STANDARD SURFACES AND NODAL CURVES IN SYMPLECTIC 4-MANIFOLDS

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Abstract. Continuing the program of [DS] and [U1], we introduce refinements of the Donaldson-Smith standard surface count which are designed to count nodal pseudoholomorphic curves and curves with a prescribed decomposition into reducible components. In cases where a corresponding analogue of the Gromov-Taubes invariant is easy to define, our invariants agree with those analogues. We also prove a vanishing result for some of the invariants that count nodal curves.

1. Introduction

Let (X, ω) be a closed symplectic 4-manifold. We assume that $[\omega] \in H^2(X, \mathbb{Z})$; however, the main theorems in this paper concern Gromov invariants, which are unchanged under deformations of the symplectic form, so since any symplectic form is deformation equivalent to an integral form there is no real loss of generality here. According to [Do], if k is large enough, taking a suitable pair of sections of a line bundle $L^{\otimes k}$ where L has Chern class $[\omega]$ and blowing X up at the common vanishing locus of these sections to obtain the new manifold X' gives rise to a symplectic Lefschetz fibration $f\colon X' \to \mathbb{C}P^1$ (the exceptional curves of the blowup $\pi\colon X' \to X$ appear as sections of f, while at other points $x' \in X'$, $f(x') \in \mathbb{C} \cup \{\infty\}$ is the ratio of the two chosen sections of $L^{\otimes k}$ at $\pi(x') \in X$). In other words, f is a fibration by Riemann surfaces over the complement of a finite set of critical values in S^2, while near its critical points f is given in smooth local complex coordinates by $f(z, w) = zw$. Results of [Sm1] show that the critical points of f may be assumed to lie in separate fibers, and all fibers of f may be assumed irreducible. Once we choose a metric on X', Donaldson's construction thus presents a suitable blowup of X as a smoothly $\mathbb{C}P^1$-parametrized family of Riemann surfaces, all but finitely of which are smooth and all of which are irreducible with at worst one ordinary double point. Where $\kappa_X = c_1(T^*X)$ is the canonical class of X, note that the adjunction formula gives the arithmetic genus of the fibers as $g = 1 + (k^2[\omega]^2 + k\kappa_X \cdot \omega)/2$.

Beginning with the work of S. Donaldson and I. Smith in [DS], some efforts have recently been made toward determining whether such a Lefschetz fibration can shed light on any questions concerning pseudoholomorphic curves in X. More specifically, for any natural number r Donaldson and Smith construct the *relative Hilbert scheme*, which is a smooth symplectic manifold $X_r(f)$ with a map $F\colon X_r(f) \to \mathbb{C}P^1$ whose fiber over a regular value t of f is the symmetric product $S^r f^{-1}(t)$. If we choose an almost complex structure j on X' with respect to which f is a pseudoholomorphic map, a j-holomorphic curve C in X' which contains no fiber components will, by the positivity of intersections between j-holomorphic curves, meet each fiber in $r := [C] \cdot [fiber]$ points, counted with multiplicities. In other words, $C \cap f^{-1}(t) \in S^r f^{-1}(t)$, so that, letting t vary, C gives rise to a section s_C of

$X_r(f)$. Conversely, a section s of $X_r(f)$ gives rise to a subset C_s of X' (namely the union of all the points appearing in the divisors $s(t)$ as t varies), and from j one may construct a (nongeneric and generally not even C^1) almost complex structure $\mathbb{J}_j$ with the property that C is a (possibly disconnected) j-holomorphic curve in X' if and only if s_C is a $\mathbb{J}_j$-holomorphic section of $X_r(f)$.

Accordingly, it seems reasonable to study pseudoholomorphic curves in X' by studying pseudoholomorphic sections of $X_r(f)$. If $\alpha \in H^2(X';\mathbb{Z})$, the *standard surface count* $\mathcal{DS}_f(\alpha)$ is defined in [Sm2] (and earlier in [DS] for $\alpha = \kappa_{X'}$) as the Gromov-Witten invariant which counts J-holomorphic sections s whose corresponding sets C_s are Poincaré dual to the class α and pass through a generic set of $d(\alpha) = \frac{1}{2}(\alpha^2 - \kappa_{X'} \cdot \alpha)$ points of X', where J is a generic almost complex structure on $X_r(f)$. Smith shows in [Sm2] that there is at most one homotopy class c_α of sections s such that C_s is Poincaré dual to α, and moreover that the complex dimension of the space of J-holomorphic sections in this homotopy class is, for generic J, the aforementioned $d(\alpha)$, which the reader may recognize as the same as the expected dimension of j-holomorphic submanifolds of X Poincaré dual to α. Further, the moduli space of J-holomorphic sections in the homotopy class c_α is compact for generic J if k is taken large enough. The moduli space in the definition of $\mathcal{DS}_f$ is therefore a finite set, and $\mathcal{DS}_f$ simply counts the members of this set with sign according to the usual (spectral-flow-based) prescription.

Donaldson and Smith have proven various results about $\mathcal{DS}$, perhaps the most notable of which is the main theorem of [Sm2], which asserts that if $\alpha \in H^2(X;\mathbb{Z})$, if $b^+(X) > b_1(X) + 1$, and if the degree k of the Lefschetz fibration is high enough, then

$$\mathcal{DS}_f(\pi^*\alpha) = \pm\mathcal{DS}_f(\pi^*(\kappa_X - \alpha)). \tag{1.1}$$

Their work has led to new, more symplectic proofs of various results in 4-dimensional symplectic topology which had previously been accessible only by Seiberg-Witten theory (as an example we mention the main theorem of [DS], according to which X admits a symplectic surface Poincaré dual to κ_X, again assuming $b^+(X) > b_1(X) + 1$). In [U1] it was shown that the invariant $\mathcal{DS}_f$ agrees with the Gromov invariant Gr which was introduced by C. Taubes in [T2] and which counts possibly-disconnected pseudoholomorphic submanifolds of X' Poincaré dual to a given cohomology class. This in particular shows that $\mathcal{DS}_f$ is independent of the choice of Lefschetz fibration structure, and, in combination with Smith's duality theorem (1.1) and the fact that under a blowup π one has $Gr(\pi^*\alpha) = Gr(\alpha)$, yields a new proof of the relation

$$Gr(\alpha) = \pm Gr(\kappa_X - \alpha)$$

if $b^+(X) > b_1(X) + 1$, a result which had previously only been known as a shadow of the charge conjugation symmetry in Seiberg-Witten theory.

The information contained in the Gromov invariants comprises only a part of the data that might be extracted from pseudoholomorphic curves in X. The present paper aims to show that many of these additional data can also be captured by Donaldson-Smith-type invariants. For instance, $Gr(\alpha)$ counts all of the curves with any decomposition into connected components whose homology classes add up (counted with multiplicities) to α. It is natural to wish to keep track of the decompositions of our curves into reducible components; accordingly we make the following:

Definition 1.1. *Let $\alpha \in H^2(X;\mathbb{Z})$. Let*

$$\alpha = \beta_1 + \cdots + \beta_m + c_1\tau_1 + \cdots + c_n\tau_n$$

be a decomposition of α into distinct summands, where none of the β_i satisfies $\beta_i^2 = \kappa_X \cdot \beta_i = 0$, while the τ_i are distinct classes which are primitive in the lattice $H^2(X;\mathbb{Z})$ and all satisfy $\tau_i^2 = \kappa_X \cdot \tau_i = 0$. Then

$$Gr(\alpha; \beta_1, \ldots, \beta_m, c_1\tau_1, \cdots, c_n\tau_n)$$

is the invariant counting ordered $(m+n)$-tuples $(C_1, \ldots, C_{m+n})$ of transversely intersecting smooth pseudoholomorphic curves in X, where

(i) *for $1 \leq i \leq m$, C_i is a connected curve Poincaré dual to β_i which passes through some prescribed generic set of $d(\beta_i)$ points;*

(ii) *for $m+1 \leq k \leq m+n$, C_k is a union of connected curves Poincaré dual to classes $l_{k,1}\tau_k, \cdots, l_{k,p}\tau_k$ decorated with positive integer multiplicities $m_{k,q}$ with the property that*

$$\sum_q m_{k,q} l_{k,q} = c_k.$$

The weight of each component of each such curve is to be determined according to the prescription given in the definition of the Gromov invariant in [T2] *(in particular, the components $C_{k,q}$ in class $l_{k,q}\tau_k$ are given the weight $r(C_{k,q}, m_{k,q})$ specified in Section 3 of* [T2]*), and the contribution of the entire curve is the product of the weights of its components.*

