Deformed Hamiltonian Floer theory, quasimorphisms, and the Hofer-Zehnder capacity

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Outline

Throughout this talk (M,ω) denotes a closed symplectic manifold, and $S^1 = \mathbb{R}/\mathbb{Z}$.

For a Hamiltonian $H\colon S^1 \times M \to \mathbb{R}$, consider, on an appropriate cover

$$\widetilde{\mathscr{L}_0 M} = \frac{\{(\gamma, v) | v\colon D^2 \to M, v|_{S^1} = \gamma\}}{\sim}$$

of the contractible loopspace, the *action functional*

$$\mathscr{A}_H([\gamma, v]) = -\int_{D^2} v^*\omega - \int_0^1 H(t, \gamma(t))dt.$$

The critical points of $\mathscr{A}_H$ are those $[\gamma, v]$ for which the loop γ is an integral curve of the Hamiltonian vector field X_H ($\iota_{X_H}\omega = dH$), so they correspond to (some of the) fixed points of the time one map $\phi_H^1\colon M \to M$.

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The function $\mathscr{A}_H$ is Morse provided that the fixed points of the time-one map ϕ_H^1 are nondegenerate; in this case, the Hamiltonian Floer complex $(CF(H), \partial)$ is, formally speaking, the Morse-Novikov complex of $\mathscr{A}_H$.

Thus for a certain Novikov field Λ_ω we have a vector space isomorphism

$$CF(H) \cong \Lambda_\omega^m$$

where m is the number of contractible closed 1-periodic orbits of ϕ_H^t, while the boundary operator "counts negative gradient flowlines" of $\mathscr{A}_H$.

These negative gradient flowlines are solutions $u \colon \mathbb{R} \times S^1 \to M$ of

$$\frac{\partial u}{\partial s} + J(t,u)\left(\frac{\partial u}{\partial t} - X_H\right) = 0$$

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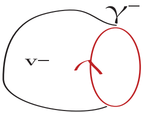
$$\partial[\gamma^-, v^-] = \sum_{[\gamma^+, v^+]} n([\gamma^-, v^-], [\gamma^+, v^+])[\gamma^+, v^+].$$



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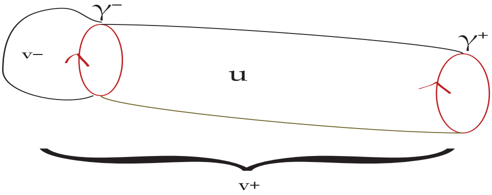

Theorem (Floer, Hofer-Salamon, Liu-Tian, Fukaya-Ono)

$\partial^2 = 0$, *and the resulting homology* $HF_*(H)$ *is isomorphic to singular homology* $H_*(M;\Lambda_\omega)$, *independently of* H.

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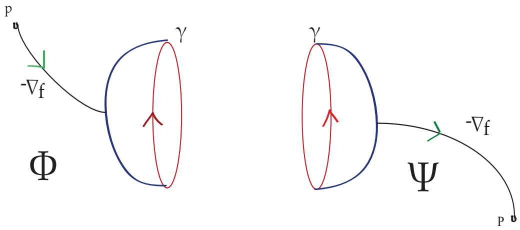
One way (due to Piunikhin-Salamon-Schwarz) of realizing the isomorphism with singular homology (equivalently, Morse homology) on chain level is by choosing a Morse function f and constructing chain maps $\Phi\colon CM(f;\Lambda_\omega) \to CF(H)$ and $\Psi\colon CF(H) \to CM(f;\Lambda_\omega)$ which count "spiked discs" (Here $CM(f;\Lambda_\omega)$ is a chain complex generated over Λ_ω by $p \in Crit(f)$, with differential counting negative gradient flowlines of f):



While the homology of the Floer chain complex $CF(H)$ is independent of the Hamiltonian H, other aspects the behavior of $CF(H)$ are in fact sensitive to H, and this can yield interesting information about properties that hold for some, but not all, Hamiltonian diffeomorphisms.

