Spectral numbers in Floer theories

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Let M be a closed manifold, and let $f\colon\ M \to \mathbb{R}$ be a Morse function.

Where

$$Crit_k(f) = \{p \in M | (df)_p = 0, ind_f p = k\}$$

write

$$CM_k(f) = \bigoplus_{p \in Crit_k(f)} \mathbb{Z}\langle p \rangle.$$

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The *Morse chain complex* of f is $CM_*(f) = \oplus_k CM_k(f)$, with a differential obtained from the negative gradient flow of f.

The differential

With respect to a generic metric on M, the following holds:
If $p, q \in Crit(f)$ with $ind_f p = ind_f q + 1$, then (up to $\mathbb{R}$-translation) there are finitely many solutions $\gamma : \mathbb{R} \to M$ to

$$\gamma'(t) = -\nabla f(\gamma(t))$$
$$\lim_{t \to -\infty} \gamma(t) = p$$
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Denote the number of such solutions (counted with appropriate signs, or mod 2) by $n(p, q)$, and define

$$\partial\colon\ CM_k(f) \to CM_{k-1}(f)$$

by

$$\partial p = \sum_q n(p, q) q.$$

The homology

Theorem (Thom, Smale,...)

$\partial^2 = 0$, *and the resulting homology groups satisfy*

$$HM_k(f) \cong H_k(M),$$

canonically.

The filtration

For any $\lambda \in \mathbb{R}$, put

$$CM_*^{\lambda}(f) = \{\sum_i n_i p_i \in CM_*(f) | n_i \neq 0 \Rightarrow f(p_i) \leq \lambda\}.$$

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Since f decreases along its negative gradient flowlines, the boundary operator restricts as

$$\partial \colon CM_*^{\lambda}(f) \to CM_*^{\lambda}(f).$$

Obviously the resulting homologies $HM_*^{\lambda}(f)$ will depend on f (unlike $HM_*(f)$); indeed

$$HM_*^{\lambda}(f) = H_*(\{f \leq \lambda\}).$$

Spectral numbers

Define

$$\begin{aligned}\rho_f :\ H_*(M) = HM_*(f) &\to \mathbb{R} \cup \{-\infty\} \\ a &\mapsto \inf\{\lambda | a \in Im(HM_*^{\lambda}(f) \to HM_*(f))\} \\ &= \inf\{\max_{i : n_i \neq 0} f(p_i) | [\sum_i n_i p_i] = a\}.\end{aligned}$$

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Obvious properties of ρ_f

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The main result of this talk is that these properties still hold in some analagous theories in which they aren't so obvious.

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But this fails, since there will sometimes be infinitely many such.

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$$I_\theta \colon \pi_1(M) \to \mathbb{R}.$$

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This has deck transformation group

$$\Gamma_\theta = \pi_1(M)/\ker I_\theta,$$

and we have, for some

$$\mathcal{A} \colon \tilde{M} \to \mathbb{R},$$

$$\pi^*\theta = d\mathcal{A}.$$

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Suppose $\gamma\colon \mathbb{R} \to \tilde{M}$ is a negative gradient flow line for $\mathcal{A}$ from p to q.
Then

$$\begin{aligned}\mathcal{A}(p) - \mathcal{A}(q) &= -\int_{-\infty}^{\infty} \frac{d}{dt}(\mathcal{A}(\gamma(t)))dt \\ &= -\int_{-\infty}^{\infty} \langle \nabla \mathcal{A}(\gamma(t)), \gamma'(t)\rangle dt \\ &= \int_{-\infty}^{\infty} |\gamma'(t)|^2 dt.\end{aligned}$$

Thus a bound on the change in $\mathcal{A}$ along a gradient flowline also bounds its $W^{1,2}$ norm, hence (on any $[-T, T]$) its $C^{1/2}$ norm, and hence, by bootstrapping, each of its C^k norms.