The objects counted by $Gr(\alpha; \alpha_1, \ldots, \alpha_n)$ will then be reducible curves with smooth irreducible components and a total of $\sum \alpha_i \cdot \alpha_j$ nodes arising from intersections between these components. $Gr(\alpha)$ is the sum over all decompositions of α into classes which are pairwise orthogonal under the cup product of the

$$\frac{d(\alpha)!}{\prod(d(\alpha_i)!)} Gr(\alpha; \alpha_1, \ldots, \alpha_n);$$

in turn, one has

$$Gr(\alpha; \alpha_1, \ldots, \alpha_n) = \prod_{i=1}^{n} Gr(\alpha_i; \alpha_i).$$

In Section 2, given a symplectic Lefschetz fibration $f\colon X \to S^2$ with sufficiently large fibers, by counting sections of a relative Hilbert scheme we construct a corresponding invariant $\widetilde{\mathcal{DS}}_f(\alpha; \alpha_1, \ldots, \alpha_n)$ provided that none of the α_i can be written as $m\beta$ where $m > 1$ and β is Poincaré dual to either a symplectic square-zero torus or a symplectic (-1)-sphere. Further:

Theorem 1.2. $\frac{(\sum d(\alpha_i))!}{\prod(d(\alpha_i)!)} Gr(\alpha; \alpha_1, \ldots, \alpha_n) = \widetilde{\mathcal{DS}}_f(\alpha; \alpha_1, \ldots, \alpha_n)$ *provided that the degree of the fibration is large enough that $\langle [\omega_X], fiber\rangle > [\omega_X] \cdot \alpha$.*

The sections s counted by $\widetilde{\mathcal{DS}}_f(\alpha; \alpha_1, \ldots, \alpha_n)$ correspond tautologically to curves $C_s = \cup C_s^i$ in X with each C_s^i Poincaré dual to α_i. The C_s^i will be symplectic, and Proposition 2.5 guarantees that they will intersect each other positively, so there will exist an almost complex structure making C_s holomorphic. However, if s_1 and s_2 are two different sections in the moduli space enumerated by $\widetilde{\mathcal{DS}}_f(\alpha; \alpha_1, \ldots, \alpha_n)$, it is unclear whether there will exist a single almost complex structure on X making both C_{s_1} and C_{s_2} holomorphic.

The almost complex structures on $X_r(f)$ used in the definition of $\widetilde{\mathcal{DS}}$ are, quite crucially, required to preserve the tangent space to the *diagonal stratum* consisting of divisors with one or more points repeated. One might hope to define analogous invariants which agree with $Gr(\alpha; \alpha_1, \dots, \alpha_n)$ using arbitrary almost complex structures on $X_r(f)$. If one could do this, though, the arguments reviewed in Section 4 would rather quickly enable one to conclude that $Gr(\alpha; \alpha_1, \dots, \alpha_n) = 0$ whenever α has larger pairing with the symplectic form than does the canonical class and $\alpha_i \cdot \alpha_j = 0$ for $i \neq j$. However, this is not the case: the manifold considered in [MT] admits a symplectic form such that, for certain primitive, orthogonal, square-zero classes α, β, γ, and δ each with positive symplectic area, the canonical class is $2(\alpha+\beta+\gamma)$ but the invariant $Gr(2(\alpha+\beta+\gamma)+\delta; \alpha, \beta, \gamma, \alpha+\beta+\gamma+\delta)$ is nonzero.

While the Gromov–Taubes invariant restricts attention to curves whose components are all covers of embedded curves which do not intersect each other, it is natural to hope for information about curves Poincaré dual to α having some number n of transverse self-intersections. One might like to define an analogue $Gr_n(\alpha)$ of the Gromov–Taubes invariant counting such curves, but as we review in Section 3, owing to issues relating to multiple covers it is somewhat unclear what the definition of such an invariant should be in general. If one imposes some rather stringent conditions on α (α should be "n-semisimple" in the sense of Definition 3.1), there is however a natural such choice.

Note that for arbitrary α and n, following [RT] one may define an invariant $RT_n(\alpha)$ which might naively be viewed as a count of *connected* pseudoholomorphic curves Poincaré dual to α with n self-intersections by enumerating solutions $u\colon \Sigma_g \to X$ of the equation $(\bar{\partial}_j\, u) = \nu(x, u(x))$ for generic j and "inhomogeneous term" ν, where the genus g of the source curve is given in accordance with the adjunction formula by $2g - 2 = \alpha^2 + \kappa_X \cdot \alpha - 2n$. (Note that the nontrivial dependence of ν on x prevents multiple cover problems from arising.) In the case $n = 0$, the main theorem of [IP1] provides a universal formula equating $Gr(\alpha)$ with a certain combination of the Ruan–Tian invariants RT_0. The proof of that theorem goes through easily to show that in the case when α is n-semisimple, there exists a similar formula equating $Gr_n(\alpha)$ with a combination of Ruan–Tian invariants. We mention also that, again as an artifact of the multiple cover problem, the Ruan–Tian invariants are obliged to take values in $\mathbb{Q}$ rather than $\mathbb{Z}$. $Gr(\alpha)$, on the other hand, is an integer-valued invariant.

By combining the approaches of [DS] and [L1], in the presence of a Lefschetz fibration $f\colon\ X \to S^2$ we construct in Section 3 an integer-valued invariant $\mathcal{FDS}^n_f(\alpha - 2\sum e_i)$ which we conjecture to be an appropriate candidate for a "nodal version" $Gr_n(\alpha)$ of the Gromov invariant for general classes α (after perhaps dividing by $n!$ to account for a symmetry in the construction). Pleasingly, the technical difficulties that often arise in defining invariants like $Gr_n(\alpha)$ do not affect $\mathcal{FDS}$: since $\mathcal{FDS}$ counts sections of a (singular) fibration, which of course necessarily represent a primitive homology class in the total space, we need not worry about multiple covers; further, the fact that any bubbles that form in the limit of a sequence of holomorphic sections must be contained in the fibers of the fibration turns out (via an easy elaboration of a dimension computation from [DS]) to generically rule out bubbling as well. In principle, though, $\mathcal{FDS}^n_f$ might depend on the choice of Lefschetz fibration f.

Note that if $\pi\colon X' \to X$ is a blowup with exceptional divisor Poincaré dual to ϵ, whenever $Gr_n(\beta)$ is defined we will have $(Gr_n)_{X'}(\beta+\epsilon) = (Gr_n)_X(\beta)$ (here and elsewhere we use the same notation for $\beta \in H^2(X;\mathbb{Z})$ and $\pi^*\beta \in H^2(X';\mathbb{Z})$), as the curves contributing to $(Gr_n)_X(\beta)$ generically miss the point being blown up, and so the unions of their proper transforms with the exceptional divisor will be precisely the curves contributing to $(Gr_n)_{X'}(\beta+\epsilon)$. With this said, we formulate the:

Conjecture 1.3. *Let (X,ω) be a symplectic 4-manifold and $\alpha \in H^2(X;\mathbb{Z})$, and $f\colon X' \to S^2$ a Lefschetz fibration obtained from a sufficiently high-degree Lefschetz pencil on X, with the exceptional divisors of the blowup $X' \to X$ Poincaré dual to the classes $\epsilon_1, \dots, \epsilon_N$. Then the family Donaldson–Smith invariants*

$$\mathcal{FDS}^n_f\left(\alpha + \sum_{i=1}^N \epsilon_i - 2\sum_{k=1}^n e_k\right)$$

are independent of the choice of f, and have a general expression in terms of the Ruan–Tian invariants of X.

Note that this conjectural general expression would then produce an integer by taking appropriate combinations of the (*a priori* only rational) Ruan–Tian invariants, similarly to the formula of [IP1]

In light of the behavior of Gr_n under blowups, Theorem 3.8 amounts to the statement that:

Theorem 1.4. *If α is strongly n-semisimple, then Conjecture 1.3 holds for α; more specifically, we have*

$$\mathcal{FDS}^n_f\left(\alpha + \sum_{i=1}^N \epsilon_i - 2\sum_{k=1}^n e_k\right) = n! Gr_n(\alpha)$$

if f has sufficiently high degree.

We also prove that $\mathcal{FDS}$ vanishes under certain circumstances. This result depends heavily on the constructions used by Smith in [Sm2] to prove his duality theorem, and so we review these constructions in Section 4. Section 5 is then devoted to a proof of the following theorem.

