The topology of symplectic four-manifolds

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Definition

A symplectic manifold is a pair (M, ω) where

1. M is a smooth manifold of some even dimension $2n$.
2. $\omega \in \Omega^2(M)$ is a two-form such that
   - ω is *closed*: $d\omega = 0$; and
   - ω is *nondegenerate*: if $p \in M$ and $0 \neq v \in T_pM$ then for some $w \in T_pM$ $\omega(v, w) \neq 0$.

By nondegeneracy, $\omega^{\wedge n}$ is nonvanishing, so symplectic manifolds always carry a natural orientation.

Classical examples

1. If V is any manifold, the **cotangent bundle** T^*V carries a natural 1-form
$$\lambda = \sum p_i dq_i,$$
and $(T^*V, d\lambda)$ is a symplectic manifold.
This is the setting for Hamiltonian mechanics.
2. $\mathbb{C}P^n$ carries a natural symplectic form ω_{FS}, and if $X \subset \mathbb{C}P^n$ is any **smooth projective variety** $(X, \omega_{FS}|_X)$ is a symplectic manifold.
More generally, a Kähler form on a complex manifold induces a symplectic structure.

Local structure

Symplectic manifolds have no local invariants!
Darboux's theorem: Any $p \in M$ has a coordinate neighborhood in which the symplectic form ω takes the form

$$\omega = \sum_{i=1}^{n} dx_i \wedge dx_{n+i}.$$

Similar results hold near certain submanifolds: $P \subset M$ is called a symplectic submanifold if $\omega|_P$ is a symplectic form on TP.

Weinstein neighborhood theorem: The symplectic structure on a neighborhood of a compact symplectic submanifold P is completely determined by $\omega|_P$ and the isomorphism class of the normal bundle to P.

Almost complex structures

An **almost complex structure** J on M is a bundle map $J : TM \to TM$ such that $J^2 = -1$.
J is called **compatible** with the symplectic form ω provided that the map $(v, w) \mapsto \omega(v, Jw)$ defines a Riemannian metric on M.

Theorem (Gromov)

The set of compatible almost complex structures on a symplectic manifold (M, ω) is nonempty and contractible.

So the tangent bundle of a symplectic manifold is, canonically up to homotopy, a complex vector bundle.
Hence we can speak of the Chern classes $c_k(M, \omega) \in H^{2k}(M; \mathbb{Z})$.

Pseudoholomorphic curves

Let Σ be a Riemann surface, with complex structure j, and let J be an almost complex structure on M.
A map $u : \Sigma \to M$ is called J-holomorphic (and its image is called a J-holomorphic curve) if its linearization $u_* : T\Sigma \to u^*TM$ is complex linear:

$$u_* \circ j = J \circ u_*, \text{ i.e., } du + J(u) \circ du \circ j = 0.$$

Gromov discovered a compactness property for such maps, which made it possible to define global symplectic invariants ("Gromov–Witten invariants") by (roughly speaking) choosing an almost complex structure J compatible with ω, counting the number of J-holomorphic curves satisfying some conditions, and showing that the result is independent of the choice of J.

The intersection form

We'll restrict attention to closed symplectic four-manifolds from now on.
A basic topological invariant of a closed oriented four-manifold is the intersection form

$$\begin{aligned} H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) &\to \mathbb{Z} \\ (a,b) &\mapsto \langle a \cup b, [M] \rangle \\ &= \langle a, PD(b) \rangle \end{aligned}$$

This is a nondegenerate, symmetric bilinear form.

A symplectic structure ω gives us *two* distinguished elements in $H^2(M;\mathbb{Z})$: $[\omega]$ and $\kappa_M = -c_1(M,\omega)$. The universe of symplectic four-manifolds splits naturally into classes based on how κ_M and $[\omega]$ behave with respect to the intersection form.

Since a J-holomorphic map $u : \Sigma \to M$ satisfies

$$\int_\Sigma u^*\omega = \int_\Sigma |du|^2,$$

homology classes represented by pseudoholomorphic curves pair positively with $[\omega]$.

Symplectic spheres

Some important topological properties of a symplectic four-manifold (M, ω) are reflected in whether M contains certain kinds of embedded spheres as symplectic submanifolds.

Theorem (McDuff)

If (M, ω) contains any embedded symplectic sphere S of nonnegative self-intersection, then (M, ω) is symplectomorphic to a Kähler manifold obtained by blowing up either

$$\mathbb{C}P^2 \text{ or a } \mathbb{C}P^1\text{-bundle over a Riemann surface.}$$

(In fact, there is a singular foliation of M by pseudoholomorphic spheres homologous to S.)

