now for the trivial cobordism, $\bar{\Omega}$ is just the pullback of the Salamon form $\Omega_{d,\omega,\tau}$ on $Y_d(f)$ to $\mathbb{R}\times Y_d(f)$. Also, using the exactness of $H_2(\partial I\times Y;\mathbb{Z})\to H_2(I\times Y;\mathbb{Z})\to H_2(I\times Y,\partial I\times Y;\mathbb{Z})\to H_1(\partial I\times Y;\mathbb{Z})$ and the fact that the first map is a surjection, the difference between any two elements of $H_{h_-,h_+}$ lies in the image of $H_2(I\times Y;\mathbb{Z})\to H_2(I\times Y,\partial I\times Y;\mathbb{Z})$ and so is zero. So in this case $H_{h_-,h_+}$ is a singleton $\{\tilde{h}\}$ and $\tilde{F}_{\mathfrak{m},\theta}=\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}$. The complex $CF(Y,f,h,c,\tau;\Lambda^R_{Nov}\otimes\Gamma_\theta)$ is generated over $\Lambda^R_{Nov}$ by the fixed points $\mathbf{x}$, and the differential just counts holomorphic sections $u$ of $\mathbb{R}\times Y_d(f)$, weighted by $T^{\int_{\mathbb{R}\times S^1}u^*\Omega_{d,\omega,\tau}}e^{\int_{\mathbb{R}\times S^1}u^*(\dagger\theta)}$. In particular,

$$\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}(1\otimes 1\otimes\cdot)\colon\ CF(\mathfrak{o};\Lambda^R_{Nov}\otimes\Gamma_\theta)\to CF(\mathfrak{o};\Lambda^R_{Nov}\otimes\Gamma_\theta)$$

is none other than the identity (corresponding to sections of form $\mathbb{R}\times\{\mathbf{x}\}$) plus the Floer boundary operator. Furthermore, by taking all the chains $A_i, D_j\in C_*(\mathbb{R}\times Y_d(f);\mathbb{Z})$ to be contained in $\{0\}\times Y_d(f)$, comparing with section 4 reveals that

$$\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}\colon\ \mathbb{Z}[U]\otimes\Lambda^*(H_1(\mathbb{R}\times Y;\mathbb{Z})/torsion)\otimes CF(\mathfrak{o};\Lambda^R_{Nov}\otimes\Gamma_\theta)\to CF(\mathfrak{o}_+;\Lambda^R_{Nov}\otimes\Gamma_\theta)$$

is a chain map which induces the $\mathbb{Z}[U]\otimes\Lambda^*(H_1(Y;\mathbb{Z})/torsion)$–module structure on $HF(\mathfrak{o};\Lambda^R_{Nov}\otimes\Gamma_\theta)$. This proves the last statement of the lemma.

To see that, for general morphisms $\mathfrak{m}$ and for each $\tilde{h}\in H_{h_-,h_+}$, $\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}$ defines a chain map, let $\partial_\pm$ be the Floer boundary operators for $\mathfrak{o}_\pm$. We can then write, for any generators $\mathbf{x}_\pm$ of $\mathfrak{o}_\pm$,

$$\langle\partial_+\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}(U^r\otimes\eta_1\wedge\cdots\wedge\eta_k\otimes\mathbf{x}_-),\mathbf{x}_+\rangle=\Theta_{\tilde{h}}(\mathbf{x}_-,\mathbf{x}_+)T^{\bar{\Omega}(\tilde{h},\mathbf{x}_-,\mathbf{x}_+)}e^{\dagger\theta(\tilde{h},\mathbf{x}_-,\mathbf{x}_+)}$$

and

$$\langle\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}(U^r\otimes\eta_1\wedge\cdots\wedge\eta_k\otimes\partial_-\mathbf{x}_-),\mathbf{x}_+\rangle=\Psi_{\tilde{h}}(\mathbf{x}_-,\mathbf{x}_+)T^{\bar{\Omega}(\tilde{h},\mathbf{x}_-,\mathbf{x}_+)}e^{\dagger\theta(\tilde{h},\mathbf{x}_-,\mathbf{x}_+)}.$$

Here all terms are zero except when $\delta(\tilde{h})=2k+l+1$, in which case $\Theta_{\tilde{h}}(\mathbf{x}_-,\mathbf{x}_+)$ is the signed count of broken sections from $\mathbf{x}_-$ to $\mathbf{x}_+$ consisting of a section of $\bar{X}_d(\bar{f})$ in the relative homotopy class $\gamma_{\tilde{h}}$ followed by a flowline for $CF(\mathfrak{o}_+)$ and which satisfy incidence conditions corresponding to $r$ and the $\eta_i$, while $\Psi_{\tilde{h}}(\mathbf{x}_-,\mathbf{x}_+)$ is the signed count of broken sections from $\mathbf{x}_-$ to $\mathbf{x}_+$ consisting of a flowline for $CF(\mathfrak{o}_-)$

followed by a section of $\bar{X}_d(\bar{f})$ in the class $\gamma_{\tilde{h}}$ which satisfy these same incidence conditions. Note that if $s_0\#s_1$ is a broken section from $\mathbf{x}_-$ to $\mathbf{x}_+$ consisting of a section $s_0$ of $\bar{X}_d(\bar{f})$ asymptotic at its negative end to $\mathbf{x}_-$ and at its positive end to a generator (say $\mathbf{y}$) of $CF(\mathfrak{o}_+)$, followed by a flowline $s_1$ for $CF(\mathfrak{o}_+)$ from $\mathbf{y}$ to $\mathbf{x}_+$ (the latter which extends to a map $[-\infty,\infty]\times S^1\to(Y_+)_d(f_+)$), then extending $s_0$ by concatenating it with $s_1|_{[-\infty,T]\times S^1}$ for $T\in\mathbb{R}\cup\{-\infty,\infty\}$ defines a homotopy rel $(Y_+)_d(f_+)$ between $s_0$ and the broken section $s_0\#s_1$. Hence for any such $s_0\#s_1$, $s_0$ belongs to the relative homotopy class $\gamma_{\tilde{h}}$ if and only if the broken section $s_0\#s_1$ does, and so $\Theta_{\tilde{h}}(\mathbf{x}_-,\mathbf{x}_+)$ may equally well be described as the signed count of broken sections from $\mathbf{x}_-$ to $\mathbf{x}_+$ consisting of a section of $\bar{X}_d(\bar{f})$ in followed by a flowline for $CF(\mathfrak{o}_+)$ such that the (concatenated) broken section belongs to $\gamma_{\tilde{h}}$ and satisfies certain incidence conditions; a similar description applies to $\Psi_{\tilde{h}}(\mathbf{x}_-,\mathbf{x}_+)$. But then a standard argument shows that $\Psi_{\tilde{h}}(\mathbf{x}_-,\mathbf{x}_+)=\Theta_{\tilde{h}}(\mathbf{x}_-,\mathbf{x}_+)$, for their difference counts the oriented number of boundary points of the 1-manifold $\mathcal{M}_{J,\mathfrak{m},\tilde{h}}(A_1,\ldots,A_r,D_1,\ldots,D_k;\mathbf{x}_-,\mathbf{x}_+)$.

Thus $\partial_+\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}=\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}\partial_-$, and, summing over $\tilde{h}\in H_{h_-,h_+}$, $\partial_+\tilde{F}_{\mathfrak{m},\theta}=\tilde{F}_{\mathfrak{m},\theta}\partial_-$. The fact that the induced maps $F_{\mathfrak{m},\theta,\tilde{h}}$ on homology are independent of $J$ and of the choices of chains giving the incidence conditions now follows by standard cobordism arguments: for example if $D_1,D_1'$ are both representatives of $PD(\uparrow PD(\eta_1))$ (say $D_1-D_1'=\partial E$), then a consideration of the boundary components of the moduli spaces $\mathcal{M}_{J,\mathfrak{m},\tilde{h}}(A_1,\ldots,A_r,E,D_2,\ldots,D_k)$ shows that replacing $D_1$ by $D_1'$ in the definition of $\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}$ does not affect the induced map $F_{\mathfrak{m},\theta,\tilde{h}}$ on homology. □

We have been assuming that our morphism $\mathfrak{m}=(X,\tilde{f},\tau)$ is a morphism between nonempty objects $\mathfrak{o}_\pm=(Y_\pm,f_\pm,h_\pm,c_\pm,\tau)$; the construction of $F_{\mathfrak{m},\theta,\tilde{h}}$ in the case in which one or both of the fibered 3–manifolds $Y_\pm$ underlying $\mathfrak{o}_\pm$ is empty is a simple modification of what we have already done. If the "incoming" boundary component $Y_-$ is empty but $Y_+\neq\varnothing$, then we are to define a map

$$F_{\mathfrak{m},\theta,\tilde{h}}\colon\ \mathbb{A}(X)\otimes\Lambda^R_{Nov}\to HF(\mathfrak{o}_+;\Lambda^R_{Nov}\otimes\Gamma_\theta);$$

this is done as before by first defining maps $\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}$ to $CF(\mathfrak{o}_+;\Lambda^R_{Nov}\otimes\Gamma_\theta)$ for classes $\tilde{h}\in H_2(X,Y_+;\mathbb{Z})$ such that $\partial\tilde{h}=h_+$ and $\delta(\tilde{h})=2r+k$ by

$$\begin{aligned}&\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}(U^r\otimes\eta_1\wedge\cdots\wedge\eta_k\otimes 1)\\&\qquad=\sum_{\mathbf{x}_+\in\mathcal{A}_+}T^{\bar{\Omega}(\tilde{h},\mathbf{x}_+)}e^{\uparrow\theta(\tilde{h},\mathbf{x}_+)}\#\mathcal{M}_{J,\mathfrak{m},\tilde{h}}(A_1,\ldots,A_r,D_1,\ldots,D_k;\mathbf{x}_+),\end{aligned}$$

where the notation is as before; of course since $B$ now only has one boundary component there is only one asymptotic condition to specify in expressions such as $\bar{\Omega}(\tilde{h},\mathbf{x}_+)$ and $\#\mathcal{M}_{J,\mathfrak{m},\tilde{h}}(A_1,\ldots,A_r,D_1,\ldots,D_k;\mathbf{x}_+)$. Consideration of the boundary of the spaces $\mathcal{M}_{J,\mathfrak{m},\tilde{h}}(A_1,\ldots,A_r,D_1,\ldots,D_k;\mathbf{x}_+)$ when $\delta(\tilde{h})=1$ reveals that $\partial_+F_{\mathfrak{m},\theta,\tilde{h}}=0$; $F_{\mathfrak{m},\theta,\tilde{h}}$ is then the map on homology induced by this sum, and is independent of the additional choices exactly as in Lemma 5.3