In particular, because $CF(H)$ is formally the Morse complex of the function $\mathscr{A}_H$, there is a natural $\mathbb{R}$-valued filtration $\{CF^{\lambda}(H)\}_{\lambda\in\mathbb{R}}$:

$$CF^{\lambda}(H) = \left\{ \sum a_i[\gamma_i, v_i] \,\middle|\, \mathscr{A}_H([\gamma_i, v_i]) \leq \lambda \right\}.$$

Since $\mathscr{A}_H$ decreases along its negative gradient flowlines, for any $\lambda \in \mathbb{R}$, $CF^{\lambda}(H)$ is a subcomplex of $CF(H)$.

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The *filtered* Floer complex is a dynamically significant invariant:

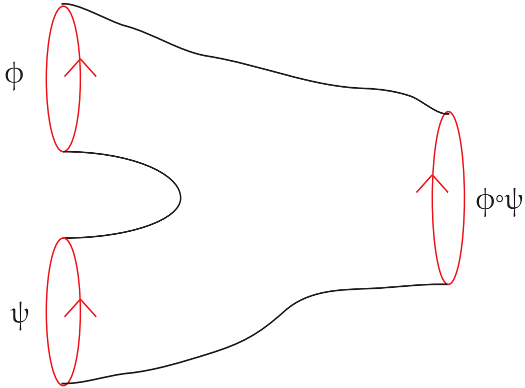
Theorem (U.)

Given $\tilde{\phi} = \tilde{\phi}_H \in \widetilde{Ham}(M, \omega)$, generated by a normalized ($\int_M H(t\cdot)\omega^n = 0$) Hamiltonian $H\colon S^1 \times M \to \mathbb{R}$, the filtered chain isomorphism type of the Floer complex $CF(H) = \cup_{\lambda\in\mathbb{R}} CF^{\lambda}(H)$ depends only on $\tilde{\phi}$, and is independent of H and of the other auxiliary data in the construction.

Accordingly we'll write $CF^{\lambda}(\tilde{\phi})$ (where $\tilde{\phi} \in \widetilde{Ham}(M, \omega)$) rather than $CF^{\lambda}(H)$.

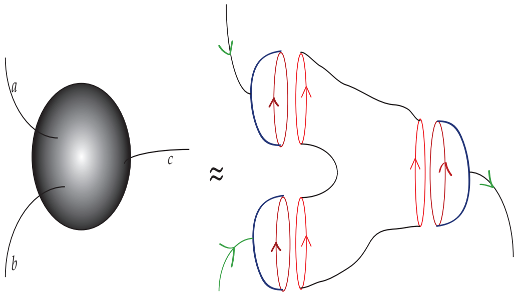
Hamiltonian Floer complexes admit a pair of pants product, which behaves compatibly with the filtrations:

$$*_{PP}\colon\ CF^{\lambda}(\tilde{\phi}) \otimes CF^{\mu}(\tilde{\psi}) \mapsto CF^{\lambda+\mu}(\tilde{\phi} \circ \tilde{\psi}).$$



Piunikhin–Salamon–Schwarz (and, by different methods, Ruan–Tian and Liu–Tian) showed that, on homology, this pair of pants product induces the small quantum product on $H_*(M) \otimes \Lambda_\omega$:

$$a * b = \sum_{A \in H_2} (a*b)_A T^{-\omega(A)} \quad \text{where } (a*b)_A \cap c = \langle a, b, c \rangle_{0,3,A}.$$



One way of extracting information from the filtration on $CF(\tilde{\phi})$ is through the Oh–Schwarz spectral invariants: where $\Phi\colon H_*(M,\Lambda_\omega)\to HF(\tilde{\phi})$ is the PSS isomorphism, for $a\in H_*(M;\Lambda_\omega)$ and $\tilde{\phi}\in\widetilde{Ham}(M,\omega)$, set

$$\rho(a,\tilde{\phi})=\inf\left\{\lambda\in\mathbb{R}\,\middle|\,\begin{array}{c}\Phi(a)\text{ is represented by a chain in }CF(\tilde{\phi})\\ \text{of filtration level }\leq\lambda\end{array}\right\}$$

