Existence and uniqueness for symplectic embeddings

Trends in Modern Geometry, University of Tokyo

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Consider $\mathbb{R}^{2n} = \{(\vec{x}, \vec{y}) | \vec{x}, \vec{y} \in \mathbb{R}^n\}$ with its standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

The structure $(\mathbb{R}^{2n}, \omega_0)$ has many symmetries ("symplectomorphisms") $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with $\phi^*\omega_0 = \omega_0$, arising *e.g.* from solutions to Hamilton's equations for any well-behaved $H\colon [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$:

$$\vec{x}'(t) = -\nabla^{\vec{y}} H(t, \vec{x}(t), \vec{y}(t))$$
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If ϕ is the map sending arbitrary initial conditions $(\vec{x}(0), \vec{y}(0))$ to $(\vec{x}(1), \vec{y}(1))$ then $\phi^*\omega_0 = \omega_0$.

Motivating questions

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Given $A, U \subset \mathbb{R}^{2n}$, is there a symplectomorphism $\phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with $\phi(A) \subset U$?

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Technical assumptions: Unless otherwise stated A, U will be star-shaped with A compact, U open and $A, \bar{U}$ both manifolds with corners.

Liouville's theorem (1838)

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So if $\phi^*\omega_0 = \omega_0$ then ϕ must be volume-preserving. Thus **a necessary condition for $\phi(A) \subset U$ is that $vol(A) \leq vol(U)$.**
Actually $vol(A) < vol(U)$ given that we assume A compact, U open.

In $\mathbb{R}^2$ this is the end of the story: under our assumptions on $A, U \subset \mathbb{R}^2$, a 1965 argument of Moser implies that A symplectically embeds into U if and only if $vol(A) < vol(U)$, and that two such embeddings are symplectically isotopic provided that they are smoothly isotopic through embeddings whose images have constant volume.

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We'll see that things are much more complicated starting in dimension 4.

Gromov's non-squeezing theorem (1985)

Consider the $2n$-dimensional ball of capacity $a > 0$

$$B^{2n}(a) = \left\{ (\vec{x}, \vec{y}) \in \mathbb{R}^{2n} \,\middle|\, \sum_j \pi(x_j^2 + y_j^2) \leq a \right\}$$

and the corresponding cylinder

$$P(a, \infty, \ldots, \infty) = \left\{ (\vec{x}, \vec{y}) \in \mathbb{R}^{2n} \,\middle|\, \pi(x_1^2 + y_1^2) \leq a \right\}.$$

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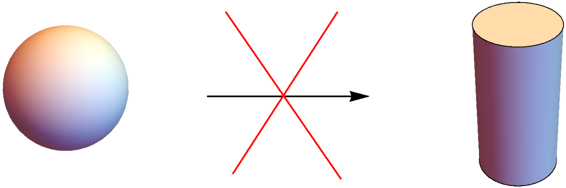
Gromov's non-squeezing theorem asserts that $B^{2n}(a)$ symplectically embeds into $P(A, \infty, \ldots, \infty)^\circ$ only if $a < A$.



A good family of test cases is provided by *toric domains* (also called "Reinhardt domains") in $\mathbb{R}^{2n}$, which are given as preimages $X_\Omega = \mu^{-1}(\Omega)$ of regions $\Omega \subset [0, \infty)^n$ under the map

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For instance we have the ellipsoid

$$E(a_1,\ldots,a_n) = \mu^{-1}\left(\left\{(t_1,\ldots,t_n)\,\middle|\,\sum \frac{t_i}{a_i} \le 1\right\}\right)$$

with $B^{2n}(a) = E(a,\ldots,a)$ and the polydisk

$$\begin{aligned} P(a_1,\ldots,a_n) &= \mu^{-1}\left(\{(t_1,\ldots,t_n)\,|0 \le t_i \le a_i\}\right) \\ &= B^2(a_1)\times\cdots\times B^2(a_n) \end{aligned}$$

Our normalization gives $vol^{2n}(X_\Omega) = vol^n(\Omega)$, so Liouville's theorem says that if we have a symplectic embedding of $2n$-dimensional toric domains $X_{\Omega_1} \hookrightarrow X^\circ_{\Omega_2}$ then $vol^n(\Omega_1) < vol^n(\Omega_2)$.