Theorem 1.5. *If $b^+(X) > b_1(X) + 4n + 1$, then for all $\alpha \in H^2(X;\mathbb{Z})$ such that $r = \langle \alpha, [\Phi]\rangle$ satisfies $r > \max\{g(\Phi) + 3n + d(\alpha), (4g(\Phi) - 11)/3\}$, either $\mathcal{FDS}^n_f(\alpha - 2\sum e_i) = 0$ or there exists an almost complex structure j on X compatible with the fibration $f\colon X \to S^2$ which simultaneously admits holomorphic curves C and D Poincaré dual to the classes α and $\kappa_X - \alpha$. In particular, $\mathcal{FDS}^n_f(\alpha - 2\sum e_i) = 0$ if α has larger pairing with the symplectic form than does κ_X.*

Note that in the Lefschetz fibrations obtained from degree-k Lefschetz pencils on some fixed symplectic manifold (X,ω), the number N of exceptional sections is $k^2[\omega]^2$ while the number $2g(\Phi) - 2$ is asymptotic to $k^2[\omega]^2$, so the invariants

$$\mathcal{FDS}^n_f\left(\alpha + \sum_{i=1}^N \epsilon_i - 2\sum_k e_k\right)$$

considered in Conjecture 1.3 all eventually satisfy the restriction on r in Theorem 1.5.

The almost complex structure in the second alternative in Theorem 1.5 cannot be taken to be regular (in the sense that the moduli spaces $\mathcal{M}_X^j(\beta)$ of j-holomorphic curves Poincaré dual to β are of the expected dimension); the most we can say appears to be that it can be taken to be a member of a regular $4n$-real-dimensional family of almost complex structures, *i.e.*, a family of almost complex structures $\{j_b\}$ parametrized by elements b of an open set in $\mathbb{R}^{4n}$ such that the spaces $\{(b, C)|C \in \mathcal{M}_X^{j_b}(\beta)\}$ are of the expected real dimension $2d(\beta) + 4n$ near each (b, C) such that C has no multiply-covered components. Also, if X is in fact Kähler and admits a compatible *integrable* complex structure j_0 with respect to which the fibration f is holomorphic, then we can take the j in Theorem 1.5 equal to j_0.

In fact, if we could take j to be regular, then we could rule out the second alternative in Theorem 1.5 entirely (when $n > 0$) using the following argument: the invariant vanishes trivially when $d(\alpha) < n$, so we can assume $d(\alpha) = -\frac{1}{2}\alpha \cdot (\kappa - \alpha) > 0$. But then our curves Poincaré dual to α and $\kappa - \alpha$ have negative intersection number, which is only possible if they share one or more components of negative square. For generic j, a virtual dimension computation shows that the only irreducible j-holomorphic curves of negative square are (-1)-spheres. Moreover whatever (-1)-spheres appear in X must be disjoint, since if they were not, blowing one of two intersecting (-1)-spheres down would cause the image of the other to be a symplectic sphere of nonnegative self-intersection, which (by a result of [McD]) would force X to have $b^+ = 1$, which we assumed it did not. Ignoring all the (-1)-spheres in C and D and taking the union of what is left over would then give a j-holomorphic curve Poincaré dual to a class $\kappa_X - \sum a_i e_i$ where the e_i are classes of (-1)-spheres with $e_i \cdot e_k = 0$ for $i \neq k$ and where at least one $a_i \geq 2$. But one easily finds $d(\kappa_X - \sum a_i e_i) < 0$, so this too is impossible for generic j. For nongeneric j, this argument breaks down because of the possibility that C and D might share components of negative square and negative expected dimension, and there is a wider diversity of possible homology classes of such curves.

The final section of the paper contains proofs of two technical results that are used in the proofs of the main theorems. First, we show that the operation of blowing up a point can be performed in the almost complex category, a fact which does not seem to appear in the literature and whose proof is perhaps more subtle than one might anticipate. The paper then closes with a proof of the following result, which is necessary for the compactness argument that we use to justify the definition of our invariant $\mathcal{FDS}$:

Theorem 1.6. *Let $F\colon \mathcal{H}_r \to D^2$ denote the r-fold relative Hilbert scheme of the map $(z, w) \mapsto zw$, ϕ_0 the partial resolution map $F^{-1}(0) \to Sym^r\{zw = 0\}$, and $\Delta \subset \mathcal{H}_r$ the diagonal stratum. At any point $p \in \Delta \cap F^{-1}(0)$ with $\phi_0(p) = \{(0, 0), \dots, (0, 0)\}$, where $T_p\Delta$ is the tangent cone to Δ at p, we have $T_p\Delta \subset T_pF^{-1}(0)$.*

We end the introduction with some remarks on the possible relation of $\mathcal{FDS}$ to (family) Seiberg–Witten theory. In [Sa] it was shown that where X is the product of $\mathbb{R}$ and a fibered three-manifold, so that X fibers over a cylinder, if one examines the Seiberg–Witten equations on X using a family of metrics for which the size of the fibers shrinks to zero, then one obtains in the adiabatic limit the equations for a holomorphic family of solutions to the symplectic vortex equations on the fibers. In turn, there is a natural isomorphism between the space of solutions to the vortex equations on a Riemann surface and the symmetric product of the surface. In other words, in this simple context the adiabatic limit of the Seiberg–Witten equations is the equation for a holomorphic family of elements of the symmetric products of the fibers of the fibration $X \to \mathbb{R} \times S^1$. As was noted in [DS], since for a Lefschetz fibration $f\colon X \to S^2$ $\mathcal{DS}_f$ precisely counts pseudoholomorphic families of elements of the symmetric products of the fibers of f, one might take inspiration from Salamon's example and hope to obtain the equivalence between $\mathcal{DS}_f$ and the Seiberg–Witten invariant by considering the Seiberg–Witten equations on X for a family of metrics with respect to which the size of the fibers shrinks to zero.

Now our invariant $\mathcal{FDS}_f^n$ is constructed by counting pseudoholomorphic families of elements of the symmetric products of the fibers of a family of Lefschetz fibrations f^b obtained by restricting a map $f_n\colon X_{n+1} \to S^2 \times X_n$ to the preimage X^b of $S^2 \times \{b\}$ as b ranges over the complement X_n' of a set of codimension 4 in X_n. In the above vein, one might hope to relate the family Seiberg–Witten invariants FSW for the family of 4-manifolds $X_{n+1} \to X_n$ (which enumerate Seiberg–Witten monopoles in the various X^b as b ranges over X_n; see, *e.g.*, [LL]) to $\mathcal{FDS}_f^n$ via an adiabatic limit argument. This would in particular yield a proof of the independence of $\mathcal{FDS}_f^n$ from f in Conjecture 1.3, and indeed may well be the most promising way to establish this independence in the absence of a suitable invariant Gr_n (or of a "family Gromov–Taubes invariant" FGr) with which $\mathcal{FDS}_f^n$ might be directly equated.

As was shown in [L2], when X is an algebraic surface and $b^+(X) = 1$ the family Seiberg–Witten invariants agree with certain curve counts in algebraic geometry. For larger values of b^+, though, the family Seiberg–Witten invariants that are hoped to correspond to nodal curve counts are expected to vanish due to the fact that symplectic manifolds have Seiberg–Witten simple type; note that Theorem 1.5 suggests that $\mathcal{FDS}_f^n$ also tends to vanish for large b^+. By contrast, there are plenty of nontrivial nodal curve counts in algebraic surfaces with $b^+ > 1$ (see [L1] for a review of some of these); these counts correspond to Liu's "algebraic Seiberg–Witten invariants" $\mathcal{ASW}$ and differ from FSW when $b^+ > 1$.

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2. Refining the standard surface count

Throughout this section, $X_r(f)$ will denote the relative Hilbert scheme constructed from some high-degree but fixed Lefschetz fibration $f\colon X \to S^2$ obtained by Donaldson's construction applied to the fixed symplectic 4-manifold (X, ω). The fiber of f over $t \in S^2$ will occasionally be denoted by Σ_t, and the homology class of the fiber by $[\Phi]$.

As has been mentioned earlier, $\mathcal{DS}_f(\alpha)$ is a count of holomorphic sections of the relative Hilbert scheme $X_r(f)$ in a certain homotopy class c_α characterized by the property that if s is a section in the class c_α then the closed set $C_s \subset X$ "swept out" by s (that is, the union over all t of the divisors $s(t) \in \Sigma_t$) is Poincaré dual to α (note that points of C_s in this interpretation may have multiplicity greater than 1). That c_α is the unique homotopy class with this property is seen in Lemma

4.1 of [Sm2]; in particular, for instance, we note that sections which descend to *connected* standard surfaces Poincaré dual to α are not distinguished at the level of homotopy from those which descend to disjoint unions of several standard surfaces which combine to represent $PD(\alpha)$.