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As a result, writing

$$\partial p = \sum_q n(p, q)q$$

as before, even though the right hand side isn't a finite sum it does belong to the *Novikov chain complex*

$$CN_*(\mathcal{A}) = \{ \sum_{q \in Crit(\mathcal{A})} a_q q | (\forall C \in \mathbb{R})(\#\{q | a_q \neq 0, \mathcal{A}(q) > C\} < \infty)\}.$$

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The deck transformation group Γ acts on $Crit(\mathcal{A})$, with

$$\mathcal{A}(gq) - \mathcal{A}(p) = \langle \theta, g \rangle.$$

Define the *Novikov ring* as

$$\Lambda_{\Gamma,\theta} = \{\sum_{g \in \Gamma} a_g g | (\forall C \in \mathbb{R})(\#\{g | a_g \neq 0, \langle \theta, g \rangle > C\} < \infty)\}.$$

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The boundary operator ∂ is a $\Lambda_{\Gamma,\theta}$-homomorphism, satisfying $\partial^2 = 0$.

Write k for the ring in which the coefficients a_g live.

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Theorem (Latour, Pazhitnov)

There is a canonical isomorphism

$$HN_*(\mathcal{A}) \cong H_*(C_*(\tilde{M}) \otimes_{k[\Gamma]} \Lambda_{\Gamma,\theta}).$$

Just as before, the subcomplex

$$CN_*^{\lambda}(\mathcal{A}) = \{\sum a_p p | a_p \neq 0 \Rightarrow \mathcal{A}(p) \leq \lambda\}$$

is preserved by ∂, giving groups

$$HN_*^{\lambda}(\mathcal{A})$$

(which, unlike $HN_*(\mathcal{A})$, depend on $\mathcal{A}$) and spectral numbers

$$\rho_{\mathcal{A}}(a) = \inf\{\lambda | a \in Im(HN_*^{\lambda}(\mathcal{A}) \to HN_*(\mathcal{A}))\}.$$

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Since the critical values of $\mathcal{A}$ will (at least for most $[\theta] \in H^1(M; \mathbb{R})$) form a dense subset of $\mathbb{R}$, the properties of ρ_f are much subtler than in the Morse homology case.

Theorem (U.)

Let $\mathcal{A}$ be as earlier, and $a \in HN_*(\mathcal{A})$. Then

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In particular, for $a \neq 0$ $\rho_{\mathcal{A}}(a)$ is a critical value of $\mathcal{A}$.
Significantly, the proof depends only on some formal algebraic properties of the Novikov complex, and as such it applies equally well to various Floer homology theories, where M is replaced by a Banach manifold and the "gradient flow" is ill-defined.

Hamiltonian flows

Let (M, ω) be a closed symplectic manifold, and

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$$d(H(t, \cdot)) = \iota_{X_H}\omega.$$

Integrating this vector field gives a *Hamiltonian isotopy*

$$\phi_H^t : M \to M \quad (t \in \mathbb{R}),$$

with each ϕ_H^t a symplectomorphism.

Let $\mathcal{L}_0 M$ be the space of contractible loops in M. Define a 1-form $\mathfrak{a}_H \in \Omega^1(\mathcal{L}_0 M)$ by

$$(\mathfrak{a}_H)_\gamma(\xi) = \int_0^1 \big(\omega(\gamma'(t), \xi(t)) - d(H(t,\cdot))(\xi(t))\big)\, dt.$$

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The zeros of $\mathfrak{a}_H$ are precisely those loops γ such that $\gamma'(t) = X_H(t, \gamma(t))$, and thus correspond to the *contractible* 1-*periodic orbits* of the Hamiltonian flow.

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Hamiltonian Floer homology is an analogue of Novikov homology for the 1-form $\mathfrak{a}_H$ on $\mathcal{L}_0 M$.