Consideration of the pair of pants product shows

$$\rho(a*b;\tilde{\phi}\circ\tilde{\psi})\leq\rho(a,\tilde{\phi})+\rho(b,\tilde{\psi}).$$

In particular, if $e*e=e$ (for example if $e=[M]$, though there are often other useful idempotents as well)

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The "big quantum homology" of a symplectic manifold (M,ω) provides a family of quantum products $*_\eta$ on $H_*(M;\Lambda_\omega)$, parametrized by $\eta \in H_{ev}(M;\mathbb{C})$:

$$a *_\eta b = \sum_{k=0}^{\infty} \sum_{i,A} \frac{1}{k!} \langle a, b, \Delta_i, \eta, \cdots, \eta \rangle_{0,k+3,A} T^{-\omega(A)} \Delta^i,$$

where $\{\Delta_i\}$ is a basis for $H_*(M;\mathbb{C})$ with Poincaré dual basis $\{\Delta^i\}$.
If $\eta \in H_{2n-2}(M;\mathbb{C})$, the divisor axiom for GW invariants shows

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Correspondingly, for $\eta \in H_{ev}(M;\mathbb{C})$ the differential ∂ on the Floer chain complex may be "deformed by η":

In case $\eta \in H_{2n-2}(M;\mathbb{C})$ this is particularly simple: choose a closed 2-form $\theta \in \Omega^2(M;\mathbb{C})$ such that $PD[\theta] = \eta$, and modify the definition of the boundary operator to

$$\partial_\eta[\gamma^-, v^-] = \sum_{u \in \mathcal{M}_1([\gamma^-,v^-],[\gamma^+,v^+])} \pm e^{\int_{\mathbb{R}\times S^1} u^*\theta}[\gamma^+, v^+].$$

This is similar to the use of "B-fields" in Lagrangian Floer theory (Fukaya, Cho) and to Heegaard Floer homology with twisted coefficients (Ozsváth-Szabó); however I am not aware any prior uses of this in Hamiltonian Floer theory.

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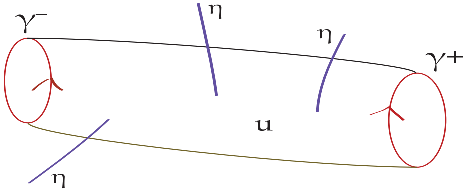

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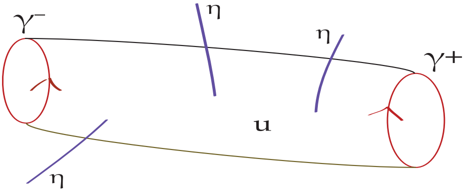

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In a very similar way, given $\eta \in H_{ev}(M;\mathbb{C})$ we may deform the PSS map $CM(f) \to CF(\tilde{\phi})$ and the pair-of-pants product $CF(\tilde{\phi}) \otimes CF(\tilde{\psi}) \to CF(\tilde{\phi} \circ \tilde{\psi})$; these deformed maps will be chain maps with respect to the deformed differentials ∂_η.

Proposition

On homology, the induced map $\Phi_\eta : H_*(M;\Lambda_\omega) \to HF(\tilde{\phi})$ induces an isomorphism of rings between the quantum homology $(H_*(M;\Lambda_\omega), *_\eta)$ and Floer homology with its η-deformed pair-of-pants product.

In particular, although (unlike in the Lagrangian "*B*-field" case) the homology $HF(\tilde{\phi})$ is independent of the deformation parameter η, the pair-of-pants ring structure *does* depend on η, and so we can hope to obtain additional information as η varies.

