Existence and uniqueness for symplectic embeddings
Trends in Modern Geometry, University of Tokyo

Michael Usher (University of Georgia)

July 11, 2017
Consider \( \mathbb{R}^{2n} = \{(\vec{x}, \vec{y})| \vec{x}, \vec{y} \in \mathbb{R}^n \} \) with its standard symplectic form
\[
\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.
\]

The structure \((\mathbb{R}^{2n}, \omega_0)\) has many symmetries ("symplectomorphisms") \(\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}\) with \(\phi^* \omega_0 = \omega_0\), arising e.g. from solutions to Hamilton’s equations for any well-behaved \(H : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}\):
\[
\begin{align*}
\vec{x}'(t) &= -\nabla^{\vec{y}} H(t, \vec{x}(t), \vec{y}(t)) \\
\vec{y}'(t) &= \nabla^{\vec{x}} H(t, x(t), y(t)).
\end{align*}
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$$\vec{y}'(t) = \nabla^\vec{x} H(t, x(t), y(t)).$$

If $\phi$ is the map sending arbitrary initial conditions $(\vec{x}(0), \vec{y}(0))$ to $(\vec{x}(1), \vec{y}(1))$ then $\phi^* \omega_0 = \omega_0$. 

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Existence and uniqueness for symplectic embeddings
Question (Existence)

Given $A, U \subset \mathbb{R}^{2n}$, is there a symplectomorphism $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with $\phi(A) \subset U$?
Motivating questions

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Given $A, U \subset \mathbb{R}^{2n}$, is there a symplectomorphism $\phi: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with $\phi(A) \subset U$?

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Given two such $\phi_1, \phi_2$, how are they related? In particular is there a symplectomorphism $\psi: U \to U$ with $\psi(\phi_1(A)) = \phi_2(A)$?
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Technical assumptions: Unless otherwise stated $A, U$ will be star-shaped with $A$ compact, $U$ open and $A, \bar{U}$ both manifolds with corners.
Liouville’s theorem (1838)

\[ \omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i. \]

Note \( \omega_0^n \) is a constant times the standard volume form

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So if \( \phi^*\omega_0 = \omega_0 \) then \( \phi \) must be volume-preserving. Thus a necessary condition for \( \phi(A) \subset U \) is that \( \text{vol}(A) \leq \text{vol}(U) \).

Actually \( \text{vol}(A) < \text{vol}(U) \) given that we assume \( A \) compact, \( U \) open.
In $\mathbb{R}^2$ this is the end of the story: under our assumptions on $A, U \subset \mathbb{R}^2$, a 1965 argument of Moser implies that $A$ symplectically embeds into $U$ if and only if $\text{vol}(A) < \text{vol}(U)$, and that two such embeddings are symplectically isotopic provided that they are smoothly isotopic through embeddings whose images have constant volume.
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Gromov’s non-squeezing theorem (1985)

Consider the $2n$-dimensional ball of capacity $a > 0$

$$B^{2n}(a) = \left\{ (\vec{x}, \vec{y}) \in \mathbb{R}^{2n} \left\| \sum_j \pi(x_j^2 + y_j^2) \leq a \right\} \right\}$$

and the corresponding cylinder

$$P(a, \infty, \ldots, \infty) = \left\{ (\vec{x}, \vec{y}) \in \mathbb{R}^{2n} \left| \pi(x_1^2 + y_1^2) \leq a \right\} \right\}.$$
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**Gromov’s non-squeezing theorem** asserts that $B^{2n}(a)$ symplectically embeds into $P(A, \infty, \ldots, \infty)^{\circ}$ only if $a < A$. 