As in Novikov homology, we pull back $\mathfrak{a}_H$ to a cover of $\mathcal{L}_0M$ so that it becomes exact; the conventional cover to use is

$$\widetilde{\mathcal{L}_0M} = \frac{\{(\gamma, w) \in \mathcal{L}_0M \times Map(D^2, M) | w|_{\partial D^2} = \gamma\}}{(\gamma, w) \sim (\gamma', w') \text{ if } \gamma = \gamma', \begin{array}{l} \int_{D^2} w^*\omega = \int_{D^2} w'^*\omega, \\ \langle c_1(M), [w' \# \bar{w}] \rangle = 0 \end{array}}.$$

A primitive for the pullback of $\mathfrak{a}_H$ is then

$$\mathcal{A}_H([\gamma, w]) = -\int_{D^2} w^*\omega - \int_0^1 H(t, \gamma(t))dt,$$

and the cover has deck transformation group

$$\Gamma = \frac{\pi_2(M)}{\ker(\langle c_1, \cdot \rangle) \cap \ker(\langle [\omega], \cdot \rangle)}.$$

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The "Morse" condition for $\mathcal{A}_H$ amounts to the requirement that the graph of $\phi_H^1 : M \to M$ be transverse to the diagonal in $M \times M$.

Formally, the "negative gradient flowlines" of $\mathcal{A}_H$ are solutions

$$u\colon\ \mathbb{R} \times S^1 \to M$$

to the perturbed Cauchy–Riemann equation

$$\frac{\partial u}{\partial s} + J(t, u(s,t))\left(\frac{\partial u}{\partial t} - X_H(t, u(s,t))\right) = 0.$$

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Theorem (Floer, Hofer-Salamon, Liu-Tian, Fukaya-Ono)

The "Novikov chain complex" for $\mathcal{A}_H$ may be defined on any symplectic manifold, and the resulting homology $HF_*(H)$ satisfies

$$HF_*(H) \cong H_*(M) \otimes \Lambda_\Gamma,$$

canonically.

Just as in Morse or Novikov homology, we have filtered groups $HF_*^{\lambda}(H)$, so we define

$$\rho_H \colon H_*(M) \otimes \Lambda \to \mathbb{R} \cup \{-\infty\}$$

by

$$\rho_H(a) = \inf\{\lambda | a \in Im(HF_*^{\lambda}(H) \to HF_*(H) \cong H_*(M) \otimes \Lambda)\}.$$

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Theorem (Oh, Schwarz)

1. *If* $a\neq 0$ *then* $\rho_H(a)>-\infty$
2. $\rho_H(a)$ *is a continuous function of* H*, with respect to the norm*

$$\|H\|=\int_0^1(\max_x H(x,t)-\min_x H(x,t))dt.$$

Theorem (U. in general, Oh in strongly semipositive case)

There is $c = \sum c_{[\gamma,w]}[\gamma, w] \in CF_*(H)$ *such that* $[c] = a$ *and*

$$\rho_H(a) = \max_{c_{[\gamma,w]} \neq 0} \mathcal{A}_H([\gamma, w]).$$

In particular $\rho_H(a)$ *belongs to the (countable) set of critical values of* $\mathcal{A}_H$.

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One reason why this is useful is that it implies that if $\{H_s\}_{s\in[0,1]}$ is a path of Hamiltonians whose sets of critical values are the same (for instance, if each $\int H_s(t, \cdot)\omega^n = 0$ and the time-1 maps $\phi^1_{H_s}$ are equal), then $\rho_{H_s}(a)$ is constant in s.

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Oh showed that this allows one to use ρ to define a "spectral norm" on $\widetilde{Ham}(M,\omega)$ which is related to Hofer's norm.

An application

Work of Entov-Polterovich shows that (given the spectrality property), the function

$$\begin{aligned}\zeta\colon\ C(M) &\to \mathbb{R} \\ F &\mapsto \lim_{k\to\infty}\frac{\rho_{kF}([M])}{k}\end{aligned}$$

defines a "partial symplectic quasi-state" on M. (In particular, $\zeta(\lambda F) = \lambda\zeta(F)$ and if $\{F, G\} = 0$ with G supported in a Hamiltonianly displaceable subset, then $\zeta(F + G) = \zeta(F)$).