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The deformed complexes $(CF(\tilde{\phi}), \partial_\eta)$ still carry standard filtrations, and so we may define η-deformed Oh-Schwarz spectral invariants:

$$\rho(a, \tilde{\phi})_\eta = \inf\left\{ \lambda \in \mathbb{R} \,\middle|\, \begin{array}{c} \Phi_\eta(a) \text{ is represented by a cycle in} \\ (CF(\tilde{\phi}), \partial_\eta) \text{ of filtration level } \leq \lambda \end{array} \right\}$$

We now have a triangle inequality

$$\rho(a *_\eta b, \tilde{\phi} \circ \tilde{\psi})_\eta \leq \rho(a, \tilde{\phi})_\eta + \rho(b, \tilde{\psi})_\eta.$$

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If (U, ω) is an *open* symplectic manifold, the universal cover $\widetilde{Ham}(U, \omega)$ of the group of compactly supported Hamiltonian diffeomorphisms admits the Calabi homomorphism

$$Cal_U \colon \widetilde{Ham}(U, \omega) \to \mathbb{R}.$$

If $\tilde{\phi} \in \widetilde{Ham}(U, \omega)$ is generated by the compactly supported Hamiltonian $H \colon [0, 1] \times U \to \mathbb{R}$, then

$$Cal_U(\tilde{\phi}) = \int_0^1 \int_M H(t, \cdot)\omega^n dt.$$

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Following Entov-Polterovich, a Calabi quasimorphism on $\widetilde{Ham}(M,\omega)$ is a quasimorphism $\mu\colon \widetilde{Ham}(M,\omega)\to\mathbb{R}$ obeying the additional properties that

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Following Entov-Polterovich, a Calabi quasimorphism on $\widetilde{Ham}(M,\omega)$ is a quasimorphism $\mu\colon \widetilde{Ham}(M,\omega) \to \mathbb{R}$ obeying the additional properties that

- $\mu(\tilde{\phi}^k) = k\mu(\tilde{\phi})$, and
- If $U \subset M$ is a *displaceable* open subset and if $\tilde{\phi} \in \widetilde{Ham}(U, \omega|_U)$ is extended trivially to an element of $\widetilde{Ham}(M,\omega)$, then

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Entov and Polterovich have shown that the existence of a Calabi quasimorphism leads to many interesting consequences, concerning for instance the commutator norm on $\widetilde{Ham}(M,\omega)$ and strong nondisplaceability results for certain subsets of (M,ω).

Moreover, Entov and Polterovich produced (starting in 2003) Calabi quasimorphisms for certain (M,ω) from the Oh-Schwarz spectral invariants; initially, the class of manifolds for which their construction worked was rather limited, but this has been improved:

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Theorem (Entov-Polterovich, as generalized by McDuff, U.)

Let (M,ω) be a closed symplectic manifold and $\eta \in H_{ev}(M;\mathbb{C})$.

- *Suppose that $e \in H_{ev}(M;\Lambda_\omega)$ obeys $e *_\eta e = e$ and that we have an estimate*

$$\rho(e,\tilde{\phi})_\eta + \rho(e,\tilde{\phi}^{-1})_\eta \le C \quad \forall \tilde{\phi} \in \widetilde{Ham}(M,\omega). \tag{1}$$

  Then

$$\mu(\tilde{\phi}) = vol(M) \lim_{k\to\infty} \frac{\rho(e,\tilde{\phi}^k)_\eta}{k}$$

  defines a Calabi quasimorphism on $\widetilde{Ham}(M,\omega)$.

- *If the algebra $(H_{ev}(M;\Lambda_\omega), *_\eta)$ splits as a direct sum*

$$F \oplus A \quad \text{where } F \text{ is a field,}$$

  then where e is the multiplicative identity of the field summand, e obeys an estimate (1).

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Theorem (U.)