Dually, if $Y_-\neq\varnothing$ but $Y_+=\varnothing$, and if $\mathbf{x}_-$ is a generator of $CF(\mathfrak{o}_-)$, we define $\tilde{F}_{\mathfrak{m},\theta,\tilde{h}}(U^r\otimes\eta_1\wedge\ldots\wedge\eta_k\otimes\mathbf{x}_-)$ by counting pseudoholomorphic sections of $\bar{X}_d(\bar{f})$ which are asymptotic at the boundary to $\mathbf{x}_-$ and which satisfy the usual incidence conditions corresponding to $r$, $\eta_1,\ldots,\eta_k$, with the usual weights $T^{\bar{\Omega}(\tilde{h},\mathbf{x}_-)}e^{\uparrow\theta(\tilde{h},\mathbf{x}_-)}$;

this is a chain map exactly as in the previous case, and $F_{\mathfrak{m},\theta,\tilde{h}}\colon \mathbb{A}(X)\otimes HF(\mathfrak{o}_-;\mathbb{Z}) \to \Lambda^R_{Nov}$ is the induced map on homology.

We pause to consider the simplest nontrivial case, where $Y_- = \varnothing$ and $Y_+ = S^1\times\Sigma$ with the obvious fibration $\pi_1$, and $h_+ = d[S^1\times pt]$ with $c$ proportional to $PD(h)$, so that $\mathfrak{m} = (D^2\times\Sigma, \pi_1, \tau)$ gives a morphism from $\mathfrak{o}_- = \varnothing$ to $\mathfrak{o}_+ = (Y_+, \pi_1, h_+, c_+, \tau)$. Now as in Proposition 3.1(ii) we have a natural surjection $\Pi\colon \mathbb{Z}[U]\otimes\Lambda^*H_1(D^2\times\Sigma;\mathbb{Z}) \twoheadrightarrow H^*(S^d\Sigma;\mathbb{Z})$. Meanwhile the monodromy of a fiberwise symplectic form on $S^1\times\Sigma$ in the class $c_+ + 2\pi PD(h_+)/\tau$ is Hamiltonian, so the Salamon monodromy $\Phi_{d,\omega,\tau}\colon S^d\Sigma \to S^d\Sigma$ is also Hamiltonian, so that the Floer homology $HF(\mathfrak{o}_+;\Lambda^R_{Nov})$ is isomorphic to $H^*(S^d\Sigma;\Lambda^R_{Nov})$. It follows directly from the relevant definitions that our map

$$F_{\mathfrak{m},0}\colon\ \mathbb{Z}[U]\otimes\Lambda^*H_1(D^2\times\Sigma;\mathbb{Z})\otimes\Lambda^R_{Nov} \to HF(\mathfrak{o}_+;\Lambda^R_{Nov})$$

factors through $\Pi\colon \mathbb{Z}[U]\otimes\Lambda^*H_1(D^2\times\Sigma;\mathbb{Z}) \twoheadrightarrow H^*(S^d\Sigma;\mathbb{Z})$ to give a map $H^*(S^d\Sigma;\Lambda^R_{Nov}) \to HF(\mathfrak{o}_+;\Lambda^R_{Nov})$ which is none other than the Piunikhin–Salamon–Schwarz isomorphism between the cohomology of $S^d\Sigma$ and its Hamiltonian Floer homology, constructed in [37]. Note that the monopole and Heegaard Floer homologies of $S^1\times\Sigma$ in the corresponding spin$^c$ structure are also known to be given by $H^*(S^d\Sigma)$ when $d < g-1$ [30],[33]. On the other hand, for $d > g-1$ we obtain $H^*(S^d\Sigma)$ rather than $H^*(S^{2g-2-d}\Sigma)$ as in [30],[33]; this discrepancy results from the fact that for this range of $d$ we cannot choose $c$ and $\tau$ to have the property that the corresponding Seiberg–Witten theory perturbation class $\eta(h,c,\tau)$ mentioned in the introduction is zero.

Finally, in the case when $Y_- = Y_+ = \varnothing$, so that the morphism $\mathfrak{m}$ corresponds to a Lefschetz fibration $\tilde{f}\colon X \to B$ on a closed manifold, we define $F_{\mathfrak{m},\theta,\tilde{h}}\colon \mathbb{A}(X)\otimes\Lambda^R_{Nov} \to \Lambda^R_{Nov}$ by once again counting pseudoholomorphic sections of the relative Hilbert scheme $\bar{X}_d(\bar{f})$; since $\partial X = \varnothing$ the relevant classes $\tilde{h}$ belong to $H_2(X;\mathbb{Z})$, and by Proposition 4.3 of [42] the virtual dimension $\delta(\tilde{h})$ of the space $\mathcal{M}_{J,\mathfrak{m},\tilde{h}}$ of pseudoholomorphic sections of $\bar{X}_f(\bar{f})$ in the homotopy class $\gamma_{\tilde{h}}$ is $\langle PD(\tilde{h})-\kappa_X, \tilde{h}\rangle$. If $\delta(\tilde{h}) \neq 2r+k$ we put $F_{\mathfrak{m},\theta,\tilde{h}}(U^r\otimes\eta_1\wedge\cdots\eta_k) = 0$; otherwise we put

$$F_{\mathfrak{m},\theta,\tilde{h}}(U^r\otimes\eta_1\wedge\cdots\eta_k) = T^{\langle[\Omega],\gamma_{\tilde{h}}\rangle}e^{\langle\uparrow[\theta],\gamma_{\tilde{h}}\rangle}\#\mathcal{M}_{J,\mathfrak{m},\tilde{h}}(A_1,\ldots,A_r,D_1,\ldots,D_k).$$

$\#\mathcal{M}_{J,\mathfrak{m},\tilde{h}}(A_1,\ldots,A_r,D_1,\ldots,D_k)$ is independent of $J$ and of the choices of $A_i$ and $D_j$, and indeed is (by definition) the Donaldson–Smith invariant

$$DS_{(X,f)}(\tilde{h};pt^r,\eta_1,\ldots,\eta_k).$$

Setting $F_{\mathfrak{m},\theta} = \sum_{\tilde{h}\in H_2(X;\mathbb{Z})} F_{\mathfrak{m},\theta,\tilde{h}}$ then completes the definition of the maps $F_{\mathfrak{m},\theta}$ in all cases, and also establishes part (iv) of Theorem 1.6 (noting that $\langle\uparrow[\theta],\gamma_{\tilde{h}}\rangle = \langle[\theta],\downarrow\gamma_{\tilde{h}}\rangle = \langle[\theta],\tilde{h}\rangle$).

Along the same lines, we could also use Lefschetz fibrations over surfaces with an arbitrary number $n$ of boundary components to obtain "quantum multiplication" maps $HF(\mathfrak{o}_1;\Lambda^R_{Nov})\otimes\cdots\otimes HF(\mathfrak{o}_{n-1};\Lambda^R_{Nov}) \to HF(\mathfrak{o}_n;\Lambda^R_{Nov})$ for suitable objects $\mathfrak{o}_i$, but we shall not develop this here.

Now that we have defined the maps $F_{\mathfrak{m},\theta}$, the various parts of Theorem 1.6 follow fairly quickly. Part (i) is obtained by standard gluing arguments: if $\mathfrak{m} = (X,\tilde{f},\tau) = \mathfrak{m}_1\circ\mathfrak{m}_0$ with $\mathfrak{m}_0 = (X_0,\tilde{f}_0,\tau)\in Mor(\mathfrak{o}_0,\mathfrak{o}_1)$, $\mathfrak{m}_1 = (X_1,\tilde{f}_1,\tau)\in Mor(\mathfrak{o}_1,\mathfrak{o}_2)$, writing $\mathfrak{o}_1 = (Y_1,f_1,h_1,c_1,\tau)$ we may choose the form $\bar{\Omega}$ on $\bar{X}_d(\bar{f})$ to restrict to a neighborhood of $(Y_1)_d(f_1)$ as the pullback of the Salamon form $\Omega_{d,\omega,\tau}$; varying the complex structure on the base $\bar{B}$ (and recalling that the almost complex structure $J$ on $\bar{X}_d(\bar{f})$ is constrained to make the projection to $\bar{B}$ pseudoholomorphic, so this also varies $J$) metrically identifies this neighborhood $N$ of $(Y_1)_d(f_1)$ with $[-T,T]\times(Y_1)_d(f_1)$ for arbitrarily large $T$. Let $J_T$ denote a generic almost complex structure obtained in this way. Choose representatives $A_1,\ldots,A_k$ of $PD(\uparrow PD[pt])\in H_{2d-2}(\bar{X}_d(\bar{f}))$ which are contained in the "$\mathfrak{m}_1$ side" $(X_1)_d(\tilde{f}_1)\subset\bar{X}_d(\bar{f})$ (and are outside the neighborhood $N$ of $(Y_1)_d(f_1)$ mentioned above), and representatives $A_{k+1},\ldots,A_{k+l}$ of $PD(\uparrow PD[pt])\in H_{2d-2}(\bar{X}_d(\bar{f}))$ which are contained in the "$\mathfrak{m}_2$ side" $(X_2)_d(\tilde{f}_2)\subset\bar{X}_d(\bar{f})$ and are also disjoint from $N$. Once $T$ is large enough, it follows from gluing theorems that in special cases date back at least to [9] (the argument in Section 3.3 of [39] can be rather directly applied to our case, the only essential difference being that here we glue at two points rather than one) that the moduli spaces $\mathcal{M}_{J_T,\mathfrak{m},\tilde{h}}(A_1,\ldots,A_{k+l};\mathbf{x}_0,\mathbf{x}_2)$ of pseudoholomorphic sections will, for generic $J_T$ and appropriate generic $J^0,J^1$ on $(\bar{X}_0)_f(\bar{f}_0),(\bar{X}_1)_f(\bar{f}_1)$, be in a one-to-one, orientation-preserving correspondence (which preserves the Novikov–ring and local–coefficient weights) with