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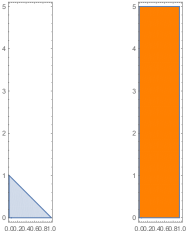
Gromov's theorem shows for instance that we cannot symplectically embed the toric domain $E(1,1)$ associated to the region at left into the toric domain $P(.95, 5)$ associated to the region at right.

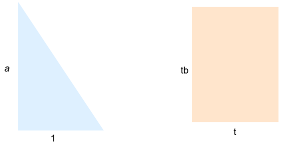


Consider the existence problem for embeddings of four-dimensional ellipsoids into polydisks:

$$E(1, a) \hookrightarrow P(c, d)^{\circ}.$$

It's conventional to fix the "aspect ratio" of the codomain equal to some $b \geq 1$ and investigate the "embedding capacity function"

$$c_b(a) = \inf\{t | (\exists \text{symplectic embedding } E(1, a) \hookrightarrow P(t, tb)^{\circ})\}.$$



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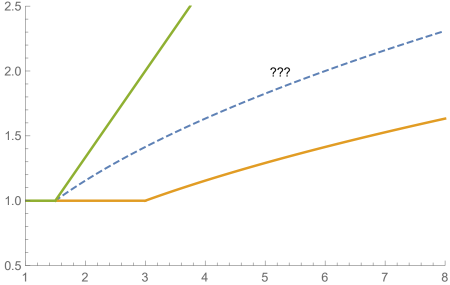
Also the inclusion gives an upper bound

$$c_b(a) \leq \max\{1, a/b\}.$$



The function c_b has remarkably intricate structure, which has been completely worked out by Frenkel–Müller for $b = 1$ (2012, modeled on work of McDuff–Schlenk for embeddings into balls) and Cristofaro-Gardiner–Frenkel–Schlenk (2016) for other integer b. The structure for non-integer b is likely at least as complicated, and not yet entirely known.

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- (McDuff-Polterovich 1994) This last condition is equivalent to the statement that there is a symplectic form on the N-fold blowup of $S^2 \times S^2$ giving areas c, d to $S^2 \times \{pt\}, \{pt\} \times S^2$ and areas $w_1(a), \ldots, w_N(a)$ to the N exceptional divisors (and homotopic to a Kähler form).

This turns our embedding problem into a problem about the cohomology classes of symplectic forms on blowups of $S^2 \times S^2$.

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We know that $c_b(1) = 1$ by non-squeezing and the fact that $E(1, 1) \subset P(1, b)$. Once one replaces $P(1, b)$ by $S^2(1) \times S^2(b)$ one can work out the largest a such that $c_b(a) = 1$ (*i.e.*, the supremum of a such that $E(1, a) \hookrightarrow S^2(1) \times S^2(b)$) by fairly direct considerations, using **toric geometry** on $S^2(1) \times S^2(b)$.

This can be made especially explicit when $b = 1$ (Gutt-U. 2017, based in part on Fukaya-Oh-Ohta-Ono 2012, Oakley-U. 2014), so that we are considering a product of equal-area spheres $S^2(1) \times S^2(1)$. Regard $S^2(1)$ as the unit sphere in $\mathbb{R}^3$, with symplectic form equal to $\frac{1}{4\pi}$ times the standard area form. The functions

$$F_1, F_2 \colon S^2(1) \times S^2(1) \to \mathbb{R}$$

$$F_1(\vec{v}, \vec{w}) = 1 - \frac{\|\vec{v} + \vec{w}\|}{2}, \quad F_2(\vec{v}, \vec{w}) = \frac{\|\vec{v} + \vec{w}\| - (v_3 + w_3)}{2}$$

have Hamiltonian flows that generate commuting $\mathbb{R}/\mathbb{Z}$-actions on $S^2 \times S^2 \setminus \{(\vec{v}, -\vec{v})\}$.