Of course, in studying standard surfaces it is natural to wish to know their connected component decompositions, so we will presently attempt to shed light on this. Suppose that we have a decomposition

$$\alpha = \alpha_1 + \cdots + \alpha_n$$

with

$$\langle \alpha, [\Phi] \rangle = r, \quad \langle \alpha_i, [\Phi] \rangle = r_i.$$

Over each $t \in S^2$ we have an obvious "divisor addition map"

$$+\colon \prod_{i=1}^n S^{r_i}\Sigma_t \to S^r\Sigma_t$$
$$(D_1, \dots, D_n) \mapsto D_1 + \cdots + D_n;$$

allowing t to vary we obtain from this a map on sections:

$$+\colon \prod_{i=1}^n \Gamma(X_{r_i}(f)) \to \Gamma(X_r(f))$$
$$(s_1, \dots, s_n) \mapsto \sum_{i=1}^n s_i.$$

As should be clear, one has

$$+(c_{\alpha_1} \times \cdots \times c_{\alpha_n}) \subset c_\alpha$$

if $\alpha = \sum \alpha_i$, since $C_{\sum \alpha_i}$ is the union of the standard surfaces C_{s_i} and hence is Poincaré dual to α if each C_{s_i} is Poincaré dual to α_i. Further, we readily observe:

Lemma 2.1. *The image $+(c_{\alpha_1} \times \cdots \times c_{\alpha_n}) \subset c_\alpha$ is closed with respect to the C^0 norm.*

Proof. Suppose we have a sequence $(s_1^m, \dots, s_n^m)_{m=1}^\infty$ in $c_{\alpha_1} \times \cdots \times c_{\alpha_n}$ such that $\sum s_i^m \to s \in c_\alpha$. Now each $S^{r_i}\Sigma_t$ is compact, so at each t, each of the sequences $s_i^m(t)$ must have subsequences converging to some $s_i^0(t)$. But then necessarily each $\sum s_i^0(t) = s(t)$, and then we can see by, for any l, fixing the subsequence used for all $i \neq l$ and varying that used for $i = l$ that in fact every subsequence of $s_l^m(t)$ must converge to $s_l^0(t)$. Letting t vary then gives sections s_i^0 such that every $s_i^m \to s_i^0$ and $\sum s_i^0 = s$; the continuity of s is readily seen to imply that of the s_i^0. □

At this point it is useful to record an elementary fact about the linearization of the divisor addition map.

Proposition 2.2. *Let Σ be a Riemann surface and $r = \sum r_i$. The linearization $+_*$ of the addition map*

$$+\colon \prod_{i=1}^n S^{r_i}\Sigma \to S^r\Sigma$$

at $(D_1, \dots, D_n)$ is an isomorphism if and only if $D_i \cap D_j = \varnothing$ for $i \neq j$. If two or more of the D_i have a point in common, then the image of $+_*$ at $(D_1, \dots, D_n)$ is *contained in* $T_{\sum D_i}\Delta$*, where* $\Delta \subset S^r\Sigma$ *is the diagonal stratum consisting of divisors with a repeated point.*

Proof. By factoring $+$ as a composition

$$S^{r_1}\Sigma \times S^{r_2}\Sigma \times \cdots \times S^{r_n}\Sigma \to S^{r_1+r_2}\Sigma \times \cdots \times S^{r_n}\Sigma \to \cdots \to S^r\Sigma$$

in the obvious way we reduce to the case $n = 2$. Now in general for a divisor $D = \sum a_i p_i \in S^d\Sigma$ where the p_i are distinct, a chart for $S^d\Sigma$ is given by $\prod S^{a_i}U_i$, where the U_i are holomorphic coordinate charts around p_i and the $S^{a_i}U_i$ use as coordinates the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_{a_i}$ in the coordinates of $U_i^{a_i}$. As such, if D_1 and D_2 are disjoint, a chart around $D_1 + D_2 \in S^{r_1+r_2}\Sigma$ is simply the Cartesian product of charts around $D_1 \in S^{r_1}\Sigma$ and $D_2 \in S^{r_2}\Sigma$, and the map $+$ takes the latter diffeomorphically (indeed, biholomorphically) onto the former, so that $(+_*)_{(D_1,D_2)}$ is an isomorphism.

On the other hand, note that

$$+\colon\ S^a\mathbb{C} \times S^b\mathbb{C} \to S^{a+b}\mathbb{C}$$

is given in terms of the local elementary symmetric polynomial coordinates around the origin by

$$(\sigma_1, \ldots, \sigma_a, \tau_1, \ldots, \tau_b) \mapsto (\sigma_1 + \tau_1, \sigma_2 + \sigma_1\tau_1 + \tau_2, \ldots, \sigma_a\tau_b),$$

and so has linearization

$$(+_*)_{(\sigma_1,\ldots,\tau_b)}(\eta_1, \ldots, \eta_a, \zeta_1, \ldots, \zeta_b) = (\eta_1 + \zeta_1, \eta_2 + \sigma_1\zeta_1 + \tau_1\eta_1 + \zeta_2, \ldots, \sigma_a\zeta_b + \tau_b\eta_a).$$

We thus see that $Im(+_*)_{(0,\ldots,0)}$ only has dimension $\max\{a, b\}$ and is contained in the image of the linearization of the smooth model

$$\begin{aligned} \mathbb{C} \times S^{a+b-2}\mathbb{C} &\to S^{a+b}\mathbb{C} \\ (z, D) &\mapsto 2z + D \end{aligned}$$

for the diagonal stratum at $(0, 0 + \cdots + 0)$. Suppose now that D_1 and D_2 contain a common point p; write $D_i = a_i p + D_i'$ where $D_i \in S^{r_i - a_i}\Sigma$ are divisors which do not contain p. Then from the commutative diagram

$$\begin{array}{ccc} S^{a_1}\Sigma \times S^{r_1-a_1}\Sigma \times S^{a_2}\Sigma \times S^{r_2-a_2}\Sigma & \longrightarrow & S^{r_1}\Sigma \times S^{r_2}\Sigma \\ \Big\downarrow & & \Big\downarrow + \\ S^{a_1+a_2}\Sigma \times S^{r_1+r_2-a_1-a_2}\Sigma & \longrightarrow & S^{r_1+r_2}\Sigma \end{array}$$

and the fact that the linearization of the top arrow at (a_1p, D_1', a_2p, D_2') is an isomorphism (by what we showed earlier, since the D_i' do not contain p), while the linearization of the composition of the left and bottom arrows at (a_1p, D_1', a_2p, D_2') has image contained in $T_{D_1+D_2}\Delta$, it follows that $(+_*)_{(D_1,D_2)}$ has image contained in $T_{D_1+D_2}\Delta$ as well, which suffices to prove the proposition. □

Corollary 2.3. *If* $s_i \in \Gamma(X_{r_i}(f))$ *are differentiable sections such that* $C_{s_i} \cap C_{s_j} \neq \varnothing$ *for some* $i \neq j$*, then* $s = \sum s_i \in \Gamma(X_r(f))$ *is tangent to the diagonal stratum of* $X_r(f)$*.*

Proof. Indeed, if $C_{s_i} \cap C_{s_j} \neq \varnothing$, then there is $x \in S^2$ such that the divisors $s_i(x)$ and $s_j(x)$ contain a point in common, and so for $v \in T_xS^2$ we have

$$s_*v = (+ \circ (s_i, s_j))_*v = +_*(s_{1*}v, s_{2*}v) \in T_{s(t)}\Delta$$

by Proposition 2.2. □

Note that it is straightforward to find cases in which the s_i are only continuous with some $C_{s_i} \cap C_{s_j}$ nonempty and the sum $s = \sum s_i$ is smooth but not tangent to the diagonal. For example, let $r = 2$, and in local coordinates let s_1 be a square root of the function $z \mapsto Re(z)$ and $s_2 = -s_1$. Then in the standard coordinates on the symmetric product we have $s(z) = (0, -Re(z))$, so that $T(Ims)$ shares only one dimension with $T\Delta$ at $z = 0$. If s is *transverse* to Δ, one can easily check that a similar situation cannot arise.

We now bring pseudoholomorphicity in the picture. Throughout this treatment, all almost complex structures on $X_r(f)$ will be assumed to agree with the standard structures on the symmetric product fibers, to make the map $F\colon X_r(f) \to S^2$ pseudoholomorphic, and, on some (*not* fixed) neighborhood of the critical fibers of F, to agree with the holomorphic model for the relative Hilbert scheme over a disc around a critical value for f provided in Section 3 of [Sm2]. Let $\mathcal{J}$ denote the space of these almost complex structures. It follows by standard arguments (see Proposition 3.4.1 of [MS1] for the general scheme of these arguments and Section 4 of [DS] for their application in the present context) that for generic $J \in \mathcal{J}$ the space $\mathcal{M}^J(c_\alpha)$ is a smooth manifold of (real) dimension $2d(\alpha) = \alpha^2 - \kappa_X \cdot \alpha$ (the dimension computation comprises Lemma 4.3 of [Sm2]); this manifold is compact, for bubbling is precluded by the arguments of Section 4 of [Sm2] assuming we have taken a sufficiently high-degree Lefschetz fibration.