$$\bigcup_{\substack{\mathbf{x}_1^-,\mathbf{x}_1^+\in\mathcal{A}_1,\\ +(\tilde{h}_0,\tilde{h}_{Y_1}\tilde{h}_1)=\tilde{h}}}\Big(\mathcal{M}_{J^0,\mathfrak{m}_0,\tilde{h}_0}(A_1,\ldots,A_k;\mathbf{x}_0,\mathbf{x}_1^-)\times\mathcal{M}_{J,\mathbb{R}\times(Y_1)_d(f),\tilde{h}_{Y_1}}(\mathbf{x}_1^-,\mathbf{x}_1^+)\times$$
$$\mathcal{M}_{J^1,\mathfrak{m}_1,\tilde{h}_1}(A_{k+1},\ldots,A_{k+l};\mathbf{x}_1^+,\mathbf{x}_2)\Big),$$

where the notation $+(\tilde{h}_0,\tilde{h}_Y,\tilde{h}_1)=\tilde{h}$ means that there exist an asymptotically constant section $s_0$ of $(\bar{X}_0)_f(\bar{f}_0)$ representing $\gamma_{\tilde{h}_0}$ (say approaching $\gamma_-\in\Gamma((Y_1)_d(f))$ at the positive end of $B_0$), a section $s_Y$ of $\mathbb{R}_t\times Y_d(f)$ representing $\gamma_{\tilde{h}_Y}$ which is asymptotic to $\gamma_\pm\in\Gamma((Y_1)_d(f))$ as $t\to\pm\infty$, and an asymptotically constant section $s_1$ of $(\bar{X}_1)_f(\bar{f}_1)$ representing $\gamma_{\tilde{h}_1}$ which approaches $\gamma_+$ at the negative end of $B_1$, and that when these sections are glued along their corresponding asymptotic limits to obtain a section $s_0\#s_Y\#s_1$ of $\bar{X}_d(\bar{f})$, $s_0\#s_Y\#s_1$ represents the homotopy class $\gamma_{\tilde{h}}$ (we allow, of course, $s_Y$ to be the trivial section $\mathbb{R}\times\gamma_-$). Summing over those $\tilde{h}$ with $h|_{X_i}=\tilde{h}_i$ $(i=0,1)$, this one-to-one correspondence then translates into the language of our cobordism maps as the statement that

$$\sum_{\substack{\tilde{h}\in H_{h_0,h_2}\\ \tilde{h}|_{X_i}=\tilde{h}_i}}\tilde{F}_{\mathfrak{m}_1\circ\mathfrak{m}_0,\theta,\tilde{h}}(U^{k+l}\otimes 1\otimes\mathbf{x})$$
$$=\tilde{F}_{\mathfrak{m}_1,\theta|_{X_1},\tilde{h}_0}\left(U^k\otimes 1\otimes(1+\partial_{\mathfrak{o}_1})\tilde{F}_{\mathfrak{m}_0,\theta|_{X_0},\tilde{h}_1}(U^l\otimes 1\otimes\mathbf{x})\right),$$

and passing to the induced maps on homology proves part (i) of Theorem 1.6.

Part (ii) of Theorem 1.6, which asserts that where $i_-:\mathbb{A}(Y_-)\to\mathbb{A}(X)$ is induced by the inclusion $Y_-\subset X$ we have

$$F_{\mathfrak{m},\theta,\tilde{h}}(1\otimes\lambda\cdot\mathbf{x})=F_{\mathfrak{m},\theta,\tilde{h}}(i_-(\lambda)\otimes\mathbf{x}),\tag{15}$$

follows from the last part of Lemma 5.3 and a similar gluing argument. Here we use the fact that $\bar{X}_d(\bar{f})$ contains a half–infinite cylinder on $(Y_-)_d(f_-)$, take the chains $A_i,D_j$ to be contained in some fixed $\{t_0\}\times(Y_-)_d(f_-)$ in this cylinder, and send the length $T$ of the cylinder to $\infty$. The right hand side of (15) counts sections satisfying the incidence conditions given by $A_i, D_j$ for any finite $T$; as $T$ becomes large these sections approach the broken sections (consisting of a section of $\mathbb{R} \times (Y_-)_d(f_-)$ followed by a section of $\bar{X}_d(\bar{f})$) counted by the left hand side of (15).

The duality statement comprising part (iii) of Theorem 1.6 follows immediately from the definition of $F_{\mathfrak{m},\theta}$: the quantities

$$\langle F_{\mathfrak{m},\theta,\tilde{h}}(\mathbf{x}_-), \mathbf{x}_+\rangle_{\mathfrak{o}_+} \text{ and } \langle \mathbf{x}_-, F_{-\mathfrak{m},\theta,\tilde{h}}(\mathbf{x}_+)\rangle$$

count precisely the same objects (namely holomorphic sections of $\bar{X}_d(\bar{f})$ asymptotic to $\mathbf{x}_\pm$ at the boundary components $(Y_\pm)_d(f_\pm)$ in the relative homotopy class $\gamma_{\tilde{h}}$), and do so with identical weights (since the relevant forms $\bar{\Omega}$ and $\uparrow \theta$ are the same for $\mathfrak{m}$ as for $-\mathfrak{m}$); hence these quantities are equal.

Since part (iv) has already been established, the proof of Theorem 1.6 is now complete.

## 6. Periodic points of symplectomorphisms and asymptotics for the parallel translation of vortices

Let us first prove Corollary 1.8 assuming Theorem 1.7. Note that for any diffeomorphism $\phi\colon \Sigma \to \Sigma$ there is a Lefschetz fibration $f\colon X \to S^2$ with fiber $\Sigma$ over some point and all fibers irreducible such that the preimage of the equator $\gamma$ in $S^2$ is isomorphic as a smooth fibration to the mapping torus $Y_\phi$ of $\phi$ (for by Theorem 2.2 of [32] one can factor the mapping class of $\phi$ as a product of right-handed Dehn twists along nonseparating loops to get a Lefschetz fibration over the disc such that the monodromy around the boundary is isotopic to $\phi$, and then complete this factorization to a factorization of the identity as a product of right-handed Dehn twists along nonseparating loops in order to complete the Lefschetz fibration to a Lefschetz fibration over the whole sphere), such that $b^+(X) > 1$ (if the initially–constructed Lefschetz fibration does not satisfy this property, then its fiber sum with itself will, by, *e.g.*, Lemma 3.1 of [43]). Now $T^{vt}X$ is a well-defined complex line bundle on the complement of a set of codimension four (namely $Crit(f)$) in $X$ and so extends from the complement of a neighborhood of $Crit(f)$ to a complex line bundle on all of $X$, and then $\langle c_1(T^{vt}X), [\Sigma]\rangle < 0$. Hence there are, by the proof of Theorem 10.2.18 of [13], symplectic forms $\beta$ on $X$ in classes of form $-c_1(T^{vt}X) + Mf^*\omega_{S^2}$ for large $M$. We wish to say that $DS(PD(\kappa_X); pt^0) \neq 0$ where $\kappa_X$ is the canonical class; if our Lefschetz fibration were obtained by blowing up a high degree Lefschetz pencil on a manifold with $b^+ > b_1 + 1$ we could deduce this directly from the main result of [7]. For more general Lefschetz fibrations, Taubes' theorems [45] show that, with respect to the symplectic structure $\beta$, we have $Gr_X(PD(\kappa_X)) = \pm 1$, and hence, by the main theorem of [46],[4] $DS(PD(\kappa_X); pt^0) = \pm 1$; meanwhile all other classes $\alpha \in H_2(X;\mathbb{Z})$ differing from $PD(\kappa_X)$ by a torsion element have $Gr_X(\alpha) = DS(\alpha; pt^0) = 0$. So

[4] In the statement of the main theorem of [46], there is a hypothesis on the area of the fiber of the Lefschetz fibration. However, for any Lefschetz fibration $f\colon X \to S^2$ with all fibers irreducible, the main theorem of [46] still applies to show that $Gr(\alpha;\cdot) = DS(\alpha;\cdot)$ for any class $\alpha \in H_2(X;\mathbb{Z})$ satisfying $d = \alpha \cap [\Sigma] > g - 1$, since then the virtual dimension of the space of pseudoholomorphic curves representing any class of form $\alpha - n[\Sigma]$ with $n > 0$ will be smaller than the virtual dimension of pseudoholomorphic representatives of $\alpha$, and so the Gromov–Taubes moduli spaces for the class $\alpha$ will, for generic almost complex structures making $f$ pseudoholomorphic, not contain any curves with fiber components; ensuring that this be the case was the only role played by the hypothesis on the area of the fiber in [46].