Gromov’s non-squeezing theorem (1985)
A good family of test cases is provided by *toric domains* (also called “Reinhardt domains”) in $\mathbb{R}^{2n}$, which are given as preimages $X_{\Omega} = \mu^{-1}(\Omega)$ of regions $\Omega \subset [0, \infty)^n$ under the map

$$\mu(\vec{x}, \vec{y}) = (\pi(x_1^2 + y_1^2), \ldots, \pi(x_n^2 + y_n^2)) .$$
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For instance we have the ellipsoid

$$E(a_1, \ldots, a_n) = \mu^{-1}\left(\left\{(t_1, \ldots, t_n) \left| \sum t_i / a_i \leq 1 \right. \right\}\right)$$

with $B^{2n}(a) = E(a, \ldots, a)$ and the polydisk

$$P(a_1, \ldots, a_n) = \mu^{-1}\left(\left\{(t_1, \ldots, t_n) \left| 0 \leq t_i \leq a_i \right. \right\}\right) = B^2(a_1) \times \cdots \times B^2(a_n)$$
Our normalization gives $\text{vol}^{2n}(X_\Omega) = \text{vol}^n(\Omega)$, so Liouville’s theorem says that if we have a symplectic embedding of $2n$-dimensional toric domains $X_{\Omega_1} \hookrightarrow X_{\Omega_2}^\circ$ then $\text{vol}^n(\Omega_1) < \text{vol}^n(\Omega_2)$. 
Our normalization gives $vol^{2n}(X_{\Omega}) = vol^n(\Omega)$, so Liouville’s theorem says that if we have a symplectic embedding of $2n$-dimensional toric domains $X_{\Omega_1} \hookrightarrow X_{\Omega_2}^\circ$ then $vol^n(\Omega_1) < vol^n(\Omega_2)$. 

Gromov’s theorem shows for instance that we cannot symplectically embed the toric domain $E(1, 1)$ associated to the region at left into the toric domain $P(.95, 5)$ associated to the region at right.
Consider the existence problem for embeddings of four-dimensional ellipsoids into polydisks:

\[ E(1, a) \hookrightarrow P(c, d)\circ. \]

It’s conventional to fix the “aspect ratio” of the codomain equal to some \( b \geq 1 \) and investigate the “embedding capacity function”

\[ c_b(a) = \inf \{ t | (\exists \text{symplectic embedding } E(1, a) \hookrightarrow P(t, tb)\circ) \}. \]
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Our classical embedding obstructions give lower bounds:
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- Liouville’s theorem (volume) \( \Rightarrow c_b(a) \geq \sqrt{\frac{a}{2b}} \)
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Also the inclusion gives an upper bound
\[ c_b(a) \leq \max\{1, a/b\}. \]
The function $c_b$ has remarkably intricate structure, which has been completely worked out by Frenkel–Müller for $b = 1$ (2012, modeled on work of McDuff–Schlenk for embeddings into balls) and Cristofaro-Gardiner–Frenkel–Schlenk (2016) for other integer $b$. The structure for non-integer $b$ is likely at least as complicated, and not yet entirely known.
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- (McDuff 2009) For $a \in \mathbb{Q}$, $E(1, a)$ symplectically embeds into $P(c, d)^\circ$ iff a certain specific disjoint union of balls $B^4(w_1(a)) \sqcup \cdots \sqcup B^4(w_N(a))$ symplectically embeds into $P(c, d)^\circ$.

- (McDuff-Polterovich 1994) Up to arbitrarily small rescaling, such a disjoint union of balls symplectically embeds into $P(c, d)^\circ$ iff it embeds into the product $S^2(c) \times S^2(d)$ of spheres of area $c$ and $d$.

- (McDuff-Polterovich 1994) This last condition is equivalent to the statement that there is a symplectic form on the $N$-fold blowup of $S^2 \times S^2$ giving areas $c$, $d$ to $S^2 \times \{\text{pt}\}$, $\{\text{pt}\} \times S^2$ and areas $w_1(a), \ldots, w_N(a)$ to the $N$ exceptional divisors (homotopic to a K"ahler form). This turns our embedding problem into a problem about the cohomology classes of symplectic forms on blowups of $S^2 \times S^2$. 
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One consequence of the previous slide is that, to understand $c_b$, we can replace the codomains of our embeddings $E(1, a) \hookrightarrow P(t, tb)$ by products of spheres $S^2(t) \times S^2(tb)$. We know that $c_b(1) = 1$ by non-squeezing and the fact that $E(1, 1) \subset P(1, b)$. Once one replaces $P(1, b)$ by $S^2(1) \times S^2(b)$ one can work out the largest $a$ such that $c_b(a) = 1$ (i.e., the supremum of $a$ such that $E(1, a) \hookrightarrow S^2(1) \times S^2(b)$) by fairly direct considerations, using toric geometry on $S^2(1) \times S^2(b)$. 