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Theorem (Entov-Polterovich+ϵ)

Let (M, ω) be any closed symplectic manifold, and let $F_1, \ldots, F_m : M \to \mathbb{R}$ be such that $\{F_i, F_j\} = 0$. Define $F = (F_1, \ldots, F_m) : M \to \mathbb{R}^m$. Then, for some $\vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}^d$, the set $F^{-1}(\vec{x})$ cannot be disjoined from itself by any Hamiltonian isotopy.

The set of critical points is acted on by a finitely generated abelian deck transformation group Γ, and there is a "period homomorphism" $\eta\colon \Gamma \to \mathbb{R}$ such that, for critical points p,

$$\mathcal{A}_H(gp) = \mathcal{A}_H(p) + \eta(g).$$

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Where

$$G = Im\,\eta,$$

the Novikov ring over which the chain complex is defined can be identified with

$$\Lambda(k[\ker\eta]; G) = \{\sum_{i=1}^{\infty} a_i T^{g_i} | a_i \in k[\ker\eta], g_i \in G, g_i \to -\infty\}.$$

This carries a natural valuation

$$\nu : \Lambda(k[\ker\eta]; G) \to G \quad \sum_i a_i T^{g_i} \mapsto \max_{i: a_i \neq 0} g_i.$$

Suppose that there are n contractible periodic orbits γ_i; then

$$CF_*(H) \cong (\Lambda(k[\ker \eta]; G))^n. \quad (1)$$

For simplicity assume that each of these has a lift $[\gamma_i, w_i]$ with $\mathcal{A}_H([\gamma_i, w_i]) = 0$. Then, with respect to (1),

$$(x_1, \ldots, x_n) \in CF_*^\lambda(H) \Leftrightarrow \bar{\nu}(\vec{x}) := \max_i \nu(x_i) \leq \lambda.$$

Hence

$$\rho_H([\vec{x}]) = \inf\{\bar{\nu}(\vec{x} - \partial\vec{w}) | \vec{w} \in CF_*(H)\}.$$

The theorem then amounts to the statement that this infimum is attained.

In fact:

Theorem (U.)

If R is Noetherian, $U \leq \Lambda(R;G)^n$ is a submodule, and $\vec{x} \in (\Lambda(R;G))^n$, then there is $\vec{u} \in U$ such that

$$\bar{\nu}(\vec{x}-\vec{u}) = \inf\{\bar{\nu}(\vec{x}-\vec{w}) | \vec{w} \in U\}.$$

(In our case, $R = k[\ker \eta]$ is Noetherian by the Hilbert basis theorem, and for the submodule U we use the image of the Floer boundary operator.)

More geometrically, on $\Lambda(R;G)^n$ we have a non-Archimedean metric given by $d(\vec{v},\vec{w}) = e^{\bar{\nu}(\vec{v}-\vec{w})}$, and the theorem is that given a submodule U and a point $\vec{x} \in \Lambda(R;G)^n$, $\vec{x}$ has a best approximation in U with respect to this metric.

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The proof proceeds as follows: pick a finite set $\vec{u}_1, \ldots, \vec{u}_m \in U$ with $\bar{\nu}(\vec{u}_i) = 0$ such that, where $\tilde{U} \leq R^n$ consists of the "leading-order terms" of elements of U, $\tilde{U}$ is spanned by the leading-order terms of the $\vec{u}_i$ (these exist since R is Noetherian).

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Starting with the given element $\vec{x}$, one then inductively produces a sequence $\vec{x}^{(n)} = \vec{x} - \sum_{i=1}^{m} \lambda_i^{(n)} \vec{u}_i$, with $\bar{\nu}(\vec{x}^{(n)})$ strictly decreasing and always contained in a particular *discrete* subset of $G \leq \mathbb{R}$.

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Either the process terminates at some finite n, in which case it does so at a minimizer; or else the $\bar{\nu}(\vec{x}^{(n)})$ diverge to $-\infty$, in which case we get

$$\vec{x} = \sum_{i=1}^{m} \lambda_i^{\infty} \vec{u}_i \in U,$$

(so $\vec{x}$ is itself a minimizer).