where $h = PD(\kappa_X|_{Y_\phi}) \in H_1(Y_\phi;\mathbb{Z})$ and $c = \beta|_{Y_\phi} = -c_1(T^{vt}Y_\phi)$, we conclude that the composition

$$\sum_{\substack{\tilde{h}\in H_2(X;\mathbb{Z}):\\ PD(\tilde{h})|_{Y_\phi}=\kappa_X|_{Y_\phi}}} F_{X,\theta,\tilde{h}}(1\otimes\cdot)\colon\ \mathbb{R} = HF(\varnothing;\mathbb{R}) \to HF(Y_\phi, f, h, c, \tau; \Gamma_{\theta|_{Y_\phi}}) \to \mathbb{R} = HF(\varnothing;\mathbb{R})$$

is nonzero for certain choices of $\theta \in \Omega^2(X)$ (and for arbitrary $\tau$), and so $HF(Y_\phi, f, h, c, \tau; \Gamma_\theta|_{Y_\phi}) \neq 0$. The monotonicity assumption on $\phi$ implies that $\omega_\phi$ (after rescaling) belongs to the class $c + \frac{2\pi}{\tau}PD(h)$, and so may be used in the definition of $HF(Y_\phi, f, h, c, \tau)$. Hence, for all $\tau > 2\pi d$, noting that $\langle h, fiber\rangle = 2g-2$, the symplectomorphism $\Phi_{2g-2,\omega_\phi,\tau}$ has a fixed point. But as $\tau \to \infty$, $\Phi_{2g-2,\omega_\phi,\tau} \to S^{2g-2}\phi$ by Theorem 1.7, so the latter map has a fixed point as well. □

As was alluded to in the introduction, the same argument reveals that for $d > g-1$ $S^d\phi$ has a fixed point whenever $\phi$ is monotone and there is a Lefschetz fibration $f\colon X \to S^2$ with irreducible fibers having monodromy around some loop isotopic to $\phi$ whose total space has the property that, for some homology class $\tilde{h} \in H_2(X;\mathbb{Z})$ having intersection number $d$ with the fibers, the sum of the Gromov–Taubes invariants in classes congruent to $\tilde{h}$ mod torsion and mod restriction to $Y_\phi$ is nonzero (with the slight modification that for the class $c$ of the fiberwise symplectic form one should use $c = -c_1(T^{vt}X) - \frac{2\pi}{\tau}PD(\tilde{h})$, with $\tau$ large enough to ensure that this class is positive on the fibers). As mentioned in footnote 4, the requirement that $d > g-1$ along with the irreducibility of the fibers suffice to replace the assumption on the symplectic areas of $\tilde{h}$ and $[\Sigma]$ in the main theorem of [46]. The same reasoning can also be applied to certain non-monotone symplectomorphisms $\phi$, provided that there is a Lefschetz fibration containing the mapping torus of $\phi$ as the preimage of some circle in the base, and carrying a symplectic form in a cohomology class which restricts appropriately to this mapping torus.

We turn finally to the proof of Theorem 1.7. We consider a symplectomorphism $\phi\colon (\Sigma,\omega) \to (\Sigma,\omega)$ of a symplectic 2–manifold. $\omega$ induces on the mapping torus $Y_\phi$ a closed fiberwise symplectic form $\omega_\phi$ in the cohomology class $c \in H^2(Y;\mathbb{R})$. In Section 2 we have, for each large enough $\tau$, chosen closed fiberwise symplectic forms $\omega^\tau$ on $Y_\phi$ representing the classes $c + 2\pi PD(h)/\tau$; since the homology will be independent of the particular forms in these classes that we choose we may as well assume that $\omega^\tau \to \omega_\phi$ as $\tau \to \infty$, and (using the Moser trick) that the restriction of $\omega^\tau$ to some fixed base fiber is proportional to $\omega$. The monodromies $\phi_\tau$ of the $\omega^\tau$ then converge to $\phi$, and so from the definition of the chain complex $CF$ it follows that Theorem 1.7 can be translated into the statement that the parallel transport map $F_{\{J_t\}}\colon S^d(\Sigma, J_0) \to S^d(\Sigma, J_1)$ of (5) approaches the identity as the parameter $\tau$ tends to $\infty$.

In proving this, we shall make use of the asymptotic properties of the vortices themselves for large $\tau$. Recall that the vortex equations are obtained by fixing a degree $d$ Hermitian line bundle $L$ on the Kähler curve $(\Sigma,\omega,J)$; they read

$$\begin{aligned} &\bar{\partial}_{J,A}\theta = 0 \\ &iF_A = \tau(1-|\theta|^2)\omega. \end{aligned} \tag{16}$$

where the unknown $(A, \theta)$ consists of a connection $A$ in $L$ and a not–identically–zero section $\theta$ of $L$. In the case that $(\Sigma, \omega, J)$ is $\mathbb{R}^2$ with its standard symplectic and complex structure, solutions for general $\tau$ can be obtained from those from the case $\tau = 1$ by pulling back via the dilation $z \mapsto \sqrt{\tau}z$. The case of the standard plane with $\tau = 1$ was exhaustively analyzed in Chapter III of [18]; in particular, according to Theorem III.8.5, the curvature satisfies an exponential decay condition which translates to the general $\tau$ case as

$$|*iF_A| \leq M\tau e^{-c\sqrt{\tau}|z|},$$

where $c$ can be taken to be any constant smaller than 2. We shall be needing analogous (though somewhat weaker) results for the vortices on general $(\Sigma, \omega, J)$. The referee has pointed out that bounds similar to what we prove (at least for a fixed $J$) can be deduced from estimates on solutions to the Seiberg–Witten equations from [44] (in particular 1.24 (6)) by specializing to the case where the four–manifold under consideration is $\Sigma \times T^2$ with a product metric; however, we shall still give our proof of these bounds because the proof is simpler (though similar in spirit) in the purely two-dimensional case, because we need to see explicitly that the estimates are uniform when we vary $J$ in a compact 1–parameter family, and because some of the necessary ingredients will reappear later when we analyze a certain Green's function. Readers familiar with the proofs of such bounds might skip Lemma 6.1 through Theorem 6.3.

Throughout our discussion, we work with a fixed Hermitian line bundle $L \to \Sigma$ of degree $d > 0$ over a fixed compact symplectic 2–manifold $(\Sigma, \omega)$. To connect this to the setup in Section 2, we should note that in that section the closed fiberwise symplectic form $\omega^\tau$ restricts to $\Sigma$ as $\frac{\langle c,[\Sigma]\rangle + 2\pi d/\tau}{\langle c,[\Sigma]\rangle}$ times $\omega$. As such, the parameter $\tau$ in (16) would be $\tau + \frac{2\pi d}{\langle c,[\Sigma]\rangle}$ in the notation of Section 2. Since we are interested here in the behavior of the vortex equations as $\tau \to \infty$ and since $\langle c, [\Sigma]\rangle > 0$, this distinction is immaterial to our present concerns and we shall henceforth suppress it.

We will also fix a smooth path $\{J_t\}_{t\in[0,1]}$ of almost complex structures on $\Sigma$; together with $\omega$, these induce metrics $g_t$. Consider the vortex equations (16) where $J$ is one of the almost complex structures appearing in the path $J_t$. We shall be making a variety of estimates on some quantities relating to solutions of these equations, which shall involve certain constants; these constants may be taken independent of $\tau$ and of $J$ *provided that $J$ is chosen from within the fixed smooth 1–parameter family* $\{J_t\}_{t\in[0,1]}$, but might not apply to an entirely arbitrary choice of $J$. More specifically, where $g$ is the metric induced by $\omega$ and $J$, the constants may depend on any or all of: the minimal or maximal curvature of the Riemannian 2–manifold $(\Sigma, g)$; the injectivity radius $r_0$ of $(\Sigma, g)$; the diameter of $(\Sigma, g)$; or the maximum of the Jacobians of the exponential maps $\exp_p^g\colon B_{r_0/2}(0) \to \Sigma$ for $p \in \Sigma$.

First, we prove a direct analogue for the case of a general Riemann surface to a pair of properties proven for the case of the flat plane in [18]. Let

$$\kappa = \max_{t\in[0,1],p\in\Sigma}\{0, -sec_{g_t}(p)\},$$

where $sec_{g_t}(p)$ is the sectional curvature of $\Sigma$ at $p$ in the metric $g_t$.

**Lemma 6.1.** *Any solution $(A, \theta)$ to (16) satisfies:*

$$w := 1 - |\theta|^2 \geq 0 \qquad |d_A\theta| \leq 2\tau^{1/2}w + 2\kappa\tau^{-1/2},$$

*provided that* $\tau \geq \kappa$.

*Proof.* First note that, for any section $\phi$ of a holomorphic line bundle $V$ with unitary connection $A$ over any Kähler manifold $M$, one has

$$\begin{aligned}(d_A\bar{\partial}_A\phi)(v,w) &= \nabla_v \iota_w \bar{\partial}_A\phi - \nabla_w \iota_v \bar{\partial}_A\phi - (\bar{\partial}_A\phi)([v,w]) \\ &= \frac{1}{2}F_A\phi(v,w) + \frac{i}{2}(\nabla^2_{v,iw}\phi - \nabla^2_{w,iv}\phi),\end{aligned}$$

as can be seen by expanding out $\iota_u\overline{\partial}_A\phi = \nabla_u\phi + i\nabla_{iu}\phi$ and then using the fact that $M$ is Kähler to move various factors of $i$ past covariant derivatives.