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The image of the map $(F_1, F_2)\colon\ S^2(1) \times S^2(1) \to \mathbb{R}$ is exactly the triangle $\{2x + y \leq 2, x, y \geq 0\}$, which is the same as the image of $E(1, 2)$ under the map $\mu\colon\ (x_1, x_2, y_1, y_2) \mapsto (\pi(x_1^2 + y_1^2), \pi(x_2^2 + y_2^2))$ (whose components likewise generate commuting $\mathbb{R}/\mathbb{Z}$-actions).

This allows us to construct "action-angle coordinates" $(F_1, F_2, \phi_1, \phi_2)$ on an open dense subset of $S^2(1) \times S^2(1)$, which symplectically identify that subset with $E(1,2)^\circ$. So we get a symplectic embedding $E(1,2)^\circ \hookrightarrow S^2(1) \times S^2(1)$; note that this fills up the entire volume of the codomain.

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$\left(\Gamma(\sqrt{4\pi}(x_1 + iy_1), \sqrt{4\pi}(x_2 + iy_2)), \Gamma(-\sqrt{4\pi}(x_1 + iy_1), \sqrt{4\pi}(x_2 + iy_2))\right)$
where

$$\Gamma(w,z) = \left(\frac{\sqrt{8-|w|^2}\left((8-2|w|^2-|z|^2)w + \bar{w}z^2\right)}{8(4-|w|^2)} + \frac{iz}{4}\sqrt{8-2|w|^2-|z|^2}, \right.$$
$$\left. 1 - \frac{|w|^2+|z|^2}{4} - \frac{\sqrt{(8-|w|^2)(8-2|w|^2-|z|^2)}}{4(4-|w|^2)} Im(w\bar{z}) \right)$$

and we think of S^2 as a subset of $\mathbb{C} \times \mathbb{R}$.

For non-integer b there is a related toric construction: writing $b = m + \epsilon$ where $m \in \mathbb{N}, 0 < \epsilon < 1$, there is a symplectomorphism between $S^2(1) \times S^2(m+\epsilon)$ and a Kähler **Hirzebruch surface** Σ_{2m}, with the sections of square $\pm 2m$ having areas $2m+\epsilon, \epsilon$ and the fiber having area 1.

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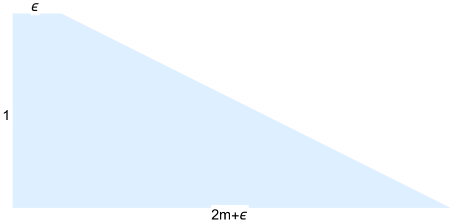
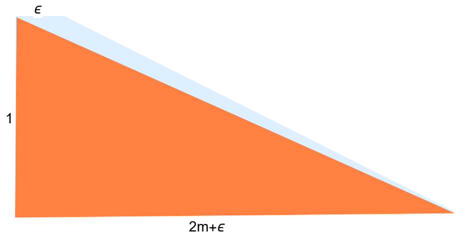
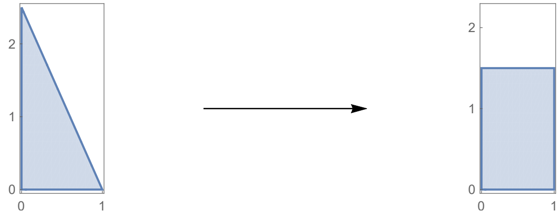
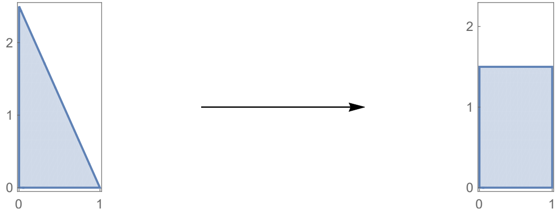




Inside this we see the usual moment image of the ellipsoid $E(2m+\epsilon, 1)^{\circ}$, from which one can infer the existence of a symplectic embedding $E(1, 2m+\epsilon)^{\circ} \hookrightarrow S^1(1) \times S^2(m+\epsilon)$.