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This can be made especially explicit when $b = 1$ (Gutt-U. 2017, based in part on Fukaya-Oh-Ohta-Ono 2012, Oakley-U. 2014), so that we are considering a product of equal-area spheres $S^2(1) \times S^2(1)$. Regard $S^2(1)$ as the unit sphere in $\mathbb{R}^3$, with symplectic form equal to $\frac{1}{4\pi}$ times the standard area form. The functions

$$F_1, F_2 : S^2(1) \times S^2(1) \to \mathbb{R}$$

$$F_1(\vec{v}, \vec{w}) = 1 - \frac{\|\vec{v} + \vec{w}\|}{2}, \quad F_2(\vec{v}, \vec{w}) = \frac{\|\vec{v} + \vec{w}\| - (v_3 + w_3)}{2}$$

have Hamiltonian flows that generate commuting $\mathbb{R}/\mathbb{Z}$-actions on $S^2 \times S^2 \setminus \{ (\vec{v}, -\vec{v}) \}$. 
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have Hamiltonian flows that generate commuting $\mathbb{R}/\mathbb{Z}$-actions on $S^2 \times S^2 \setminus \{(\vec{v}, -\vec{v})\}$.

The image of the map $(F_1, F_2): \ S^2(1) \times S^2(1) \to \mathbb{R}$ is exactly the triangle $\{2x + y \leq 2, x, y \geq 0\}$, which is the same as the image of $E(1, 2)$ under the map

$$\mu: (x_1, x_2, y_1, y_2) \mapsto (\pi(x_1^2 + y_1^2), \pi(x_2^2 + y_2^2))$$

(whose components likewise generate commuting $\mathbb{R}/\mathbb{Z}$-actions).
This allows us to construct “action-angle coordinates” 
$(F_1, F_2, \phi_1, \phi_2)$ on an open dense subset of $S^2(1) \times S^2(1)$, which symplectically identify that subset with $E(1, 2)$. So we get a symplectic embedding $E(1, 2) \hookrightarrow S^2(1) \times S^2(1)$; note that this fills up the entire volume of the codomain.
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\((F_1, F_2, \phi_1, \phi_2)\) on an open dense subset of \(S^2(1) \times S^2(1)\), which symplectically identify that subset with \(E(1, 2)^\circ\). So we get a symplectic embedding \(E(1, 2)^\circ \hookrightarrow S^2(1) \times S^2(1)\); note that this fills up the entire volume of the codomain. One can even write a formula: 
\[(x_1, x_2, y_1, y_2) \mapsto (\Gamma(\sqrt{4\pi}(x_1 + iy_1), \sqrt{4\pi}(x_2 + iy_2)), \Gamma(-\sqrt{4\pi}(x_1 + iy_1), \sqrt{4\pi}(x_2 + iy_2)))\]
where 
\[
\Gamma(w, z) = \begin{pmatrix}
\sqrt{8 - |w|^2}((8 - 2|w|^2 - |z|^2)w + \bar{w}z^2) \\
\frac{8(4 - |w|^2)}{4|z|^2 - \sqrt{(8 - |w|^2)(8 - 2|w|^2 - |z|^2)}} \operatorname{Im}(w\bar{z})
\end{pmatrix}
\]
and we think of \(S^2\) as a subset of \(\mathbb{C} \times \mathbb{R}\).
For non-integer $b$ there is a related toric construction: writing $b = m + \epsilon$ where $m \in \mathbb{N}$, $0 < \epsilon < 1$, there is a symplectomorphism between $S^2(1) \times S^2(m + \epsilon)$ and a Kähler Hirzebruch surface $\Sigma_{2m}$, with the sections of square $\pm 2m$ having areas $2m + \epsilon$, $\epsilon$ and the fiber having area 1.
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Inside this we see the usual moment image of the ellipsoid $E(2m + \epsilon, 1)^\circ$, from which one can infer the existence of a symplectic embedding $E(1, 2m + \epsilon)^\circ \hookrightarrow S^1(1) \times S^2(m + \epsilon)$. 
This is enough to show that, if \( b = m + \epsilon \) with \( m \in \mathbb{N}, 0 \leq \epsilon < 1 \), then \( c_b(a) = 1 \) for \( 1 \leq a \leq 2m + \epsilon \).
This is enough to show that, if \( b = m + \epsilon \) with \( m \in \mathbb{N}, 0 \leq \epsilon < 1 \), then \( c_b(a) = 1 \) for \( 1 \leq a \leq 2m + \epsilon \).