In particular

$$d_A\bar{\partial}_A\phi(v,iv) = \frac{1}{2}F_A\phi(v,iv) - \frac{i}{2}(\nabla^2_{v,v}\phi + \nabla^2_{iv,iv}\phi),$$

so that if $M$ is 1–complex dimensional we see that

$$*d_A\bar{\partial}_A\phi = \frac{1}{2} * F_A\phi - \frac{i}{2}\Delta\phi,$$

where $*$ is the Hodge star operator induced by the metric and $\Delta = *d_A * d_A$ is the (negative) Laplacian on sections of $V$ induced by $A$. Applying this to our vortex $(A,\theta)$ on $\Sigma$, since $\bar{\partial}_A\theta = 0$ and $*iF_A = \tau(1-|\theta|^2)$, we see

$$\Delta\theta + \tau(1-|\theta|^2)\theta = 0. \tag{17}$$

Now

$$\Delta|\theta|^2 = 2Re\langle\Delta\theta,\theta\rangle + 2|d_A\theta|^2,$$

while (17) implies that $\langle\Delta\theta,\theta\rangle$ is real, so that

$$\langle\Delta\theta,\theta\rangle = \Delta|\theta|^2/2 - |d_A\theta|^2.$$

Hence taking the inner product of (17) with $\theta$ and setting $w = 1-|\theta|^2$ yields

$$-\Delta w + 2\tau|\theta|^2 w = 2|d_A\theta|^2. \tag{18}$$

In particular if $z_0 \in \Sigma$ were such that $w(z_0) < 0$, we would have $(-\Delta w)(z_0) > 0$, so that $z_0$ could not be a local minimum for $w$. So since $\Sigma$ is compact and $w\colon \Sigma \to \mathbb{R}$ cannot attain a negative local minimum, we have $w \geq 0$ everywhere.

Now $\Delta\theta = *d_A * d_A\theta = - * iF_A\theta = -\tau w\theta$, where we have used that, since $d_A\theta$ has type $(1,0)$, $*d_A\theta = -id_A\theta$. So setting $h = d_A\theta \in \Omega^{1,0}(L)$, we see

$$\begin{aligned}d_A * d_A * h = d_A(-\tau w\theta) &= -\tau(wh + \theta dw) = -\tau(wh - \theta(\bar{\theta}h + \theta\bar{h})) \\ &= \tau h(1-2w) + \tau\theta\theta\bar{h},\end{aligned}$$

while

$$\begin{aligned}*d_A * d_A h = *d_A * d_A d_A\theta &= -i\tau * d_A(w\theta) \\ &= -i\tau * (\theta(-\bar{\theta}h - \theta\bar{h}) + wh) = -i\tau * ((2w-1)h - \theta\theta\bar{h}) \\ &= \tau(1-2w)h - \tau\theta\theta\bar{h},\end{aligned}$$

where in the last equality we have used that since $h$ has type $(1,0)$, $*h = -ih$ and $*\bar{h} = i\bar{h}$. So following Section III.6 of [18] by writing $\Delta_A = *d_A * d_A + d_A * d_A *$, we see

$$\Delta_A h = 2\tau h(1-2w).$$

Further, on $L$–valued 1–forms there is a Weitzenböck formula (see, *e.g.*, [18], III.6.15; [36], Chap. 7)

$$tr\nabla_A^2 = \Delta_A + (*F_A) * + sec,$$

so since $h$ has type $(1,0)$ and so $(*F_A) * h = (- * iF_A)h = -\tau wh$, we obtain

$$tr\nabla_A^2 h = \tau(2 - 5w + sec/\tau)h,$$

from which Kato's inequality ([18], III.6.20) provides

$$|h|\Delta|h| \geq \tau(2 - \kappa/\tau - 5w)|h|^2.$$

Hence

$$\begin{aligned}
|h|\Delta(2\sqrt{\tau}(w+\kappa/\tau) - |h|) &\leq 2\sqrt{\tau}|h|\left(2\tau w(1-w) - 2|h|^2\right) + (5w + \kappa/\tau - 2)\tau|h|^2 \\
&= \sqrt{\tau}|h|\left(4\tau(1-w)w + \sqrt{\tau}|h|(5w+\kappa/\tau - 2) - 4|h|^2\right) \\
&= \sqrt{\tau}|h|\left((2\sqrt{\tau}(w+\kappa/\tau) - |h|)\left(2\sqrt{\tau}\frac{w(1-w)}{w+\kappa/\tau} + 4|h|\right)\right) \\
&\quad - \tau|h|^2\left((2-\kappa/\tau - 5w) - \left(\frac{2w(1-w)}{w+\kappa/\tau} - 8(w+\kappa/\tau)\right)\right) \\
&\leq \sqrt{\tau}|h|\left((2\sqrt{\tau}(w+\kappa/\tau) - |h|)\left(2\sqrt{\tau}\frac{w(1-w)}{w+\kappa/\tau} + 4|h|\right)\right),
\end{aligned}$$

where we have used the fact that, since $0 \leq w \leq 1$ and $\kappa/\tau \leq 1$, we have $\frac{2w}{w+\kappa/\tau} \leq \frac{2}{1+\kappa/\tau} \leq 2 - \kappa/\tau$.

So we see that wherever $2\tau^{1/2}w + 2\kappa\tau^{-1/2} - |h|$ is negative (which forces $|h| > 0$ since we've already shown that $w \geq 0$ everywhere) we have $\Delta(2\tau^{1/2}w + 2\kappa\tau^{-1/2} - |h|) < 0$. But then $2\tau^{1/2}w + 2\kappa\tau^{-1/2} - |h| \colon \Sigma \to \mathbb{R}$ cannot attain a negative local minimum, which by the compactness of $\Sigma$ forces $|d_A\theta| = |h| \leq 2\tau^{1/2}w + 2\kappa\tau^{-1/2}$ everywhere. $\square$

**Proposition 6.2.** *There is a constant $C > 0$ with the property that, for all sufficiently large $\tau$ if $(A, \theta)$ is a solution to (16) with $J \in \{J_t\}_{t\in[0,1]}$ and $w = 1 - |\theta|^2$, we have, for each $z_0 \in \Sigma$,*

$$w(z_0)\min\{d(z_0,p)|\theta(p) = 0\} \leq \frac{C}{\sqrt{\tau}},$$

*where $d(\cdot,\cdot)$ denotes the distance measured in the metric $g$ induced by $J$ and $\omega$.*

*Proof.* First note that the first statement of Lemma 6.1 shows that $|\theta| \leq 1$, so $|d(w+\kappa/\tau)| = |2Re\langle\theta, d_A\theta\rangle| \leq 2|d_A\theta| \leq 4\sqrt{\tau}(w+\kappa/\tau)$. So if $\gamma$ is an arc–length parametrized path in $\Sigma$, say from $z$ to $z'$ and having length $l_\gamma$, we have

$$\begin{aligned}
\log\left(\frac{w(z')+\kappa/\tau}{w(z)+\kappa/\tau}\right) &= \int_0^{l_\gamma}\frac{d}{dt}\log(w(\gamma(t))+\kappa/\tau)\,dt \leq \int_0^{l_\gamma}\frac{|d(w+\kappa/\tau)|}{|w+\kappa/\tau|}\,dt \\
&\leq 4\sqrt{\tau}l_\gamma.
\end{aligned}$$

Thus, for any $z, z' \in \Sigma$,

$$w(z') + \kappa/\tau \geq (w(z)+\kappa/\tau)e^{-4\sqrt{\tau}d(z,z')}.$$

We claim now that $w = 1 - |\theta|^2$ is equal to either 0 or 1 at each of its local maxima. Indeed, note that, again writing $h = d_A\theta \in \Omega^{1,0}(L)$, we have $dw = -d|\theta|^2 = -(\bar{\theta}h + \theta\bar{h})$, and $*dw = -*(\bar{\theta}h + \theta\bar{h}) = i\bar{\theta}h - i\theta\bar{h}$, so that

$$dw + i * dw = -2\bar{\theta}h,$$

so at a putative local maximum $z$ of $w$ with $w(z) \notin \{0,1\}$ (so $\theta(z) \neq 0$), we necessarily have $h(z) = 0$. Meanwhile since $w$ takes values only in $[0,1]$ we must also have $w(z)(1-w(z)) > 0$, so recalling the equation

$$-\Delta w + 2\tau w(1-w) = |h|^2$$

we see that $\Delta w(z) > 0$, in contradiction with the fact that $z$ was taken to be a local maximum.

Suppose now that

$$N\tau^{-1/2} \geq \min\{d(z_0,p)|\theta(p) = 0\} \geq (N-1)\tau^{-1/2} \quad (N \in \mathbb{N}).$$

Since $w$ is everywhere nonnegative and is strictly less than 1 on $B_{(N-1)\tau^{-1/2}}(z_0)$, we deduce that for $k = 1, \ldots, N-1$, $\sup_{B_{k/\sqrt{\tau}}(z_0)} w$ must be attained at some point $z_k$ with $d(z_k, z_0) = k\tau^{-1/2}$; in particular $w(z_k) \geq w(z_0)$. This together with the conclusion of the first paragraph of the proof shows that, on each of the disjoint balls $B_k = B_{\frac{1}{2\sqrt{\tau}}}(z_k)$ $(k = 0, \ldots, N-1)$,

$$w|_{B_k} \geq e^{-2}w(z_0) - \kappa/\tau.$$

Now for some constant $A$ (related to the Jacobian of the exponential map on balls of radius smaller than the injectivity radius, if $\tau$ is large enough) we have $vol(B_k) \geq A\tau^{-1}$ for each $k$, and so since $w \geq 0$ throughout $\Sigma$

$$\int_\Sigma w\omega \geq \sum_{k=0}^{N-1} \int_{B_k} w\omega \geq ANe^{-2}w(z_0)\tau^{-1} - A\kappa\tau^{-2}.$$

But the original vortex equations imply that

$$\int_\Sigma w\omega = \tau^{-1}\int_\Sigma *iF_A = 2\pi d\tau^{-1}.$$

Thus

$$w(z_0)\min\{d(z_0,p)|\theta(p) = 0\} \leq w(z_0)N\tau^{-1/2} \leq C\tau^{-1/2}$$

for an appropriate choice of $C$. □

Having taken these first steps, we can now prove a basic exponential decay estimate for vortices.