Inside $\mathcal{M}^J(c_\alpha)$ we have the set $\mathcal{M}^J(c_{\alpha_1} \times \cdots \times c_{\alpha_n})$ consisting of holomorphic sections which lie in the image $+(c_{\alpha_1} \times \cdots \times c_{\alpha_n})$. By Lemma 2.1 and the compactness of $\mathcal{M}^J(c_\alpha)$, $\mathcal{M}^J(c_{\alpha_1} \times \cdots \times c_{\alpha_n})$ is evidently compact; however, the question of its dimension or even whether it is a manifold appears to be a more subtle issue in general.

Let us pause to consider what we would like the dimension of $\mathcal{M}^J(c_{\alpha_1} \times \cdots \times c_{\alpha_n})$ to be. The objects in $\mathcal{M}^J(c_{\alpha_1} \times \cdots \times c_{\alpha_n})$ are expected to correspond in some way to unions of holomorphic curves Poincaré dual to α_i. Accordingly, assume we have chosen the α_i so that $d(\alpha_i) = \frac{1}{2}(\alpha_i^2 - \kappa_X \cdot \alpha_i) \geq 0$ (for otherwise we would expect $\mathcal{M}^J(c_{\alpha_1} \times \cdots \times c_{\alpha_n})$ to be empty). Holomorphic curves in these classes will intersect positively as long as they do not share any components of negative square; for a generic almost complex structure the only such components that can arise are (-1)-spheres, so if we choose the α_i to not share any (-1)-sphere components (*i.e.*, if the α_i are chosen so that there is no class E represented by a symplectic (-1)-sphere such that $\langle \alpha_i, E\rangle < 0$ for more than one α_i), then it would also be sensible to assume that $\alpha_i \cdot \alpha_j \geq 0$ for $i \neq j$.

The above naive interpretation of $\mathcal{M}^J(c_{\alpha_1} \times \cdots \times c_{\alpha_n})$ would suggest that its dimension ought to be $\sum d(\alpha_i)$. Note that

$$d(\alpha) = d(\sum \alpha_i) = \sum d(\alpha_i) + \sum_{i>j} \alpha_i \cdot \alpha_j,$$

so under the assumptions on the α_i from the last paragraph we have that the expected dimension of $\mathcal{M}^J(c_{\alpha_1} \times \cdots \times c_{\alpha_n})$ is at most the actual dimension of $\mathcal{M}^J(c_\alpha)$ (as we would hope, given that the former is a subset of the latter), with equality if and only if $\alpha_i \cdot \alpha_j = 0$ whenever $i \neq j$.

As usual, we will find it convenient to cut down the dimensions of our moduli spaces by imposing incidence conditions, so we shall fix a set Ω of points $z \in X$ and consider the space $\mathcal{M}^{J,\Omega}(c_{\alpha_1} \times \cdots \times c_{\alpha_n})$ of elements $s \in \mathcal{M}^J(c_{\alpha_1} \times \cdots \times c_{\alpha_n})$ such that C_s passes through each of the points z (or, working more explicitly in

$X_r(f)$, such that s meets each divisor $z+S^{r-1}\Sigma_t$, Σ_t being the fiber which contains z). $\mathcal{M}^{J,\Omega}(c_\alpha)$ is defined similarly, and standard arguments show that for generic choices of Ω $\mathcal{M}^{J,\Omega}(c_\alpha)$ will be a compact manifold of dimension

$$2(d(\alpha)-\#\Omega).$$

We wish to count J-holomorphic sections s of $X_r(f)$ such that the reducible components of C_s are Poincaré dual to the α_i. If we impose $\sum d(\alpha_i)$ incidence conditions, then according to the above discussion $\mathcal{M}^{J,\Omega}(c_\alpha)$ will be a smooth manifold of dimension $2\sum_{i>j}\alpha_i\cdot\alpha_j$. A section $\sum s_i\in +(c_{\alpha_1}\times\cdots c_{\alpha_n})$ whose summands are all differentiable would then, by Corollary 2.3, have one tangency to the diagonal Δ for each of the intersections between the C_{s_i}, of which the total expected number is $\sum_{i>j}\alpha_i\cdot\alpha_j$. This suggests that the sections we wish to count should be found among those elements of $\mathcal{M}^{J,\Omega}(c_\alpha)$ which have $\sum_{i>j}\alpha_i\cdot\alpha_j$ tangencies to Δ, where Ω is a set of $\sum d(\alpha_i)$ points.

To count pseudoholomorphic curves tangent to a symplectic subvariety it is necessary to restrict to almost complex structures which preserve the tangent space to the subvariety (see [IP2] for the general theory when the subvariety is a submanifold). Accordingly, we shall restrict attention to the class of almost complex structures J on $X_r(f)$ which are *compatible with the strata* in the sense to be explained presently (for more details, see Section 6 of [DS], in which the notion was introduced).

Within Δ, there are various strata χ_π indexed by partitions $\pi : r=\sum a_i n_i$ with at least one $a_i>1$; these strata are the images of the maps

$$\begin{aligned} p_\chi\colon\ X_{n_1}(f)\times_{S^2}\cdots\times_{S^2}X_{n_k}(f)&\to X_r(f)\\ (D_1,\dots,D_k)&\mapsto \sum a_iD_i; \end{aligned}$$

in particular, $\Delta=\chi_{r=2\cdot 1+1\cdot(r-2)}$. An almost complex structure J on $X_r(f)$ is said to be compatible with the strata if the maps p_χ are (J',J)-holomorphic for suitable almost complex structures J' on their domains.

Denoting by Y_χ the domain of p_χ, Lemma 7.4 of [DS] and the discussion preceding it show:

Lemma 2.4 ([DS])**.** *For almost complex structures J on $X_r(f)$ which are compatible with the strata, each J-holomorphic section s of $X_r(f)$ lies in some unique minimal stratum χ and meets all strata contained in χ in isolated points. In this case, there is a J'-holomorphic section s' of Y_χ such that $s=p_\chi\circ s'$. Furthermore, for generic J among those compatible with the strata, the actual dimension of the space of all such sections s is equal to the expected dimension of the space of J'-holomorphic sections s' lying over s.*

We note the following analogue for standard surfaces of the positivity of intersections of pseudoholomorphic curves.

Proposition 2.5. *Let $s=m_1s_1+\cdots+m_ks_k$ be a J-holomorphic section of $X_r(f)$, where the $s_i\in c_{\alpha_i}\subset\Gamma(X_{r_i}(f))$ are each not contained in the diagonal stratum of $X_{r_i}(f)$, and where the almost complex structure J on $X_r(f)$ is compatible with the strata. Assume that the s_i are all differentiable. Then all isolated intersection points of C_{s_i} and C_{s_j} contribute positively to the intersection number $\alpha_i\cdot\alpha_j$.*

Proof. We shall prove the lemma for the case $k = 2$, the general case being only notationally more complicated. The analysis is somewhat easier if the points of $C_{s_1} \cap C_{s_2} \subset X$ at issue only lie over $t \in S^2$ for which $s_1(t)$ and $s_2(t)$ both miss the diagonal of $X_{r_1}(f)$ and $X_{r_2}(f)$, respectively, so we first argue that we can reduce to this case. Let χ be the minimal stratum (possibly all of $X_r(f)$) in which $s = m_1 s_1 + m_2 s_2$ is contained, so that all intersections of s with lower strata are isolated. Let $p \in X$ be an isolated intersection point of C_{s_1} and C_{s_2} lying over $0 \in S^2$, and let $\delta > 0$ be small enough that there are no other intersections of s with any substrata of χ (and so in particular no other points of $C_{s_1} \cap C_{s_2}$) lying over $D_{2\delta}(0) \subset S^2$. We may then perturb $s = m_1 s_1 + m_2 s_2$ to $\tilde{s} = m_1 \tilde{s_1} + m_2 \tilde{s_2}$, still lying in χ, such that

(i) Over $D_\delta(0)$, $\tilde{s}$ is J-holomorphic and disjoint from all substrata having real codimension larger than 2 in χ, and the divisors $\tilde{s_1}(0)$ and $\tilde{s_2}(0)$ both still contain p;
(ii) Over the complement of $D_{2\delta}(0)$, $\tilde{s}$ agrees with s; and
(iii) Over $D_{2\delta}(0) \setminus D_\delta(0)$, $\tilde{s}$ need not be J-holomorphic but is connected to s by a family of sections s_t contained in χ which miss all substrata of χ

(it may be necessary to decrease δ to find such $\tilde{s}$, but after doing so such $\tilde{s}$ will exist by virtue of the abundance of J-holomorphic sections over the small disc $D_\delta(0)$ which are close to $s|_{D_\delta(0)}$). The contribution of p to the intersection number $\alpha_1 \cdot \alpha_2$ will then be equal to the total contribution of all the intersections of $C_{\tilde{s_1}}$ and $C_{\tilde{s_2}}$ lying over $D_\delta(0)$, and the fact that $\tilde{s}$ misses all substrata with codimension larger than 2 in χ is easily seen to imply that these intersections (of which there is at least one, at p) are all at points where $\tilde{s_1}$ and $\tilde{s_2}$ miss the diagonals in $X_{r_1}(f)$ and $X_{r_2}(f)$.

