Lefschetz Fibrations and nodal pseudoholomorphic curves

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Let (X, ω) be a symplectic 4-manifold, and j an ω-tame almost complex structure on X ($j^2 = -1\,, \omega(\cdot, J\cdot) > 0$).

Definition 1. *$u\colon \Sigma_g \to X$ is a j-holomorphic map if $\bar\partial_j u := u_* \circ i - j \circ u_* = 0$. $C = Im\, u$ is then a j-holomorphic curve of genus g.*

Facts: 1) For $\alpha \in H^2(X;\mathbb{Z})$ the expected complex dimension of the space $\mathcal{M}_g^j(\alpha)$ of unparametrized j-holomorphic curves $C = Im\, u$ of genus g with $PD(u_*[\Sigma_g]) = \alpha$ is

$$d(g, \alpha) = g - 1 - \kappa_X \cdot \alpha$$

($\kappa_X = c_1(T^*X)$).

2) For generic j, $\mathcal{M}_g^j(\alpha)$ is a smooth manifold of the expected dimension near each $C = Im\, u$ for which u is somewhere (hence generically) injective.

3) (Gromov compactness) If $j_n \to j$, any sequence $C_n \in \mathcal{M}_g^{j_n}(\alpha)$ has a subsequence which either bubbles off a sphere or converges to some $C \in \mathcal{M}_g^j(\alpha)$.

If $\mathcal{M}_g^j(\alpha)$ were generically a compact manifold of (real) dimension $2d(g,\alpha)$, and if for generic paths j_t $\{(t,C)|C \in \mathcal{M}_g^{j_t}(\alpha)\}$ were a compact manifold of dimension $2d(g,\alpha)+1$, we could define invariants by picking a generic set Ω of $d(g,\alpha)$ points (to reduce the dimension of the moduli space to zero) and counting those $C \in \mathcal{M}_g^j(\alpha)$ which pass through Ω with appropriate signs.

$\mathcal{M}_g^j(\alpha)$ isn't a compact manifold, due to:

- bubbling, but in dimension 4 the dimensions of the moduli spaces are such that this doesn't create problems; and
- the fact that a sequence of generically injective maps might converge to a multiple cover.

Ruan–Tian: Define $RT_g(\alpha)$ by counting solutions to $\bar{\partial}_j u(x) = \nu(x, u(x))$ for a generic "inhomogeneous term" ν. Owing to the multiple cover issue and the way ν is defined, the Ruan–Tian invariants take values in $\mathbb{Q}$.

If we restrict to *embedded* curves, so that g is given by the adjunction formula

$$2g_\alpha - 2 = \alpha^2 + \kappa_X \cdot \alpha,$$

the dimensions $d(g_\alpha, \alpha)$ are such that the only possible noncompactness issue arises from embedded square-zero tori converging to multiple covers of other square-zero tori.

Taubes found a way of assigning integer weights with which multiple covers of square-zero tori should contribute to an invariant $Gr(\alpha)$ which enumerates non-self-intersecting j-holomorphic curves (possibly disconnected with some of its square-zero toroidal components having multiplicity > 1) passing through $\frac{1}{2}(\alpha^2 - \kappa_X \cdot \alpha)$ points. $Gr(\alpha)$ is integer-valued and independent of the almost complex structure j used to define it; it also agrees with the Seiberg–Witten invariant for a corresponding $spin^c$ structure.

Theorem 1. *(Ionel-Parker) There is a universal formula for $Gr(\alpha)$ in terms of the invariants $RT_g(\beta)$.*

Question 1. *Given $n > 0$, is there a similar integer-valued invariant "$Gr_n(\alpha)$" counting possibly-reducible curves PD to α having n self-intersections, passing through $\frac{1}{2}(\alpha^2 - \kappa_X \cdot \alpha) - n$ points?*

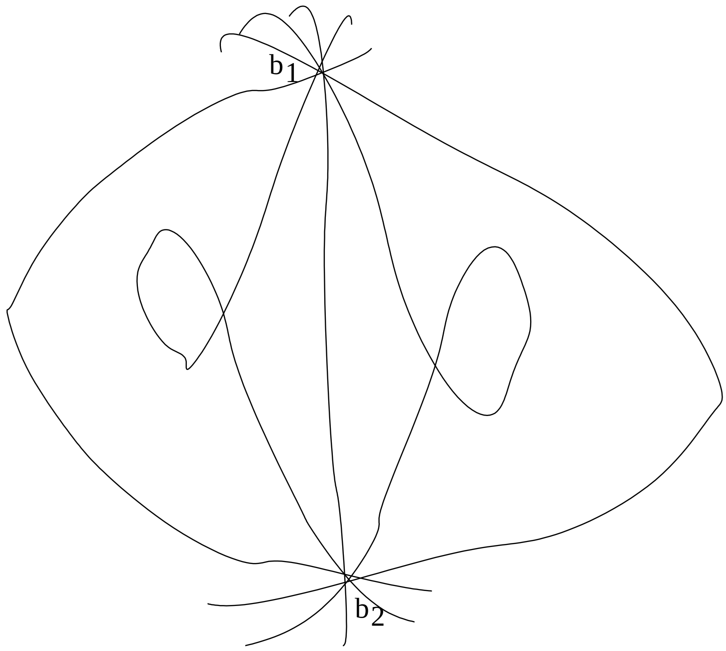
Definition 2. *A* Lefschetz pencil *on* X^4 *is a map* $f\colon X \setminus \{b_1, \dots, b_N\} \to S^2$ *such that*