**Theorem 6.3.** *There are constants $R, M, \tau_0 > 0$ with the property that, if $(A,\theta)$ is a solution to (16) with $J \in \{J_t\}_{t\in[0,1]}$ and $w = 1 - |\theta|^2$, we have, for each $z \in \Sigma$ and for $\tau \geq \tau_0$,*

$$w(z) := 1 - |\theta(z)|^2 \leq \kappa\tau^{-1} + M \sum_{\{p:\theta(p)=0\}} \left(e^{-\sqrt{\tau}d(z,p)} + e^{-R\sqrt{\tau}}\right).$$

*Proof.* According to Lemma 6.1 and Equation 18, we have

$$-\Delta w + 2\tau w(1-w) = 2|d_A\theta|^2 \leq 8\tau(w + \kappa/\tau)^2,$$

so that

$$-\Delta w \leq \tau w(34w - 2) \text{ wherever } w > \kappa/\tau.$$

Meanwhile, according to Proposition 6.2, where

$$V = \{z \in \Sigma : d(z, \theta^{-1}(0)) \geq 68C\tau^{-1/2}\},$$

at each $z \in V$ we have $w(z) \leq 1/68$, and so

$$-\Delta w \leq -\frac{3}{2}\tau w \text{ on } V \cap \{w > \kappa/\tau\}.$$

Now note that if $q \in \Sigma$ and $\alpha\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$, the function $u_{q,\alpha}(z) = e^{-\sqrt{\tau}\alpha(d(q,z))}$ satisfies (wherever all terms exist)

$$\Delta u_{q,\alpha}(z) = \left(\tau\alpha'(d(q,z))^2 - \sqrt{\tau}(\alpha''(d(q,z)) - \sqrt{\tau}\alpha'(d(q,z))\Delta(d(q,z))\right) u_{q,\alpha}(z).$$

Also, if the curvature of $(\Sigma, g)$ is bounded above by $K > 0$, then it follows from Theorem 6.2.1 and the discussion before Lemma 9.1.1 in [36] that where $R$ is the minimum of $\frac{\pi}{2\sqrt{K}}$ and the injectivity radius of $(\Sigma, g)$, we have $\Delta(d(q,z)) \geq 0$ as long as $d(q,z) \leq R$.

Take for $\alpha$ a smooth function with the following properties:

(i) $\alpha(0) = 0$,
(ii) $0 \leq \alpha'(t) \leq 6/5$, with $\alpha'(t) = 6/5$ for $t < R/2$,
(iii) $\alpha(t) = R$ for $t \geq R$,
(iv) $-3/R \leq \alpha''(t) \leq 0$.

Then since $u_{q,\alpha}$ is positive everywhere and is constant outside the region that $\Delta d(q, \cdot)$ is known to be nonnegative, we have

$$\begin{aligned}\Delta u_{q,\alpha} &\leq \left(\frac{36}{25}\tau + \frac{3\sqrt{\tau}}{R}\right) u_{q,\alpha} \\ &\leq \frac{3}{2}\tau u_{q,\alpha}\end{aligned}$$

provided that $\sqrt{\tau} \geq 50/R$.

Also, assuming that $\tau$ is large enough that $68C\tau^{-1/2} \leq R/2$, if $d(q,z) = 68C\tau^{-1/2}$ then $u_{q,\alpha}(z) = e^{-408C/5}$. Hence setting $M = e^{408C/5}$ and

$$u = M \sum_{p \in \theta^{-1}(0)} u_{p,\alpha},$$

we have

$$u|_{\partial V} \geq 1 > \frac{1}{68} \geq w|_{\partial V}$$

and

$$\Delta(u - w) \leq \frac{3}{2}\tau(u - w) \text{ on } V \cap \{w > \kappa/\tau\}.$$

But then, as usual, $\Delta(u - w) < 0$ anywhere on $V \cap \{w > \kappa/\tau\}$ that $u - w$ is negative, so that $u - w$ cannot attain a negative local minimum on $V \cap \{w > \kappa/\tau\}$. In particular, then, $u + \kappa/\tau - w$ also cannot attain a negative local minimum on $V \cap \{w > \kappa/\tau\}$, so since $u + \kappa/\tau - w$ is obviously positive where $w \leq \kappa/\tau$, $u + \kappa/\tau - w$ cannot attain a negative local minimum anywhere on $V$. So since $u - w > 0$ on $\partial V$ we deduce that $w \leq u + \kappa/\tau$ througout $V$, and indeed throughout $\Sigma$ since away from $V$ we have $u \geq 1 \geq w$. The proof is then completed by noting that the construction of $\alpha$ ensures that, for each $p \in \theta^{-1}(0)$, we have

$$u_{p,\alpha}(z) \leq e^{-\sqrt{\tau}d(p,z)} + e^{-R\sqrt{\tau}}.$$

$\square$

Now the parallel translation inducing the map $F_{\{J_t\}}\colon S^d(\Sigma, J_0) \to S^d(\Sigma, J_1)$ that we are investigating is, by Theorem 5.1 of [38], given by

$$F_{\{J_t\}}([A,\theta]) = [A(1), \theta(1)]$$

where $(A(t), \theta(t))$ solves the ODE

$$i\dot{A}(t) = 2\tau Re\langle \theta(t), \eta(t)\rangle \qquad i\dot{\theta}(t) = \bar{\partial}^*_{A(t)}\eta(t)$$

with initial condition $(A(0), \theta(0)) = (A, \theta)$ where $\eta(t) \in \Omega^{0,1}(L)$ is the unique solution to

$$\bar{\partial}_{J_t,A(t)}\bar{\partial}^*_{J_t,A(t)}\eta(t) + \tau|\theta(t)|^2\eta = \frac{1}{2}(d_{A(t)}\theta(t)) \circ \dot{J}(t). \tag{19}$$

(Recall that the isomorphism between the set of gauge equivalence classes of vortices takes $[A,\theta]$ to the vanishing locus of $\theta$. Also, to compare to [38], our terms $\theta$ and $\eta$ are $1/\sqrt{2\tau}$ times the corresponding terms $\Theta_0, \Theta_1$, respectively, in [38]. The reader may calculate directly or consult the proof of Theorem 5.1 of [38] to see that $(A(t), \theta(t))$ so defined does indeed satisfy (16) with $J = J_t$ for all $t$ and that this recipe is consistent with the symplectic parallel transport description discussed in Section 2.)

Our goal is to show that $F_{\{J_t\}}$ is close to the identity; we shall accomplish this by obtaining upper bounds on $|\bar{\partial}^*_{J_t,A(t)}\eta(t)|$ where $\eta(t)$ solves (19). Now where

$$G(x,p)\colon\ \Omega^{0,1}(L)|_p \to \Omega^{0,1}(L)|_x$$

denotes the Green's kernel for the operator

$$\bar{\partial}_{J,A}\bar{\partial}^*_{J,A} + \tau|\theta|^2\colon\ \Omega^{0,1}(L) \to \Omega^{0,1}(L)$$

we have

$$\eta(t)(x) = \frac{1}{2}\int_\Sigma G(x,p)\left((d_{A(t)}\theta(t,p)) \circ \dot{J}(t,p)\right)\omega_p;$$

the desired upper bounds on $|\bar{\partial}^*_{J_t,A(t)}\eta(t)|$ will follow from our already–obtained exponential decay bounds on $w(t) = 1 - |\theta(t)|^2$ (and hence on $|d_{A(t)}\theta(t)|$ by Lemma 6.1), together with bounds on the derivatives of the Green's kernel

We now set about deriving these Green's kernel estimates. Let $(A, \theta)$ be an arbitrary solution to (16) (with $J$ taken from the path $\{J_t\}_{t\in[0,1]}$; with this $J$ understood, we shall just write $\bar{\partial}_A$ for $\bar{\partial}_{J,A}$). Note that $\bar{\partial}_A\bar{\partial}^*_A + \tau|\theta|^2$ is manifestly positive definite, and in fact the Weitzenböck formula used in the proof of Lemma 6.4 below allows us to rewrite this operator as $\frac{1}{2}(\nabla^*_A\nabla_A + \tau(1+|\theta|^2) + sec)$, and so as long as $\tau > 4\kappa$ (as we shall assume hereinafter) its spectrum is bounded below by $\tau/4$.