Blowing up and down

McDuff also showed that the operation of *blowing up* can be performed in the symplectic category: replace a small ball with a tubular neighborhood of a symplectic sphere of self-intersection -1.

Conversely, if (M, ω) contains a symplectic sphere *S* of self-intersection -1, we can get a new symplectic manifold $(N, \bar{\omega})$ by *blowing down S* (replace its tubular neighborhood with a ball).

Minimality

If M is obtained by blowing up N, then, as smooth oriented manifolds,

$$M = N \# \overline{\mathbb{C}P^2}.$$

A symplectic manifold is called *minimal* if it contains no symplectic spheres of square -1; evidently, then, nonminimal symplectic four-manifolds decompose as connected sums. Any symplectic four-manifold has a *minimal model*, obtained from it by blowing down a maximal collection of spheres of square -1.

The Seiberg–Witten invariants

The *Seiberg–Witten equations* are a natural elliptic system of PDE's that can be written down on any closed oriented four-manifold M. Counting their solutions gives rise to a powerful invariant, which in case M is symplectic takes the form of a map

$$SW : H^2(M;\mathbb{Z}) \to \mathbb{Z}.$$

In monumental work in the mid-1990's, Taubes showed that, when (M,ω) is symplectic,

1. If $b^+(M) > 1$, then $SW(\kappa_M) = \pm 1$.
2. For all $\alpha \in H^2(M;\mathbb{Z})$, $SW(\alpha)$ can be expressed as a combination of Gromov invariants which count pseudoholomorphic curves Poincaré dual to α.

Taubes' work had many consequences, including:

Theorem (Kotschick)

If (M,ω) is minimal and $\pi_1(M)$ is residually finite, then M is irreducible (i.e. if M is diffeomorphic to a connected sum $X \# Y$ then X or Y is a homotopy S^4).

Theorem (Taubes, Liu)

- *If $\kappa_M \cdot [\omega] < 0$, then M is a blowup of $\mathbb{C}P^2$ or a $\mathbb{C}P^1$-bundle.*
- *If $\kappa_M^2 < 0$ and M is minimal then M is a $\mathbb{C}P^1$-bundle.*

Kodaira dimension

The Kodaira dimension of a *minimal* symplectic 4-manifold (M, ω) is

$$\kappa(M,\omega) = \begin{cases} -\infty & \text{if } \kappa_M \cdot [\omega] < 0 \text{ or } \kappa_M^2 < 0 \\ 0 & \text{if } \kappa_M \cdot [\omega] = \kappa_M^2 = 0 \\ 1 & \text{if } \kappa_M \cdot [\omega] > 0 \text{ and } \kappa_M^2 = 0 \\ 2 & \text{if } \kappa_M \cdot [\omega] > 0 \text{ and } \kappa_M^2 > 0 \end{cases}$$

If (M, ω) is not minimal, its Kodaira dimension is defined to be that of any of its minimal models.

If (M, ω) is a Kähler surface, this coincides with the definition from algebraic geometry.

Theorem (Li, et al.)

- *The Kodaira dimension of* (M,ω) *is completely determined by the diffeomorphism type of* M*.*
- $\kappa(M,\omega)=-\infty$ *if and only if* (M,ω) *is symplectomorphic to a blowup of either* $\mathbb{C}P^2$ *or a* $\mathbb{C}P^1$*-bundle.*
- *If* M *is minimal,* $\kappa(M,\omega)=0$ *if and only if* κ_M *is torsion.*

We've seen that, if $k \geq -1$, the presence of symplectic spheres of square k is equivalent to certain basic topological properties (reducibility if $k = -1$, being a blowup of $\mathbb{C}P^2$ or a $\mathbb{C}P^1$-bundle if $k \geq 0$).
By contrast:

Theorem (Li-U.)

If (M, ω_0) is a symplectic four-manifold and $S \subset M$ is a symplectic sphere of self-intersection $k < -1$, then there is a path of symplectic forms $\{\omega_t\}_{t \in [0,1]}$ such that the symplectic manifold (M, ω_1) admits no symplectic spheres homologous to S.

Most of what I've discussed so far about the topology of symplectic four-manifolds applies equally well to the topology of Kähler surfaces.

But symplectic four-manifolds form a vastly more diverse category than Kähler surfaces; the symplectic sum (Gompf, McCarthy-Wolfson) has been the most powerful tool for demonstrating this.