(i) Near each of the base points b_i*,* f *is modelled in orientation-preserving complex coordinates by* $f(z, w) = \frac{z}{w}$*, and*

(ii) f *has just finitely many critical points, near each of which it is modelled by* $f(z, w) = zw$*.*

Letting $X' \to X$ be the blowup of X along the finite base locus $B = \{b_1, \dots, b_N\}$, f lifts to a Lefschetz *fibration* $f\colon X' \to S^2$, with the exceptional divisors of the blowup appearing as sections.

Theorem 2. *(Donaldson) Any symplectic manifold* (X, ω) *admits Lefschetz pencils, the closures of whose smooth fibers are symplectic submanifolds of* X *of arbitrarily large symplectic area.*



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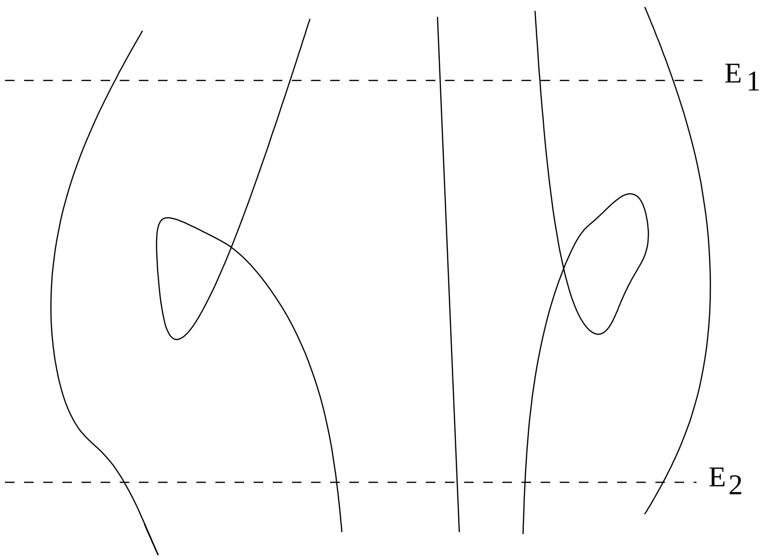

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Given a Lefschetz fibration $f\colon X \to S^2$, we can find almost complex structures j with respect to which f is a pseudoholomorphic map; a curve C contributing to $Gr(\alpha)$ will then intersect each fiber $r = \langle \alpha, [fiber] \rangle$ times, counted with (positive) multiplicities:

$$C \cap f^{-1}(t) \in Sym^r f^{-1}(t) \ \forall t$$

Donaldson–Smith: Form a "relative Hilbert scheme" $F\colon X_r(f) \to S^2$ such that $F^{-1}(t) = Sym^r f^{-1}(t)$ when t is a regular value of f. A curve C contributing to $Gr(\alpha)$ would then correspond to a *section* s_C of $X_r(f)$. Conversely, a section s gives rise to a surface $C_s \subset X$.
As long as f has no fibers with more than one critical point, $X_r(f)$ is a smooth manifold which admits a symplectic structure, and so we can define an invariant $DS_f(\alpha)$ as a count of J-holomorphic sections s of $X_r(f)$ such that $[C_s] = PD(\alpha)$, for J a generic almost complex structure on $X_r(f)$.

Theorem 3. *(U.) Provided that the Lefschetz fibration $f\colon X \to S^2$ has fibers of large enough symplectic area,*

$$DS_f(\alpha) = Gr(\alpha).$$

Consequences: 1) DS_f is independent of f provided that f has large enough fibers.
2) (Using results of Donaldson and Smith about DS) If $b^+(X) > b_1(X) + 1$ then $Gr(\kappa_X) = \pm 1$ and $Gr(\alpha) = \pm Gr(\kappa_X - \alpha)$. This was previously known, but only as a consequence of Taubes-Seiberg-Witten theory.