We first obtain estimates on $G(x,p)$ for $p$ close to $x$. In this direction, consider the effect of replacing the metric $g$ induced by $J$ and $\omega$ by $\tilde{g} = \tau g$. Then, since on 1–forms we have $\bar{\partial}^{*_{\tilde{g}}}_A = -*_{\tilde{g}}\,\partial_A *_{\tilde{g}} = \tau^{-1}\bar{\partial}^{*_g}_A$ we see that $G$ is also the Green's kernel (using the metric $\tilde{g}$) for the operator

$$\bar{\partial}_A\bar{\partial}^{*_{\tilde{g}}}_A + |\theta|^2\colon\ \Omega^{0,1}(L) \to \Omega^{0,1}(L),$$

where by Lemma 6.1 we have $\tau$–independent bounds

$$0 \le |\theta|^2 \le 1, \quad |d|\theta|^2|_{\tilde{g}} \le 3/2$$

for the potential term $|\theta|^2$. Furthermore (18) gives

$$\Delta^{\tilde{g}}|\theta|^2 + 2|\theta|^2(1-|\theta|^2) = |d_A\theta|^2_{\tilde{g}};$$

differentiating this and repeatedly using the bounds of Lemma 6.1 and the fact that $*iF_A = \tau(1-|\theta|^2)$ gives, for all $k$, $\tau$–independent constants $C_k$ such that

$$|(\Delta^{\tilde{g}})^k|\theta|^2| \leq C_k, \quad |d(\Delta^{\tilde{g}})^k|\theta|^2|_{\tilde{g}} \leq C_k.$$

Using the approach of Chapitre III, E.III, of [3] (adapted from the case of the Laplacian on functions to that of a more general Laplace type operator on sections of a vector bundle as in [12]) one then finds, for a fixed $c < \inf injrad(\Sigma, \tilde{g})$, uniform–in–$\tau$ estimates on the $c$–neighborhood of the diagonal in $(\Sigma, \tilde{g}) \times (\Sigma, \tilde{g})$ for the $C^1$–accuracy of the third-order asymptotic approximation $S_3(t,x,y)$ to the heat kernel $S(t,x,y)$ for $\bar{\partial}_A \bar{\partial}_A^{*\tilde{g}} + |\theta|^2$. (Note that since the functions $K_k$ of Lemme E.III.6 of Chapitre III of [3] vanish outside the $c$–neighborhood of the diagonal the term $V$ on p. 212 can be replaced by the maximal volume of a ball of radius $c$ in the $(\Sigma, \tilde{g})$, which is bounded independently of $\tau$.) Since the spectrum of $\bar{\partial}_A \bar{\partial}_A^{*\tilde{g}} + |\theta|^2$ is bounded below by $1/4$, we may then integrate with respect to $t$ to see that these estimates imply a uniform bound on the $C^1$–norm of the difference between a cut-off version of the third-order Hadamard expansion of the Green's kernel and the actual kernel $G$ (see, *e.g.*, section II.2. of [2]; the relevant coefficients in the expansion may be found on p. 336 of [12], and the salient point for our purposes is that the $k$th derivatives of the potential term $|\theta|^2$ only contribute a correction factor proportional to the $2(k+1)$th power of the distance and so do not substantially affect the rate at which $G(x,p)$ diverges near the diagonal). As a result, there is a $\tau$–independent constant $C > 0$ such that, whenever $dist_{\tilde{g}}(x,p) \leq c$, we have

$$|G(x,p)| \leq C(1 + |\log d_{\tilde{g}}(x,p)|), \qquad |\bar{\partial}_A^{*\tilde{g}} G(x,p)|_{\tilde{g}} \leq \frac{C}{d_{\tilde{g}}(x,p)},$$

where in the second formula we are viewing $p$ as fixed, so that $x \mapsto G(x,p)$ is an element of $(\Omega^{0,1}(L)|_p)^* \otimes \Omega^{0,1}(L)$, and then taking $\bar{\partial}_A^{*\tilde{g}}$ of this section (with respect to $x$). Scaling back, these relations translate to:

$$|G(x,p)| \leq C(1 + |\log \tau^{1/2} d_g(x,p)|), \quad |\bar{\partial}_A^{*g} G(x,p)|_g \leq \frac{C}{d_g(x,p)} \quad \text{whenever } d_g(x,p) \leq c\tau^{-1/2} \tag{20}$$

since the $\tilde{g} = \tau g$–norm of a given element of $Hom(\Omega^{0,1}(L)|_p, L|_x)$ is $\tau^{1/2}$ times its $g$–norm. Hereinafter we use $g$ to measure all distances and norms and to take all adjoints, and so we shall drop $g$ from notations such as $\bar{\partial}_A^{*g}$.

**Lemma 6.4.** *Fix $p \in \Sigma$, view $G(p,\cdot)$ as a section of $Hom(T_p^{0,1}\Sigma \otimes L|_p, \Lambda^{0,1}\Sigma \otimes L)$, and write*

$$\beta = \bar{\partial}_A^* G(p,\cdot) \in \Gamma(Hom(T_p^{0,1}\Sigma \otimes L|_p, L)).$$

*Then provided that $\tau \geq \kappa/8$ we have the differential inequality*

$$\Delta\left(4|\beta|^2 + 81\tau|G|^2\right) \geq \frac{3}{2}\tau\left(4|\beta|^2 + 81\tau|G|^2\right) \text{ on } \Sigma \setminus \{p\}.$$

*Proof.* First, on $\Sigma \setminus \{p\}$, we have by the definition of $G$

$$\bar{\partial}_A \bar{\partial}_A^* G + \tau|\theta|^2 G = 0. \tag{21}$$

Now

$$\langle \bar{\partial}_A \bar{\partial}_A^* G, G\rangle = \langle -\frac{1}{2}\Delta G, G\rangle = -\frac{1}{2}\left(\langle tr\nabla_A^2 G, G\rangle - (*iF_A + sec)|G|^2\right)$$
$$= -\frac{1}{4}\Delta|G|^2 + \frac{1}{2}|\nabla_A G|^2 + \frac{\tau}{2}(1 + sec/\tau - |\theta|^2)|G|^2,$$

where in the second equality we have used the Weitzenböck formula on $L$–valued 1–forms $\Delta = tr\nabla_A^2 - (*F_A)* - sec$ and the fact that $*G = iG$ since $G$ has type $(0,1)$. Hence taking the inner product of (21) with $G$ gives

$$-\frac{1}{4}\Delta|G|^2 + \frac{\tau}{2}(1 + sec/\tau + |\theta|^2)|G|^2 + \frac{1}{2}|\nabla_A G|^2 = 0 \tag{22}$$

Meanwhile, applying $\bar{\partial}_A^*$ to (21) gives

$$\bar{\partial}_A^* \bar{\partial}_A \beta + \tau|\theta|^2\beta = i\tau * (\bar{\theta}\partial_A\theta \wedge G) \tag{23}$$

(we have used that $\partial_A\bar{\theta} = 0$ here). Now since $\Delta = tr\nabla_A^2$ on sections of $L$, we have

$$Re\langle \bar{\partial}_A^*\bar{\partial}_A\beta, \beta\rangle = -\frac{1}{2}Re\langle tr\nabla_A^2\beta, \beta\rangle = -\frac{1}{4}\Delta|\beta|^2 + \frac{1}{2}|\nabla_A\beta|^2,$$

while

$$|*(\bar{\theta}\partial_A\theta \wedge G)| \leq 2\tau^{1/2}|\theta|(9/8 - |\theta|^2)|G|$$

by Lemma 6.1 and the assumption $\kappa\tau^{-1} \leq 1/8$, so taking the real part of the inner product of (23) with $\beta$ shows

$$-\frac{1}{4}\Delta|\beta|^2 + \tau|\theta|^2\beta + \frac{1}{2}|\nabla_A\beta|^2 \leq 2\tau^{3/2}|\theta|(9/8 - |\theta|^2)|G||\beta|. \tag{24}$$

But now note that $|\nabla_A G|^2 \geq |\bar{\partial}_A^* G|^2 = |\beta|^2$, while $|\nabla_A\beta|^2 \geq |\bar{\partial}_A\beta|^2 = |\bar{\partial}_A\bar{\partial}_A^* G|^2 = \tau^2|\theta|^4|G|^2$ by (21). Substituting these relations into (22) and (24) and using that, by assumption, $sec \geq -\kappa \geq -\tau/8$, yields

$$\frac{1}{4}\Delta|G|^2 \geq \frac{\tau}{2}(\frac{7}{8} + |\theta|^2)|G|^2 + \frac{1}{2}|\beta|^2 \tag{25}$$
$$\frac{1}{4}\Delta|\beta|^2 \geq \frac{\tau^2}{2}|\theta|^4|G|^2 + \tau|\theta|^2|\beta|^2 - \frac{9}{4}\tau^{3/2}|\theta||G||\beta|$$

But since

$$\frac{81}{16}\tau^2|G|^2 + 4\tau|\theta|^2|\beta|^2 \geq 9\tau^{3/2}|\theta||G||\beta|,$$

adding $81\tau$ times the first inequality of (25) to 4 times the second gives

$$\Delta\left(\frac{81}{4}\tau|G|^2 + |\beta|^2\right) \geq \tau^2\left(\frac{243}{8} + \frac{81}{2}|\theta|^2 + 2|\theta|^4\right)|G|^2 + \frac{81}{2}\tau|\beta|^2$$
$$\geq \frac{3}{2}\tau\left(\frac{81}{4}\tau|G|^2 + |\beta|^2\right),$$

as desired □

**Corollary 6.5.** *Where $R$ is the constant from Theorem 6.3, there is a constant $K$ such that, for all sufficiently large $\tau$ and for all $p, x \in \Sigma$ with $d(p,x) \geq c\tau^{-1/2}$ we have*

$$4|\beta(x,p)|^2 + 81\tau|G(x,p)|^2 \leq K(e^{-\sqrt{\tau}d(x,p)} + e^{-R\sqrt{\tau}})\tau\log\tau.$$