This is enough to show that, if $b = m + \epsilon$ with $m \in \mathbb{N}, 0 \leq \epsilon < 1$, then $c_b(a) = 1$ for $1 \leq a \leq 2m + \epsilon$.



This is enough to show that, if $b = m + \epsilon$ with $m \in \mathbb{N}, 0 \leq \epsilon < 1$, then $c_b(a) = 1$ for $1 \leq a \leq 2m + \epsilon$.
In fact the conversion to a problem about blowups of $S^2 \times S^2$ implies that this interval of a's is sharp, or equivalently that the ellipsoid $E(1, 2m + \alpha)$ does not embed into arbitarily small dilates of $P(1, m + \epsilon)^\circ$ if $\alpha > \epsilon$.

The general source of obstructions to embeddings of this nature consists of symplectic *exceptional classes* in blowups of $S^2 \times S^2$—second homology classes of self-intersection -1 and Chern number 1 which are represented by smoothly embedded spheres. **Any** symplectic form in the standard deformation class must evaluate positively on such a class.

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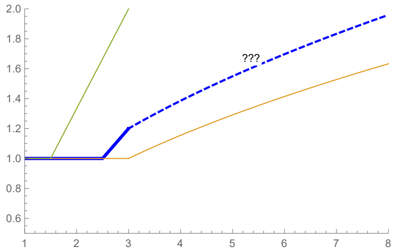
One such class is

$$F_m = m[S^2 \times \{pt\}] + [\{pt\} \times S^2] - \sum_{i=1}^{2m+1} E_i.$$

If we could embed $E(1, 2m + \alpha)$ into $P(t, t(m + \epsilon))^\circ$ McDuff's results imply that there would be a symplectic form evaluating as t on $[S^2 \times \{pt\}]$, $t(m + \epsilon)$ on $[\{pt\} \times S^2]$, 1 on $E_1, \ldots, E_{2m}$, and α on E_{2m+1}. But in order for this form to evaluate positively on F_m, if $\alpha > \epsilon$ we would need to bound t away from 1.

By thinking about rescalinigs of the above embeddings to get an upper bound, and applying similar reasoning with the exceptional class F_m to get the lower bound, one can obtain the following picture for the initial part of the graph of c_b (together with the simpler bounds from the start of the talk):



To go much further than this one seems to need deeper results about blowups of $S^2 \times S^2$, taking one into arguments that are more remote from explicit constructions. Work of Li-Liu (2001) and Li-Li (2002) based on Taubes–Seiberg–Witten theory shows that a cohomology class can be represented by a symplectic form in the standard deformation class if and only if it has positive square and evaluates positively on every exceptional class. Thus the *only* obstructions that prevent $c_b(a)$ from coinciding with the volume bound $\sqrt{\frac{a}{2b}}$ come from exceptional classes like F_m.