In fact the conversion to a problem about blowups of \( S^2 \times S^2 \) implies that this interval of \( a \)'s is sharp, or equivalently that the ellipsoid \( E(1, 2m + \alpha) \) does not embed into arbitrarily small dilates of \( P(1, m + \epsilon)^o \) if \( \alpha > \epsilon \).
The general source of obstructions to embeddings of this nature consists of symplectic exceptional classes in blowups of $S^2 \times S^2$—second homology classes of self-intersection $-1$ and Chern number $1$ which are represented by smoothly embedded spheres. Any symplectic form in the standard deformation class must evaluate positively on such a class.
The general source of obstructions to embeddings of this nature consists of symplectic *exceptional classes* in blowups of $S^2 \times S^2$—second homology classes of self-intersection $-1$ and Chern number $1$ which are represented by smoothly embedded spheres. *Any* symplectic form in the standard deformation class must evaluate positively on such a class.

One such class is

$$F_m = m[S^2 \times \{pt\}] + [\{pt\} \times S^2] - \sum_{i=1}^{2m+1} E_i.$$ 

If we could embed $E(1, 2m + \alpha)$ into $P(t, t(m + \epsilon))$ McDuff’s results imply that there would be a symplectic form evaluating as $t$ on $[S^2 \times \{pt\}]$, $t(m + \epsilon)$ on $[\{pt\} \times S^2]$, $1$ on $E_1, \ldots, E_{2m}$, and $\alpha$ on $E_{2m+1}$. But in order for this form to evaluate positively on $F_m$, if $\alpha > \epsilon$ we would need to bound $t$ away from $1$. 

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By thinking about rescalings of the above embeddings to get an upper bound, and applying similar reasoning with the exceptional class $F_m$ to get the lower bound, one can obtain the following picture for the initial part of the graph of $c_b$ (together with the simpler bounds from the start of the talk):
To go much further than this one seems to need deeper results about blowups of $S^2 \times S^2$, taking one into arguments that are more remote from explicit constructions. Work of Li-Liu (2001) and Li-Li (2002) based on Taubes–Seiberg–Witten theory shows that a cohomology class can be represented by a symplectic form in the standard deformation class if and only if it has positive square and evaluates positively on every exceptional class. Thus the only obstructions that prevent $c_b(a)$ from coinciding with the volume bound $\sqrt{\frac{a}{2b}}$ come from exceptional classes like $F_m$. 
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Introduction
Embedding four-dimensional ellipsoids into polydisks
(Non-)Uniqueness

Setup and key ingredients
Explicit embeddings
Exceptional sphere obstructions

Scaling required to embed $E(1,a)$ into $P(1,3/2)$
Volume bound

<table>
<thead>
<tr>
<th>$a$</th>
<th>0.824</th>
<th>0.887</th>
<th>0.951</th>
<th>1.014</th>
<th>1.078</th>
<th>1.141</th>
<th>1.204</th>
<th>1.268</th>
<th>1.331</th>
<th>1.394</th>
<th>1.458</th>
<th>1.521</th>
<th>1.585</th>
<th>1.648</th>
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</thead>
<tbody>
<tr>
<td>Volume bound</td>
<td>1.8</td>
<td>2.4</td>
<td>3</td>
<td>3.6</td>
<td>4.2</td>
<td>4.8</td>
<td>5.4</td>
<td>6</td>
<td>6.6</td>
<td>7.2</td>
<td>7.8</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Existence and uniqueness for symplectic embeddings
Note the complicated behavior for $a \in [6, 7]$, which has not yet been completely understood:

![Graph showing the scaling required to embed $E(1,a)$ into $P(1,3/2)$ and the volume bound](image)

- Blue line: Scaling required to embed $E(1,a)$ into $P(1,3/2)$
- Orange line: Volume bound
As mentioned earlier, the functions $c_b$ have been completely worked out when $b \in \mathbb{N}$. $b = 1$ turns out to be the richest case: analogously to work of McDuff-Schlenk for embeddings of ellipsoids into balls, Frenkel–Müller found the graph of $c_1$ to be given on the interval $[1, (1 + \sqrt{2})^2]$ by an **infinite staircase** arising from exceptional class obstructions near a sequence of $a$ related to the Pell numbers and the half-companion Pell numbers (each of which satisfy a recurrence $x_{n+2} = x_{n+1} + 2x_n$):

$$a = \frac{3}{1}, \frac{5}{1}, \frac{17}{3}, \frac{29}{5}, \frac{99}{17}, \frac{169}{29}, \ldots$$
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$$a = \frac{3}{1}, \frac{5}{1}, \frac{17}{3}, \frac{29}{5}, \frac{99}{17}, \frac{169}{29}, \cdots$$

For $a > 7\frac{1}{32}$, $c_1(a)$ coincides with the volume bound $\sqrt{\frac{a}{2}}$, and for $a \in [(1 + \sqrt{2})^2, 7\frac{1}{32}]$, $c_1(a)$ is the maximum of the volume bound and a collection of seven exceptional class obstructions that have been determined explicitly.
Introduction

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(Non-)Uniqueness

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Exceptional sphere obstructions

Scaling required to embed $E(1,a)$ into $P(1,1)$

Volume bound

$1.8$ $2.4$ $3$ $3.6$ $4.2$ $4.8$ $5.4$ $6$ $6.6$ $7.2$ $7.8$}

$1$ $1.1$ $1.2$ $1.3$ $1.4$ $1.5$ $1.6$ $1.7$ $1.8$ $1.9$ $2$

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Existence and uniqueness for symplectic embeddings
For integer $b \geq 2$, Cristofaro-Gardiner–Frenkel–Schlenk showed that the only obstructions come from the exceptional classes $F_m$ mentioned earlier for a finite subset of $m$ depending on $b$, together with one additional exceptional class $G_b$. In particular the Frenkel–Müller infinite staircase somehow disappears as $b$ increases from 1 to 2.
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This leads to the question of what happens for $b \in \mathbb{R}$ very close to 1. We don’t entirely know, but:

**Theorem (U., in progress)**

*Fix $b > 1$. The only exceptional class obstructions for $a < (1 + \sqrt{2})^2$ are given by a finite subset of size $O(\log(1/(b - 1)))$ of the classes from the Frenkel–Müller infinite staircase. However, new obstructions arise for $a, b$ arbitrarily close to (and larger than) $(1 + \sqrt{2})^2, 1$.***
We now discuss the extent to which symplectic embeddings $A \hookrightarrow U$ are unique up to some notion of equivalence.
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McDuff’s 2009 work extended this result to the case that \( A \) and \( U \) are each ellipsoids in \( \mathbb{R}^4 \). Cristofaro-Gardiner later observed that her argument extends more generally to the case that \( A \) is an ellipsoid and \( U \) is a “convex toric domain,” e.g. a polydisk.
On the other hand, Floer–Hofer–Wysocki showed in 1994 that, if \( a, b < 1 \) and \( a + b > 1 \), then the two embeddings

\[
P(a, b) \hookrightarrow P(1, 1)^°
\]

\[
(x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, y_1, y_2)
\]

are not symplectically isotopic inside \( P(1, 1)^° \).