*Proof.* Let $u(x) = e^{-\sqrt{\tau}\alpha(d(p,x))}$ where $\alpha\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the same function as in the proof of Theorem 6.3 (so that in particular, we have $\Delta u \leq \frac{3}{2}\tau u$ and $u(x) \leq e^{-\sqrt{\tau}d(p,x)} + e^{-R\sqrt{\tau}}$). Now if $d(p,x) = c\tau^{-1/2}$, the local estimates (20) show that

$$4|\beta(x,p)|^2 + 81\tau|G(x,p)|^2 \leq A\tau\log\tau$$

for an appropriate constant $A$, so we can choose $K$ independently of $\tau$, $p$ such that, when $d(p,x) = c\tau^{-1/2}$, $Ku(x)\tau\log\tau = Ke^{-6c/5}\tau\log\tau \geq 4|\beta(x,p)|^2 + 81\tau|G(x,p)|^2$. So since

$$\Delta\left(Ku\tau\log\tau - \left(4|\beta|^2 + 81\tau|G|^2\right)\right) \leq \frac{3}{2}\tau\left(Ku\tau\log\tau - \left(4|\beta|^2 + 81\tau|G|^2\right)\right)$$

on $\Sigma \setminus B_{c\tau^{-1/2}}(p)$ and

$$Ku\tau\log\tau|_{\partial B_{c\tau^{-1/2}}(p)} \geq 4|\beta|^2 + 81\tau|G|^2\big|_{\partial B_{c\tau^{-1/2}}(p)},$$

we deduce by the usual argument that $Ku\tau\log\tau - \left(4|\beta|^2 + 81\tau|G|^2\right)$ cannot attain a negative local minimum and hence must be nonnegative throughout $\Sigma \setminus B_{c\tau^{-1/2}}(p)$. □

In particular, after renaming $K$, we have

$$|\beta(x,p)| \leq K(e^{-\sqrt{\tau}d(x,p)/2} + e^{-R\sqrt{\tau}/2})\tau^{1/2}\log\tau$$

when $d(x,p) \geq c\tau^{-1/2}$.

In light of this corollary, together with Theorem 6.3, the local Green's kernel bound (20) and the parallel transport prescription (19), we can get bounds on

$$|\dot{\theta}(x)| \leq \left(\sup\frac{|\dot{J}|}{2}\right)\int_\Sigma |\beta(x,y)||d_A\theta(y)|\omega_y$$

from simple bounds on integrals over $\Sigma$ of various expressions involving functions of form $e^{-a\sqrt{\tau}d(q,\cdot)}$ for $a$ a constant and $q \in \Sigma$. Namely, note first that if $r < injrad(\Sigma)$ and $C > 0$ are constants such that for each $z \in \Sigma$, $\exp_z\colon \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < r^2\} \to \Sigma$ is a diffeomorphism onto its image with Jacobian at most $C$, then we have, for any $z \in \Sigma$,

$$\begin{aligned}\int_\Sigma e^{-a\sqrt{\tau}d(z,x)}\omega_x &\leq vol(\Sigma)e^{-a\sqrt{\tau}r} + C\int_0^{2\pi}\int_0^r e^{-a\sqrt{\tau}\rho}\rho d\rho d\theta \\ &\leq vol(\Sigma)e^{-a\sqrt{\tau}r} + \frac{2\pi C}{a^2\tau}.\end{aligned}$$

Along the same lines, if $x, z \in \Sigma$ are two given points, for any $y \in \Sigma$, adding the equations $d(z,y) + d(x,z) \geq d(x,y)$, and $2(d(x,y) + d(z,y)) \geq 2d(x,z)$ shows that

$$3(d(x,y) + d(z,y)) \geq 2d(x,y) + d(x,z),$$

and so

$$\begin{aligned}\int_\Sigma e^{-a\sqrt{\tau}d(x,y)}e^{-a\sqrt{\tau}d(z,y)}\omega_y &\leq e^{-a\sqrt{\tau}d(x,z)/3}\int_\Sigma e^{-2a\sqrt{\tau}d(x,y)/3}\omega_y \\ &\leq e^{-a\sqrt{\tau}d(x,z)/3}\left(\frac{C'}{a^2\tau} + Be^{-ar\sqrt{\tau}}\right)\end{aligned} \tag{26}$$

for certain constants $B$, $C'$.

Finally, noting that given $x \in \Sigma$, where $C$ is the same Jacobian bound as earlier, and we assume that $\tau$ is large enough that $c\tau^{-1/2} < r$, we have

$$\int_{B_{c\tau^{-1/2}}(x)} \frac{1}{d(x,y)}\omega_y \leq C\int_0^{2\pi}\int_0^{c\tau^{-1/2}} \frac{\rho d\rho d\theta}{\rho} = \frac{2\pi Cc}{\sqrt{\tau}}.$$

So if $d(x,z) \geq 2c\tau^{-1/2}$, so that $d(y,z) \geq 2d(x,z)$ for each $y \in B_{c\tau^{-1/2}}(x)$, we get

$$\int_{B_{c\tau^{-1/2}}(x)} \frac{e^{-\sqrt{\tau}d(y,z)}}{d(x,y)}\omega_y \leq D\tau^{-1/2}e^{-\sqrt{\tau}d(x,z)/2}$$

for some constant $D$ while if $d(x,z) \leq 2c\tau^{-1/2}$ then $e^{-\sqrt{\tau}d(x,z)/2} \geq e^{-c}$, so that

$$\int_{B_{c\tau^{-1/2}}(x)} \frac{e^{-\sqrt{\tau}d(y,z)}}{d(x,y)}\omega_y \leq D\tau^{-1/2}e^{-\sqrt{\tau}d(x,z)/2}$$

still holds, possibly after increasing the (still $x, z$, and $\tau$–independent) constant $D$.

So recalling our estimates

$$|\beta(x,y)| \leq \frac{C}{d(x,y)} \text{ when } d(x,y) \leq c\tau^{-1/2}$$

$$|\beta(x,y)| \leq K(e^{-\sqrt{\tau}d(x,y)/2} + e^{-R\sqrt{\tau}/2})\tau^{1/2}\log\tau \text{ when } d(x,y) \geq c\tau^{-1/2}$$

$$|d_A\theta(y)| \leq 2\tau^{1/2}w(y)+2\kappa\tau^{-1/2} \leq 4\kappa\tau^{-1/2}+2M\tau^{1/2}\sum_{\{p:\theta(p)=0\}}\left(e^{-\sqrt{\tau}d(y,p)} + e^{-R\sqrt{\tau}}\right)$$

we deduce

**Corollary 6.6.** *There are constants $L, b > 0$, depending only on the path of almost complex structures $\{J_t\}_{t\in[0,1]}$, such that if $\tau$ is sufficiently large the path $(A(t), \theta(t))$ of $J_t$–vortices is obtained by (19), we have, for all $t$,*

$$|\dot{\theta}(t)(x)| \leq L\left(\tau^{-1} + (\log\tau)\sum_{p:\theta(t)(p)=0}(e^{-b\sqrt{\tau}d(x,p)} + \tau e^{-b\sqrt{\tau}})\right).$$

In particular, since $\theta(t)$, being a not–identically–zero section of a degree $d$ holomorphic line bundle, vanishes at no more than $d$ points, we have, where $w(t,x) = 1 - |\theta(t,x)|^2 \geq 0$,

$$\left|\frac{\partial w}{\partial t}\right| \leq |\theta||\dot{\theta}| \leq 2dL\log\tau$$

everywhere (we restrict here to $\tau$ large enough that $\tau^{-1} + d\tau e^{-b\sqrt{\tau}} \leq d$). So if $|h| \leq (4dL\log\tau)^{-1}$ and $x \in \Sigma$ is such that $w(t+h,x) = 1$, we must have had $w(t,x) \geq 1/2$. Referring back to the notation in Theorem 6.3, assuming that $\tau$ is large enough that $\kappa\tau^{-1}+Mde^{-R\sqrt{\tau}} \leq 1/4$, this implies that one of the $d$ expressions $e^{-\sqrt{\tau}d(x,p)}$ for $p \in \theta(t)^{-1}(0)$ must be at least $(4dM)^{-1}$, so that $d(x,p) \leq B\tau^{-1/2}$ where the constant $B$ is independent of $t$. So since the points where $w(t+h,\cdot)$ is equal to 1 are those where $\theta(t+h)$ vanishes, we deduce that *for all $t \in [0,1]$, if $|h| \leq (4dL\log\tau)^{-1}$, then each zero of $\theta(t+h)$ is a distance at most $B\tau^{-1/2}$ from a zero of $\theta(t)$, and vice versa*, where the "vice versa" part comes from just replacing $t$ by $t+h$ and $h$ by $-h$. But then we can subdivide $[0,1]$ into at most $(5dL\log\tau)$ intervals each of length at most $(4dL\log\tau)^{-1}$ and apply this fact to the endpoints of each interval to deduce that

**Corollary 6.7.** *Where $N = 5dLB$, each zero of $\theta(1)$ lies a distance at most $N\tau^{-1/2}\log\tau$ from some zero of $\theta(0)$, and vice versa.*

Thus since the parallel transport map

$$F_{\{J_t\}}\colon\ S^d(\Sigma, J_0) \to S^d(\Sigma, \phi_\omega^* J_0) \quad (\phi_\omega^* J_0 = J_1)$$

sends the zero set of $\theta(0)$ to that of $\theta(1)$, we deduce that, as $\tau \to \infty$, $F_{\{J_t\}}$ converges in $C^0$ norm to the identity, and so the $\Omega_{d,\omega,\tau}$–symplectomorphisms $\Phi_{d,\omega,\tau} = S^d\phi_\omega \circ F_{\{J_t\}}$ converge in $C^0$-norm to $S^d\phi_\omega\colon\ S^d\Sigma \to S^d\Sigma$ as $\tau \to \infty$.

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