Idea of proof: If j is such that $f\colon X \to S^2$ is pseudoholomorphic, there is an almost complex structure J_j on $X_r(f)$ such that

$$C \subset X \text{ is a } j\text{-holomorphic curve } \Leftrightarrow \bar{\partial}_{J_j} s_C = 0.$$

However J_j is only Hölder-continuous near sections s_{mult} corresponding to curves C_{mult} having multiply-covered torus components; J_j therefore can't be used directly to evaluate DS. We perturb J_j to a nearby generic smooth J (which can be used to evaluate DS); owing to Gromov compactness and the dimension formula there will be finitely many J-holomorphic sections close to s_{mult}. One then shows the weight with which C_{mult} contributes to the Gromov invariant in Taubes' definition agrees with the signed count of the J-holomorphic sections near s_{mult}. □

Note that as far as DS is concerned the individual sections near s_{mult} aren't distinguished in any way from other J-holomorphic sections; in this way, DS gives a way of counting holomorphic-curve-like objects without needing to treat multiple covers differently, much like the Ruan-Tian approach of counting solutions to an inhomogeneous PDE. But unlike the Ruan-Tian invariants, DS is manifestly integer-valued.

Counting nodal curves

Suppose $C \subset X$ is a j-holomorphic curve PD to α with a single transverse self-intersection, say at q. Consider

$$X_1 = Bl_\Delta(X \times X);$$

this fibers over X with fiber over p equal to the blowup $Bl_p(X)$ of X at p.

Proposition 1. *For each p, there is a unique Lipschitz almost complex structure j_p on $Bl_p(X) \to X$ such that the blowdown $\pi_p\colon\ Bl_p(X)$ is (j_p, j)-holomorphic.*

The proper transform of C under π_p will then be an embedded j_p-holomorphic curve in class $\pi_p^*\alpha - 2e$ exactly when $p = q$.

Iterating this construction, we get families $X_n \to X_{n-1}$ whose fiber X^b over $b \in X_{n-1}$ is X blown up n times, with b parametrizing the sites of the blowups.

This suggests that instead of looking for n-nodal curves representing $PD(\alpha) \in H^2(X;\mathbb{Z})$, we could look for embedded curves representing Poincaré dual to

$$\pi_b^*\alpha - 2\sum_{i=1}^{n} e_i \in H^2(X^b;\mathbb{Z})$$

as b ranges over X_{n-1}. If we try to do this directly, though, we run into problems with both multiple covers and bubbling.

Let $f\colon X \to S^2$ be a Lefschetz fibration. For $p \in X$, $f \circ \pi_p$ is then also a Lefschetz fibration as long as p isn't a critical point of f. Iterating, where $X'_{n-1} \subset X_{n-1}$ is the complement of a codimension-four set, for $b \in X'_{n-1}$

$$f^b = f \circ \pi_b\colon X^b \to S^2$$

is a Lefschetz fibration, and so we can form the relative Hilbert scheme $X_r^b(f^b)$.

Write

$$\mathcal{X}_r^n(f) = \{(b, D)|D \in X_r^b(f^b)\}.$$

Proposition 2. *$\mathcal{X}_r^n(f)$ is a smooth manifold and admits symplectic structures.*

Theorem 4. *For generic J within an appropriate class of complex structures on $\mathcal{X}_r^n(f)$, the space of J-holomorphic sections of $X_r^b(f^b)$ corresponding to surfaces in X^b PD to $\pi_b^*\alpha - 2\sum_{i=1}^n e_i$ as b ranges over X_n' is compact.*

This allows us to define an integer-valued invariant $FDS_f^n(\alpha)$ counting such sections.

Define the class α to be *simple* if whenever $\alpha = \alpha_1 + \cdots + \alpha_k$ and each α_i is PD to a symplectic immersion, α_i is primitive in $H^2(X;\mathbb{Z})$. In such a case, there are no concerns with multiple covers and one may easily define an invariant $Gr_n(\alpha)$ counting possible reducible curves PD to α with n self-intersections.

Theorem 5. *If* α *is simple then* $FDS^n_f(\alpha) = Gr_n(\alpha)$ *provided that the fibers of* f *have large enough symplectic area.*

Conjecture 1. FDS^n_f *is independent of* f *and has a universal expression in terms of the Ruan–Tian invariants of* X.

Theorem 6. *If* $b^+(X) > b_1(X) + 1 + 4n$ *then either* $FDS^n_f(\alpha) = 0$ *or there exists an almost complex structure* j *on* X *admitting (possibly reducible, non-reduced)* j*-holomorphic curves Poincaré dual to* α *and to* $\kappa_X - \alpha$.

For α such that $FDS^n_f(\alpha)$ does not immediately vanish for dimensional reasons, one can show that the second alternative in Theorem 6 can't hold for *generic* j, but it can't be ruled out for all j.