As such, it suffices to prove the lemma for intersection points at which s_1 and s_2 both miss the diagonal. In this case, in a coordinate neighborhood U around p, the C_{s_i} can be written as graphs $C_{s_i} \cap U = \{w = g_i(z)\}$, where w is the holomorphic coordinate on the fibers of X, z is the pullback of the holomorphic coordinate on S^2, and g_i is a differentiable complex-valued function which vanishes at $z = 0$. Suppose first that $m_1 = m_2 = 1$. Then near $s(0)$, we may use coordinates $(z, \sigma_1, \sigma_2, y_3, \dots, y_r)$ for $X_r(f)$ obtained from the splitting $T_0 S^2 \oplus T_{2p} S^2 \Sigma_0 \oplus T_{s(t)-2p} S^{r-2} \Sigma_0$, and the first two vertical coordinates of $s(z) = (s_1 + s_2)(z)$ with respect to this splitting are $(g_1(z) + g_2(z), g_1(z) g_2(z))$. Now s is J-holomorphic and meets the J-holomorphic diagonal stratum Δ at $(0, s(0))$, and at this point Δ is tangent to the hyperplane $\sigma_2 = 0$, so it follows from Lemma 3.4 of [IP2] that the Taylor expansion of $g_1(z) g_2(z)$ has form $a_0 z^d + O(d+1)$. But then the Taylor expansions of $g_1(z)$ and $g_2(z)$ begin, respectively, $a_1 z^{d_1} + O(d_1 + 1)$ and $a_2 z^{d_2} + O(d_2 + 1)$, with $d_1 + d_2 = d$. Then since $C_{s_i} \cap U = \{w = g_i(z)\}$, it follows immediately that the C_{s_i} have intersection multiplicity $\max\{d_1, d_2\} > 0$ at p.

There remains the case where one or both of the m_i is larger than 1. In this case, where $Y_\chi = X_{r_1}(f) \times_{S^2} X_{r_2}(f)$ is the smooth model for χ, because J is compatible with the strata, (s_1, s_2) is a J'-holomorphic section of Y_χ for an almost complex structure J' such that $p_\chi \colon Y_\chi \to X_r(f)$ is (J', J)-holomorphic. Now where $\tilde{\Delta} = \{(D_1, D_2) \in Y_\chi | D_1 \cap D_2 \neq \varnothing\}$, compatibility with the strata implies that $\tilde{\Delta}$ will be J'-holomorphic. In a neighborhood V around $(s_1(z), s_2(z))$, we have, in appropriate coordinates, $\tilde{\Delta} \cap V = \{(z, w, w, D_1, D_2) | w \in \Sigma_z\}$,while $(s_1(z), s_2(z))$ has first three coordinates $(z, g_1(z), g_2(z))$. From this it follows by Lemma 3.4 of [IP2] that

$$g_1(z) - g_2(z) = a_0 z^d + O(d+1)$$

for some d, in which case C_{s_1} and C_{s_2} have intersection multiplicity $d > 0$ at p. □

Definition 2.6. *Let Ω be a set of $\sum d(\alpha_i)$ points and let J be an almost complex structure compatible with the strata. $\mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ shall denote the set of J-holomorphic sections $s \in c_\alpha$ with $\Omega \subset C_s$ such that there exist C^1 sections $s_i \in c_{\alpha_i}$ with $s = \sum s_i$, while the s_i themselves do not admit nontrivial decompositions as sums of C^1 sections.*

We would like to assert that for generic J and Ω, the space $\mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ does not include any sections contained within the strata. This is not true in full generality; rather we need the following assumption in order to rule out the effects of multiple covers of square-zero tori and (-1)-spheres in X.

Assumption 2.7. *None of the α_i can be written as $\alpha_i = m\beta$ where $m > 1$ and either $\beta^2 = \kappa_X \cdot \beta = 0$ or $\beta^2 = \kappa_X \cdot \beta = -1$.*

Under this assumption, we note that if $s = \sum s_i \in \mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ were contained in Δ, then since the α_i and hence the s_i are distinct we can write each s_i as $s_i = m_i \tilde{s_i}$ with at least one $m_i > 1$. The minimal stratum of s will then be χ_π where $\pi = \left\{ r = \sum m_i \left(\frac{r_i}{m_i} \right) \right\}$ and $s' = (\tilde{s_1}, \ldots, \tilde{s_n})$ will be a J'-holomorphic section of Y_χ with $s = p_\chi \circ s'$, in the homotopy class $[c_{\alpha_1/m_1} \times \cdots \times c_{\alpha_n/m_n}]$.

If any of the $d(\alpha_i/m_i) < 0$, then Lemma 2.4 implies that there will be no such sections s' at all; otherwise (again by Lemma 2.4) the real dimension of the space of such sections (taking into account the incidence conditions) will be

$$2\left(\sum d(\alpha_i/m_i) - \sum d(\alpha_i)\right). \tag{2.1}$$

But an easy manipulation of the general formula for $d(\beta)$ and the adjunction formula (which applies here because the standard surface corresponding to a section of $X_r(f)$ which meets Δ positively will be symplectic; c.f. Lemma 2.8 of [DS]) shows that if $d(\beta) \geq 0$ and $m \geq 2$ then $d(m\beta) > d(\beta)$ unless either $\beta^2 = \kappa_X \cdot \beta = 0$ or $\beta^2 = \kappa_X \cdot \beta = -1$, and these are ruled out in this context by (i) and (ii) above, respectively. So Assumption 2.7 implies that the dimension in Equation 2.1 is negative, so no such s' will exist for generic J. This proves part of the following:

Proposition 2.8. *Under Assumption 2.7, for generic pairs (J, Ω) where J is compatible with the strata and $\#\Omega = \sum d(\alpha_i)$, $\mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ is a finite set consisting only of sections not contained in Δ.*

Proof. That no member of $\mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ is contained in Δ follows from the above discussion. As for the dimension of our moduli space, note that any $s = \sum s_i \in \mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ has one tangency (counted with multiplicity) to Δ for each of the intersections of the C_{s_i}, of which there are $\sum \alpha_i \cdot \alpha_j$ (counted with multiplicity; this multiplicity will always be positive by Proposition 2.5). By the results of section 6 of [IP2], the space $\mathcal{M}_{\delta,\Delta}^{J,\Omega}(c_\alpha)$ of J-holomorphic sections in the class c_α having δ tangencies to Δ and whose descendant surfaces pass through Ω will, for generic (J, Ω), be a manifold of dimension

$$2(d(\alpha) - \sum d(\alpha_i) - \delta) = 2(\sum \alpha_i \cdot \alpha_j - \delta),$$

which is equal to zero in the case $\delta = \sum \alpha_i \cdot \alpha_j$ of present relevance to us.

Let us now show that $\mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ is compact. Now since $+(c_{\alpha_1} \times \cdots \times c_{\alpha_n})$ is C^0-closed in c_α, by Gromov compactness any sequence $s^{(m)} = \sum_{i=1}^n s_i^{(m)}$ in $\mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ has (after passing to a subsequence) a J-holomorphic limit $s = \sum s_i$ where the $s_i \in c_{\alpha_i}$ are at least continuous. We claim that, at least for generic (J, Ω), we can guarantee the s_i to be C^1. In light of Proposition 2.2, the differentiability of the s_i is obvious at all points where s misses the diagonal, since s is smooth by elliptic regularity and the divisor addition map induces an isomorphism on the tangent spaces away from the diagonal. Now each $s^{(m)}$ has $\sum \alpha_i \cdot \alpha_j$ tangencies to the diagonal, corresponding to points $t \in S^2$ at which some pair of the divisors $s_i^{(m)}(t)$ share a point in common. The limit s will then likewise have n tangencies to the diagonal; the dimension formulas in [IP2] ensure that for generic (J, Ω) no two of the tangencies will coalesce into a higher order tangency to the smooth part of Δ in the limit, and all of the intersecions of $Im\, s$ with the smooth part of the diagonal other than these n tangencies will be transverse. Furthermore, one may easily show (using for instance an argument similar to the one used in Lemma 2.1 of [U1] to preclude generic 0-dimensional moduli spaces of pseudoholomorphic curves in a Lefschetz fibration from meeting the critical points) that since the singular locus of Δ has codimension 4 in $X_r(f)$, if J has been chosen generically then s will not meet Δ^{sing}, and so no $s(t)$ will contain more than one repeated point (and that point cannot appear with multiplicity larger than two). In light of this, each tangency of s to Δ will occur at a point $s(t)$ where some pair $s_i(t)$ and $s_j(t)$ have some point p in common, and all other points contained in any $s_k(t)$ are distinct from each other and from p. Thanks to Proposition 2.2, this effectively reduces us to the case $r = 2$, with $s = s_1 + s_2$ a sum of continuous sections with $s_1(0) = s_2(0) = 0$ which is holomorphic with respect to an almost complex structure which preserves the diagonal stratum Δ in $D^2 \times Sym^2 D^2$, such that s is tangent to Δ. Then letting $\delta(t) = (s_1 + s_2)^2(t) - 4s_1(z)s_2(t)$ be the discriminant, that s is tangent to the diagonal stratum implies, using Lemma 3.4 of [IP2], that $\delta(t) = at^2 + O(3)$ for some constant a; in particular $\delta(t)$ has two C^1 square roots $\pm r(t)$. Since s is smooth, so is its first coordinate $t \mapsto s_1(t) + s_2(t)$; adding this smooth function to the C^1 functions $\pm r(t)$ and dividing by two then recovers the functions $s_1(t)$ and $s_2(t)$ and verifies that they are C^1 at $t = 0$.