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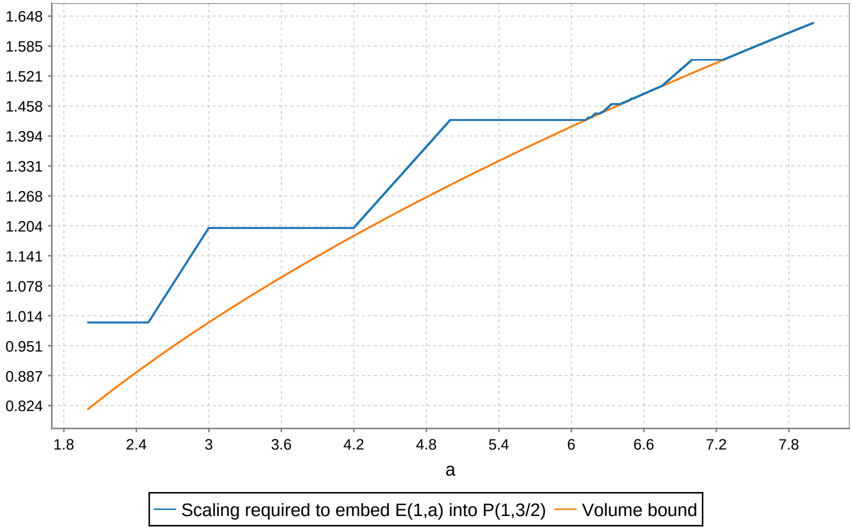
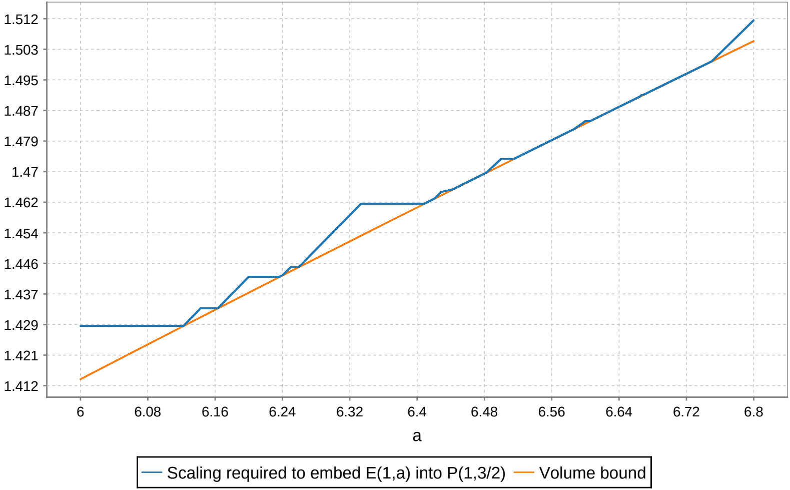

1.648
1.585
1.521
1.458
1.394
1.331
1.268
1.204
1.141
1.078
1.014
0.951
0.887
0.824
1.8
2.4
3
3.6
4.2
4.8
5.4
6
6.6
7.2
7.8
a
Scaling required to embed E(1,a) into P(1,3/2)
Volume bound

Note the complicated behavior for $a \in [6, 7]$, which has not yet been completely understood:



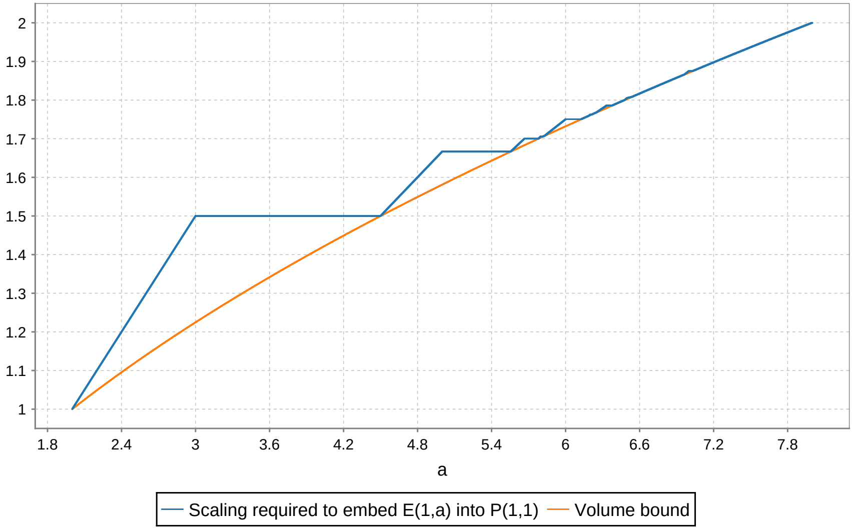
As mentioned earlier, the functions c_b have been completely worked out when $b \in \mathbb{N}$. $b = 1$ turns out to be the richest case: analogously to work of McDuff-Schlenk for embeddings of ellipsoids into balls, Frenkel–Müller found the graph of c_1 to be given on the interval $[1, (1+\sqrt{2})^2]$ by an **infinite staircase** arising from exceptional class obstructions near a sequence of a related to the Pell numbers and the half-companion Pell numbers (each of which satisfy a recurrence $x_{n+2} = x_{n+1} + 2x_n$):

$$a = \frac{3}{1}, \frac{5}{1}, \frac{17}{3}, \frac{29}{5}, \frac{99}{17}, \frac{169}{29}, \ldots$$