Note that this example illustrates that it matters what equivalence relation one uses to study uniqueness; the embeddings aren’t isotopic but they are obviously related by a symplectomorphism of the codomain \( P(1, 1)^° \).
Gutt-U. (2017) provide a broad class of examples of symplectic embeddings $\phi: A \hookrightarrow U$ where $A \subset U \subset \mathbb{R}^4$ which are distinct from the inclusion in the strong sense that there is no symplectomorphism of $U$ mapping $A$ to $\phi(A)$; for example we can take $A = P(a, a)$ and $U = P(1, 1)^\circ$ for $\frac{1}{2} < a < \frac{2}{3}$. 
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In general, our strategy is to choose a toric domain $X_{\Omega} = \{(\pi(x_1^2 + y_1^2), \pi(x_2^2 + y_2^2)) \in \Omega\}$ for a suitable $\Omega \in \mathbb{R}^2$ that is star-shaped with respect to the origin, and take $A = X_{\Omega}$ and $U = X_{\lambda \Omega}^\circ$, where $\lambda > 1$ is chosen just large enough for there to exist an ellipsoid $E$ that contains $A$ and symplectically embeds into $U$. 
The main tool that we use to prove that these embeddings are distinct from the inclusion is the filtered version of the $S^1$-equivariant symplectic homology of Viterbo and Bourgeois–Oancea. This associates to a suitable domain $X \subset \mathbb{R}^{2n}$ a $\mathbb{R}$-filtered chain complex, whose filtered homology $\{CH^L(X)\}_{L \in \mathbb{R}}$ is functorial with respect to symplectic embeddings.
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In such a situation, if we can use methods like those discussed earlier in the talk to prove the existence of a composition $\phi: A \hookrightarrow E \hookrightarrow U$ of symplectic embeddings, then $\phi(A)$ can be shown to be inequivalent to $A$ under symplectomorphisms of $U$. 
This works for a wide class of domains:

**Theorem (Gutt-U. 2017)**

Let \( \Omega \subset [0, \infty)^2 \) be any convex set strictly containing \( \{x + y \leq c\} \) and contained in \([0, c] \times [0, c]\) for some \( c > 0 \), so that the associated toric domain \( X_\Omega \) obeys \( B^4(c) \subsetneq X_\Omega \subset P(c, c) \). Then there is \( \lambda > 1 \) and a symplectic embedding \( \phi: X_\Omega \hookrightarrow X_{\lambda \Omega}^\circ \) such that \( \phi(X_\Omega) \) cannot be mapped to \( X_\Omega \) by any symplectomorphism of \( X_{\lambda \Omega}^\circ \). The same conclusion also holds with \( X_\Omega \) equal to any polydisk \( P(a, b) \) with \( a \leq b < 2a \).
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Let $\Omega \subset [0, \infty)^2$ be any convex set strictly containing $\{x + y \leq c\}$ and contained in $[0, c] \times [0, c]$ for some $c > 0$, so that the associated toric domain $X_\Omega$ obeys $B^4(c) \subsetneq X_\Omega \subset P(c, c)$. Then there is $\lambda > 1$ and a symplectic embedding $\phi: X_\Omega \hookrightarrow X_\lambda^\circ$ such that $\phi(X_\Omega)$ cannot be mapped to $X_\Omega$ by any symplectomorphism of $X_\lambda^\circ$. The same conclusion also holds with $X_\Omega$ equal to any polydisk $P(a, b)$ with $a \leq b < 2a$.

Note that $X_\Omega$ can be taken to be arbitrarily close to a ball $B^4(c)$; on the other hand recall McDuff’s 1991 theorem that implies that the result cannot hold for $X_\Omega$ exactly equal to a ball.
In general, our embeddings are constructed using the rather indirect ellipsoid embedding machinery discussed earlier in the talk; however in the case of mapping $P(c, c) \hookrightarrow P(1, 1)$, our explicit embedding $E(1, 2) \hookrightarrow S^2(1) \times S^2(1)$ makes it possible to obtain something much more concrete. Here are pictures of how the image of our embedding intersects $P(1, 1) \cap \{y_2 = a\}$ for various constants $a$: 
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![Image of intersecting sets](image-url)
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![Image of embedding intersection]

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Thank you!