A certain (complicated) algebraic criterion on the Gromov–Witten invariants of (M,ω), depending only on the deformation class of ω, is equivalent to the statement that the algebra $(H_{ev}(M;\Lambda_\omega),*_\eta)$ has a field direct summand for an open dense set of $\eta\in H_{ev}(M;\mathbb{C})$. Moreover, this criterion holds for the following classes of (M,ω), which therefore admit Calabi quasimorphisms:

- *A blowup at a point of any closed symplectic manifold.*
- *Any algebraic manifold whose quantum homology is "generically semisimple" in the sense considered by Dubrovin, Manin, et al. In particular (by a theorem of Iritani) this includes all symplectic toric manifolds.*

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Previous work:

- McDuff showed that point blowups of *non-uniruled* symplectic manifolds admit Calabi quasimorphisms.
- Fukaya-Oh-Ohta-Ono and Ostrover-Tyomkin showed that generic toric symplectic structures on toric Fano manifolds admit Calabi quasimorphisms.
- Fukaya-Oh-Ohta-Ono also have work in preparation constructing infinite families of Calabi quasimorphisms on some toric manifolds.

If (M,ω) is a closed symplectic manifold, its π_1-sensitive Hofer–Zehnder capacity $c^\circ_{HZ}(M,\omega)$ is the quantity

$$\sup\left\{\max H-\min H \;\middle|\; \begin{array}{c} H\colon M\to\mathbb{R},\text{ all nonconstant contractible} \\ \text{periodic orbits of } X_H \text{ have period} > 1 \end{array}\right\}.$$

For many (M,ω), $c^{\circ}_{HZ}(M,\omega)$ is infinite; however it may sometimes be bounded by using Gromov–Witten invariants and/or Floer theory. For instance:

Theorem (Lu, cf. also Hofer-Viterbo, Liu-Tian)

If there is a nonzero Gromov–Witten invariant $\langle [pt],[pt],a_1,\ldots,a_k\rangle^{K}_{0,k+2,A}$ (where $K\in H_*(\overline{\mathscr{M}}_{0,k+2};\mathbb{Z})$ is arbitrary and $A\in H_2(M;\mathbb{Z})$), then

$$c^{\circ}_{HZ}(M,\omega)\leq\omega(A).$$

This estimate is sharp for a number of examples, such as $\mathbb{C}P^n$ and $S^2\times S^2$ with an arbitrary symplectic form.

Theorem (U.)

- *Suppose that $\eta \in H_{ev}(M;\mathbb{C})$ and $C > 0$ and that we have either of the following two estimates, for all $\tilde{\phi} \in \widetilde{Ham}(M,\omega)$:*

$$\rho([M],\tilde{\phi})_\eta + \rho([M],\tilde{\phi}^{-1})_\eta \leq C \quad \text{or} \tag{2}$$

$$\rho([pt];\tilde{\phi})_\eta + \rho([pt];\tilde{\phi}^{-1})_\eta \geq -C. \tag{3}$$

Then

$$c^\circ_{HZ}(M,\omega) \leq C.$$

- *The estimate (3) holds for generic choices of $\eta \in H_{ev}(M;\mathbb{C})$, with $C = \omega(A)$, whenever the class $A \in \frac{H_2(M;\mathbb{Z})}{tors}$ has a nonzero Gromov–Witten invariant of the form*

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In particular this gives a Floer-theoretic interpretation of Lu's result, at least for Gromov–Witten invariants of a somewhat more limited type.

Note that the estimate $\rho([M],\tilde{\phi})_\eta+\rho([M],\tilde{\phi}^{-1})_\eta\leq C$ is the same type of estimate as is used to obtain Calabi quasimorphisms. The estimates

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are equivalent if the minimal Chern number of (M,ω) is larger than $\frac{1}{2}\dim M$. Thus in the case of, for example, $\mathbb{C}P^n$, the same estimate that gives rise to a Calabi quasimorphism also gives rise to a sharp capacity bound!

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