As mentioned earlier, the functions c_b have been completely worked out when $b \in \mathbb{N}$. $b = 1$ turns out to be the richest case: analogously to work of McDuff-Schlenk for embeddings of ellipsoids into balls, Frenkel–Müller found the graph of c_1 to be given on the interval $[1, (1+\sqrt{2})^2]$ by an **infinite staircase** arising from exceptional class obstructions near a sequence of a related to the Pell numbers and the half-companion Pell numbers (each of which satisfy a recurrence $x_{n+2} = x_{n+1} + 2x_n$):

$$a = \frac{3}{1}, \frac{5}{1}, \frac{17}{3}, \frac{29}{5}, \frac{99}{17}, \frac{169}{29}, \ldots$$

For $a > 7\frac{1}{32}$, $c_1(a)$ coincides with the volume bound $\sqrt{\frac{a}{2}}$, and for $a \in [(1+\sqrt{2})^2, 7\frac{1}{32}]$, $c_1(a)$ is the maximum of the volume bound and a collection of seven exceptional class obstructions that have been determined explicitly.


2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
1.8
2.4
3
3.6
4.2
4.8
5.4
6
6.6
7.2
7.8
a
Scaling required to embed E(1,a) into P(1,1)
Volume bound

For integer $b \geq 2$, Cristofaro-Gardiner–Frenkel–Schlenk showed that the only obstructions come from the exceptional classes F_m mentioned earlier for a finite subset of m depending on b, together with one additional exceptional class G_b. In particular the Frenkel–Müller infinite staircase somehow disappears as b increases from 1 to 2.

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This leads to the question of what happens for $b \in \mathbb{R}$ very close to 1. We don't entirely know, but:

Theorem (U., in progress)

Fix $b > 1$. The only exceptional class obstructions for $a < (1+\sqrt{2})^2$ are given by a finite subset of size $O(\log(1/(b-1)))$ of the classes from the Frenkel–Müller infinite staircase. However, new obstructions arise for a, b arbitrarily close to (and larger than) $(1+\sqrt{2})^2, 1$.

We now discuss the extent to which symplectic embeddings $A \hookrightarrow U$ are unique up to some notion of equivalence.

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By using the relation between ball embeddings and symplectic blowups, McDuff showed in 1991 that if A and U are both balls in $\mathbb{R}^4$, then the space of symplectic embeddings of A into U is connected. Since there is a symplectic version of the isotopy extension theorem, this implies that any two embeddings of A into U are related by a compactly supported ambient symplectic isotopy of U.

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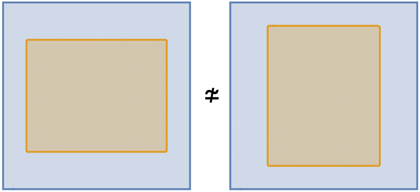
McDuff's 2009 work extended this result to the case that A and U are each ellipsoids in $\mathbb{R}^4$. Cristofaro-Gardiner later observed that her argument extends more generally to the case that A is an ellipsoid and U is a "convex toric domain," *e.g.* a polydisk.

On the other hand, Floer–Hofer–Wysocki showed in 1994 that, if $a, b < 1$ and $a + b > 1$, then the two embeddings

$$\begin{array}{cc} P(a,b) \hookrightarrow P(1,1)^\circ & P(a,b) \hookrightarrow P(1,1)^\circ \\ (x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, y_1, y_2) & (x_1, x_2, y_1, y_2) \mapsto (x_2, x_1, y_2, y_1) \end{array}$$

are **not** symplectically isotopic inside $P(1,1)^\circ$.



Note that this example illustrates that it matters what equivalence relation one uses to study uniqueness; the embeddings aren't isotopic but they are obviously related by a symplectomorphism of the codomain $P(1,1)^\circ$.

Gutt-U. (2017) provide a broad class of examples of symplectic embeddings $\phi\colon A \hookrightarrow U$ where $A \subset U \subset \mathbb{R}^4$ which are distinct from the inclusion in the strong sense that there is no symplectomorphism of U mapping A to $\phi(A)$; for example we can take $A = P(a,a)$ and $U = P(1,1)^\circ$ for $\frac{1}{2} < a < \frac{2}{3}$.