We have thus shown that the s_i are all C^1 at the points where $s = \sum s_i$ is tangent to Δ. Where s is transverse to Δ, one sees easily that the s_i are pairwise disjoint, with one s_i transverse to the diagonal in $X_{r_i}(f)$ and all others missing their diagonals, so the differentiability of the s_i is clear. This indeed verifies that the limit $s = \sum s_i$ is a sum of C^1 sections s_i, since our generic choice of J is such that the only intersections of $Im\, s$ with Δ only are either transverse or of second order.

Now each of the $C_{s_i^{(m)}}$ is connected, so C_{s_i} is connected as well. *A priori*, it is possible that s might not lie in $\mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ because some of the s_i might decompose further, say as $s_i = m_1 u_{i_1} + \cdots + m_l u_{i_l}$ where $u_{i_j} \in c_{\beta_{i_j}}$ are C^1. But since C_{s_i} is connected, the $C_{u_{i_j}}$ cannot all be disjoint, and by Corollary 2.3 any intersection between two of them would give rise to an additional tangency of s to Δ, over and above the n tangencies arising from the intersections between distinct C_{s_i}. Once again, this is ruled out for generic J by the dimension formulas of [IP2]. This proves that (for generic J) the summands s_i in a sequence $s = \sum s_i$ occurring as a limit point of $\mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ cannot decompose further and hence themselves lie in $\mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$, so that $\mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ is compact.

Since we have already shown that $\mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ is zero-dimensional, the proposition follows.

□

Proposition 2.9. *For generic (J_0, Ω_0) and (J_1, Ω_1) as in Proposition 2.8 and generic paths (J_t, Ω_t) connecting them, the space*

$$\mathcal{PM}_0(\alpha_1, \ldots, \alpha_n) = \{(t, s) | s \in \mathcal{M}_0^{J_t,\Omega_t}(\alpha_1, \ldots, \alpha_n)\}$$

is a compact one-dimensional manifold.

Proof. This follows immediately from the above discussion, noting that in the proof of Proposition 2.8 we saw that any possible boundary components of $\mathcal{M}_0^J(\alpha_1, \ldots, \alpha_n)$ have real codimension 2 and so will not appear in our one-dimensional parametrized moduli space. □

Note that we can orient these moduli spaces by using the spectral flow of the linearization of the $\bar{\partial}$ operator at an element $s \in \mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ acting on sections of $s^*T^{vt}X_r(f)$ which preserve the incidence conditions and the tangencies to Δ; $\mathcal{PM}_0(\alpha_1, \ldots, \alpha_n)$ will then be an oriented cobordism between $\mathcal{M}_0^{J_0,\Omega_0}(\alpha_1, \ldots, \alpha_n)$ and $\mathcal{M}_0^{J_1,\Omega_1}(\alpha_1, \ldots, \alpha_n)$. Accordingly, we may make the following definition.

Definition 2.10. *Let $\alpha = \alpha_1 + \cdots + \alpha_n$ be a decomposition of $\alpha \in H^2(X, \mathbb{Z})$ which satisfies Assumption 2.7. Then*

$$\widetilde{\mathcal{DS}}_f(\alpha; \alpha_1, \ldots, \alpha_n)$$

is defined as the number of points, counted with sign according to orientation, in the space $\mathcal{M}_0^{J,\Omega}(\alpha_1, \ldots, \alpha_n)$ for generic (J, Ω) as in Proposition 2.8.

Theorem 2.11. *If $\alpha = \alpha_1 + \cdots \alpha_n$ is a decomposition satisfying Assumption 2.7 then*

$$\frac{(\sum d(\alpha_i))!}{\prod(d(\alpha_i)!)} Gr(\alpha; \alpha_1, \ldots, \alpha_n) = \widetilde{\mathcal{DS}}_f(\alpha; \alpha_1, \ldots, \alpha_n)$$

provided that the degree of the fibration is large enough that $\langle [\omega_X], [\Phi] \rangle > [\omega_X] \cdot \alpha$.

Proof. Let j be an almost complex structure on X generic among those compatible with the fibration $f \colon X \to S^2$, and Ω a generic set of $\sum d(\alpha_i)$ points. The curves in X contributing to $Gr(\alpha; \alpha_1, \cdots, \alpha_n)$ are unions

$$C = \bigcup_{i=1}^n C^i$$

of embedded j-holomorphic curves C^i which are Poincaré dual to α_i (note that Assumption 2.7 implies that none of these curves will be multiple covers) with $\Omega_i \subset C^i$ for some fixed generic sets Ω_i of $d(\alpha_i)$ points. In Section 3 of [U1] it was shown that there is no loss of generality in assuming that j is integrable near

$\cup_i Crit(f|_{C^i})$, so let us assume that this is the case. Where s_C is the section of $X_r(f)$ tautologically corresponding to C, in the context of [U1] this local integrability condition was enough to ensure that the almost complex structure $\mathbb{J}_j$ on $X_r(f)$ constructed from j was smooth on a neighborhood of s_C. Here that is not quite the case, for $\mathbb{J}_j$ might only be Hölder continuous at the points of $Im(s_C)$ tautologically corresponding to the intersection points of the various C^i.

However, just as in Section 5 of [U1], we can still define the contribution $r'(C)$ to $\widetilde{DS}_f(\alpha_1, \dots, \alpha_n)$ by perturbing $\mathbb{J}_j$ to a generic almost complex structure J which is compatible with the strata and Hölder-close to $\mathbb{J}_j$, and then counting with sign the elements of $\mathcal{M}_0^{J,\Omega}(\alpha_1, \dots, \alpha_n)$ which lie near s_C; since the curves C which contribute to $Gr(\alpha_1, \dots, \alpha_n)$ are isolated, and since the members of $\mathcal{M}_0^{\mathbb{J}_j,\Omega}(\alpha_1, \dots, \alpha_n)$ are precisely the s_C corresponding to the curves C, it follows from Gromov compactness that for sufficiently small perturbations J of $\mathbb{J}_j$ all elements of $\mathcal{M}_0^{J,\Omega}(\alpha_1, \dots, \alpha_n)$ will be close to one and only one of the s_C. Thus

$$\widetilde{\mathcal{DS}}_f(\alpha_1, \dots, \alpha_n) = \sum_{\pi \in p(\Omega)} \sum_{C \in \mathcal{M}^{j,\Omega,\pi}(\alpha_1, \dots, \alpha_n)} r'(C)$$

where $p(\Omega)$ is the set of partitions of Ω into subsets Ω_i of cardinality $d(\alpha_i)$ and, writing $\pi = (\Omega_1, \dots, \Omega_n)$, $\mathcal{M}^{j,\Omega,\pi}(\alpha_1, \dots, \alpha_n)$ is the space of curves $C = \cup C^i$ contributing to $Gr(\alpha; \alpha_1, \dots, \alpha_n)$ with C^i passing through Ω_i. Meanwhile, for any π, we have

$$Gr(\alpha; \alpha_1, \dots, \alpha_n) = \sum_{C \in \mathcal{M}^{j,\Omega,\pi}(\alpha_1, \dots, \alpha_n)} r(C),$$

$r(C)$ being the product of the spectral flows of the linearizations of $\bar{\partial}_j$ at the embeddings of the C^i where $C = \cup C^i$. The theorem will thus be proven if we show that $r'(C) = r(C)$, which we now set about doing.