Construction

Let (X, ω), (Y, ω') be symplectic four-manifolds, $F \subset X$, $G \subset Y$ two-dimensional symplectic submanifolds of

- equal genus and area
- opposite self-interesection, so there's an orientation-reversing bundle isomorphism $\psi : N_F X \to N_G Y$

Then the normal connect sum

$$X \#_{F=G} Y := (X \setminus \nu F) \cup_{\psi|_{\partial \nu F}} (Y \setminus \nu G)$$

admits a natural isotopy class of symplectic structures.

Examples of families of manifolds obtained by the symplectic sum

1. Gompf: If G is any finitely presented group, infinitely many symplectic 4-manifolds with fundamental group G.
2. Gompf: Simply connected symplectic 4-manifolds X realizing infinitely many intersection forms $H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to \mathbb{Z}$ which can't be intersection forms of complex surfaces.
3. Fintushel-Stern: Infinitely many symplectic four-manifolds homeomorphic to the $K3$ surface but not diffeomorphic to any complex manifold.

Constraints on symplectic sums

While the symplectic sum gives rise to a diverse array of new symplectic four-manifolds, there are interesting restrictions on what one can obtain by it:

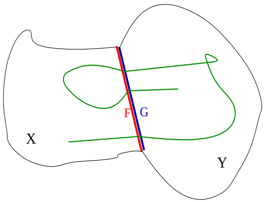
Theorem (U.)

Let $F \subset X$, $G \subset Y$ be symplectic surfaces of equal positive genus and opposite self-intersection, and let $Z = X\#_{F=G}Y$ be their (smoothly nontrivial) symplectic sum.

1. *Z is minimal if and only if both $X \setminus F$ and $Y \setminus G$ contain no symplectic spheres of square -1.*
2. *Z does not have Kodaira dimension $-\infty$.*

Idea of the proof

If a symplectic sphere of self-intersection ≥ -1 existed in Z, then in the singular space $X \cup_{F=G} Y$ we'd get a configuration of pseudoholomorphic spheres:



But such a configuration is impossible by a new version of the (almost-)positivity of the canonical class: if (M, ω) is *any* symplectic four-manifold and $F \subset M$ a positive genus surface (other than a section of an S^2-bundle) meeting all (-1)-spheres, then $\kappa_M + PD[F]$ evaluates nonnegatively on all pseudoholomorphic spheres.

Constraints on Kodaira dimension zero symplectic sums

Theorem (U.)

Where $Z = X_{F=G}Y$ as earlier, if Z has Kodaira dimension zero, then (modulo blowdowns away from F and G), X, Y, and Z are, up to diffeomorphism, as in the following table.

X	Y	Z
rational surface	*rational surface*	*K3 surface*
rational surface	*ruled surface over* T^2	*Enriques surface*
ruled surface over T^2	*ruled surface over* T^2	T^2*-bundle over* T^2

Find “small” exotic symplectic four-manifolds

What is the smallest k such that there is a symplectic (or even smooth) four-manifold homeomorphic but not diffeomorphic to $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$?
Current record: For $k = 5$, Park-Stipsicz-Szabo, Fintushel-Stern (2004) constructed infinitely many such smooth four manifolds, but at most one (and possibly none) of these is symplectic.

Akhmedov (2006) used symplectic sum to obtain a symplectic 4-manifold homeomorphic to $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$ which can be smoothly distinguished from it using theorems discussed earlier in this talk.

Classify all (minimal) symplectic four-manifolds of Kodaira dimension zero

In spite of all the ways we have of constructing new symplectic manifolds, the only known ones with Kodaira dimension zero are blowups of $K3$, the Enriques surface, or a T^2-bundle over T^2.

U. (2006): Symplectic sum along surfaces of positive genus can't give any new examples (up to diffeomorphism).

Li, Bauer (2006): Every minimal symplectic four-manifold of Kodaira dimension zero has the same rational homology as one of the known examples (in particular $b_1 \leq 4$, $\chi \in \{0, 12, 24\}$, $\sigma \in \{-16, -8, 0\}$).

How is deformation equivalence of symplectic four-manifolds related to diffeomorphism?

McMullen-Taubes, Smith, Vidussi (1999-2000): Examples of deformation inequivalent symplectic structures on the same smooth four-manifold, distinguished by Chern classes.

Ruan (1994): In dimensions 6 and higher, examples of deformation inequivalent symplectic structures on the same smooth manifold with the same Chern classes.

But Taubes' results imply that Ruan's methods can't work in dimension four, and there are still no known examples.

Can one obtain such examples by using different framings in a symplectic sum? (one would probably need new invariants to distinguish them)