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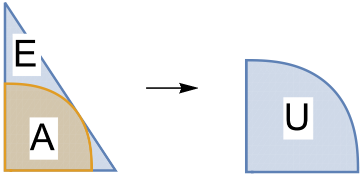
In general, our strategy is to choose a toric domain $X_\Omega = \left\{\left(\pi(x_1^2 + y_1^2), \pi(x_2^2 + y_2^2)\right) \in \Omega\right\}$ for a suitable $\Omega \in \mathbb{R}^2$ that is star-shaped with respect to the origin, and take $A = X_\Omega$ and $U = X^\circ_{\lambda\Omega}$, where $\lambda > 1$ is chosen just large enough for there to exist an ellipsoid E that contains A and symplectically embeds into U.



The main tool that we use to prove that these embeddings are distinct from the inclusion is the filtered version of the S^1-equivariant symplectic homology of Viterbo and Bourgeois–Oancea. This associates to a suitable domain $X \subset \mathbb{R}^{2n}$ a $\mathbb{R}$-filtered chain complex, whose filtered homology $\{CH^L(X)\}_{L\in\mathbb{R}}$ is functorial with respect to symplectic embeddings.

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In such a situation, if we can use methods like those discussed earlier in the talk to prove the existence of a composition $\phi\colon A \hookrightarrow E \hookrightarrow U$ of symplectic embeddings, then $\phi(A)$ can be shown to be inequivalent to A under symplectomorphisms of U.

This works for a wide class of domains:

Theorem (Gutt-U. 2017)

Let $\Omega \subset [0,\infty)^2$ be any convex set strictly containing $\{x+y\leq c\}$ and contained in $[0,c]\times[0,c]$ for some $c>0$, so that the associated toric domain X_{Ω} obeys $B^4(c)\subsetneq X_{\Omega}\subset P(c,c)$. Then there is $\lambda>1$ and a symplectic embedding $\phi\colon X_{\Omega}\hookrightarrow X^{\circ}_{\lambda\Omega}$ such that $\phi(X_{\Omega})$ cannot be mapped to X_{Ω} by any symplectomorphism of $X^{\circ}_{\lambda\Omega}$. The same conclusion also holds with X_{Ω} equal to any polydisk $P(a,b)$ with $a\leq b<2a$.

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Note that X_Ω can be taken to be arbitrarily close to a ball $B^4(c)$; on the other hand recall McDuff's 1991 theorem that implies that the result cannot hold for X_Ω exactly equal to a ball.

In general, our embeddings are constructed using the rather indirect ellipsoid embedding machinery discussed earlier in the talk; however in the case of mapping $P(c,c) \hookrightarrow P(1,1)^{\circ}$ our explicit embedding $E(1,2)^{\circ} \hookrightarrow S^2(1) \times S^2(1)$ makes it possible to obtain something much more concrete. Here are pictures of how the image of our embedding intersects $P(1,1) \cap \{y_2 = a\}$ for various constants a:

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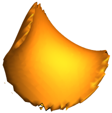

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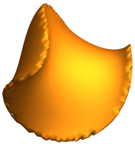

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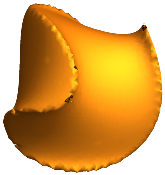

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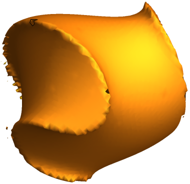

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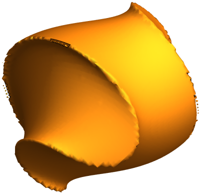

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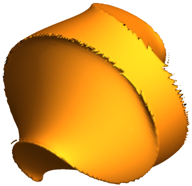

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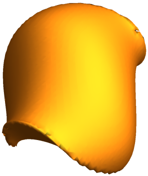

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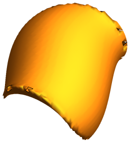

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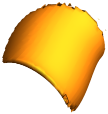

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Thank you!