So let $C = \cup C^i \in \mathcal{M}^{j,\Omega,\pi}(\alpha_1, \dots, \alpha_n)$. Taking j generically, we may assume that all intersections of the C^i are transverse and occur away from $crit(f|_{C^i})$ (this follows from the arguments of Lemma 2.1 of [U1]). Let $p \in C^i \cap C^k$. In a coordinate neighborhood U around p, where w is a holomorphic coordinate on the fibers and z the pullback of the coordinate on S^2, we may write

$$C^i \cap U = \{w = g(z)\} \qquad C^k \cap U = \{w = h(z)\}.$$

If the almost complex structure j is given in U by

$$T_j^{0,1} = \langle \partial_{\bar{z}} + b(z, w)\partial_w, \partial_{\bar{w}} \rangle \tag{2.2}$$

(note that we may choose the horizontal tangent space so that $b(0, 0) = 0$), that C^i and C^k are j-holomorphic amounts to the statement that

$$\partial_{\bar{z}} g(z) = b(z, g(z)) \qquad \partial_{\bar{z}} h(z) = b(z, h(z));$$

in particular, we have $g_{\bar{z}}(0) = h_{\bar{z}}(0) = 0$. Since $C^i \pitchfork C^k$, we have $(g - h)_z(0) \neq 0$, and by the inverse function theorem $(g - h)\colon \mathbb{C} \to \mathbb{C}$ is invertible on some disc $D_{2\delta}(0)$. Let g_t and h_t ($t \in [0, 1]$) be one-parameter families of functions satisfying

(i) $g_0 = g$, $h_0 = h$;
(ii) On $D_{2\delta}(0)$, $g_t - h_t$ is invertible as a complex-valued smooth function, with inverse p_t;
(iii) g_t and h_t agree with g and h, respectively, outside $D_{2\delta}(0)$;
(iv) $g_t(0) = h_t(0) = \partial_{\bar{z}} g_t(0) = \partial_{\bar{z}} h_t(0) = 0$; and

(v) $g_1(z)$ and $h_1(z)$ are both holomorphic on $D_\delta(0)$.

Let

$$C_t^i = (C^i \cap (X \setminus U)) \cup \{w = g_t(z)\} \text{ and } C_t^k = (C^k \cap (X \setminus U)) \cup \{w = h_t(z)\}.$$

Now set

$$b_t(z, w) = (\partial_{\bar{z}} h_t)(z) + \partial_{\bar{z}} (g_t - h_t) (p_t(w - h_t(z))).$$

Then, since $p_t = (g_t - h_t)^{-1}$,

$$b_t(z, h_t(z)) = \partial_{\bar{z}} h_t(z) + \partial_{\bar{z}} (g_t - h_t)(0) = \partial_{\bar{z}} h_t(z)$$

while

$$b_t(z, g_t(z)) = \partial_{\bar{z}} h_t(z) + \partial_{\bar{z}} (g_t - h_t)(z) = \partial_{\bar{z}} g_t(z).$$

Let b'_t agree with b_t near $\{(z, w) \in C_t^i \cup C_t^k | z \in D_{2\delta}(0)\}$ and with b sufficiently far from the origin in U. Then defining j'_t by $T^{0,1}_{j'_t} = \langle \partial_{\bar{z}} + b'_t \partial_w, \partial_{\bar{w}} \rangle$, j'_t agrees with j near ∂U and makes $C_t^i \cup C_t^k$ holomorphic. Further, we see that $b_1(z, w) \equiv 0$ for $z \in D_\delta(0)$, from which a Nijenhuis tensor computation shows that j'_1 is integrable on a neighborhood of the unique point p of $C_1^i \cap C_1^k \cap U$.

Carrying out this construction near all intersection points of the C^i, we obtain curves $C_t = \cup C_t^i$ and almost complex structures j'_t on X such that j'_1 is integrable near all intersection points of the C_1^i. Since j'_1 agrees with j and C_1^i with C^i away from small neighborhoods of these intersection points, j'_1 is also integrable on a neighborhood of $crit(f|_{C_i^1})$ for each i.

If p is a point of C_1 near which j'_1 is not already integrable, then in a neighborhood U of p we have $C_1 \cap U = \{w = g(z)\}$, and so the condition for an almost complex structure j' given by $T^{0,1}_{j'} = \langle \partial_{\bar{z}} + b\partial_w, \partial_{\bar{w}} \rangle$ to make C_1 holomorphic near p is just that $\partial_{\bar{z}} g(z) = b(z, g(z))$, while the condition for j' to be integrable in the neighborhood is that $\partial_{\bar{w}} b(z, w) = 0$. As in Lemmas 4.1 and 4.4 of [U1], then, we may easily find a path of almost complex structures j'_t $(1 \le t \le 2)$ such that each j'_t makes C_1 holomorphic and j'_2 is integrable on a neighborhood of C_1. So, changing notation slightly, we have proven:

Lemma 2.12. *There exists an isotopy (C_t, j_t) of pairs consisting of almost complex structures j_t compatible with the fibration $f \colon X \to S^2$ and j_t-holomorphic curves C_t such that $(C_0, j_0) = (C, j)$ and j_1 is integrable on a neighborhood of C_1.*

In the situation of the above lemma, $\mathbb{J}_{j_1}$ is not only smooth but also integrable on a neighborhood of C_1; Lemma 4.2 of [U1] shows that if j_1 is chosen generically among almost complex structures which make both C_1 and f pseudoholomorphic and are integrable near C_1 the linearization of $\bar{\partial}_{\mathbb{J}_{j_1}}$ at s_C will be surjective, as will the linearizations of $\bar{\partial}_{j_1}$ at the embeddings of each of the C_1^i. We now fix the isotopy C_t and the almost complex structure j_1 which is nondegenerate in the above sense; Lemma 2.12 then gives a path j_t from $j = j_0$ to j_1 such that each C_t is j_t-holomorphic. We may then define $r'_{j_t}(C_t)$ in the same way as $r'(C)$, by counting J-holomorphic sections close to s_{C_t} for some J Hölder-close to $\mathbb{J}_{j_t}$. Meanwhile, if the linearization $D\bar{\partial}_{j_t}$ is surjective at the embeddings of the C_t^i, its spectral flow gives a number $r_{j_t}(C_t)$, and our goal is to show that $r_{j_0}(C_0) = r'_{j_0}(C_0)$. To this end, we see from Lemma 5.5, Corollary 5.6, and their proofs in [U1] that:

Lemma 2.13. *For generic paths j_t from j_0 to j_1 as above such that C_t is j_t-holomorphic, the following statements hold. $D\bar{\partial}_{j_t}$ is surjective at the embeddings* *of the C_t^i for all but finitely many values of t. For t near any value t_0 for which $D\bar{\partial}_{j_{t_0}}$ fails to be surjective, the set of elements of $\mathcal{M}^{j_t,\Omega}(\alpha_1,\ldots,\alpha_n)$ in a tubular neighborhood of C_t is given by $\{C_t, \tilde{C}_t\}$ for a smooth family of curves $\tilde{C}_t$ with $\tilde{C}_{t_0} = C_{t_0}$. Further, for small $\epsilon > 0$, we have*

$$r'_{j_{t_0+\epsilon}}(C_{t_0+\epsilon}) = r'_{j_{t_0-\epsilon}}(\tilde{C}_{t_0-\epsilon}) = -r'_{j_{t_0-\epsilon}}(C_{t_0-\epsilon})$$

and

$$r_{j_{t_0+\epsilon}}(C_{t_0+\epsilon}) = r_{j_{t_0-\epsilon}}(\tilde{C}_{t_0-\epsilon}) = -r_{j_{t_0-\epsilon}}(C_{t_0-\epsilon}).$$

Moreover, on intervals not containing any t_0 for which j_{t_0} has a non-surjective linearization, $r'_{j_t}(C_t)$ and $r_{j_t}(C_t)$ both remain constant.

Since (for generic paths j_t), $r'_{j_t}(C_t)$ and $r_{j_t}(C_t)$ stay constant except for finitely many points at which they both change sign, to show that $r'_{j_0}(C_0) = r_{j_0}(C_0)$ it is enough to see that $r'_{j_1}(C_1) = r_{j_1}(C_1)$. But since j_1 is *integrable* and nondegenerate near C_1, as is $\mathbb{J}_{j_1}$ near s_{C_1}, we immediately see that $r'_{j_1}(C_1) = r_{j_1}(C_1) = 1$, and the theorem follows. □

Remark 2.14. The above proof suggests a simplification of the proof that $\mathcal{DS} = Gr$ in [U1]. As mentioned above, in Section 3 of [U1] it is shown that we can take the almost complex structure j to be integrable on neighborhoods of the critical points of the various $f|_C$ for C contributing to $Gr(\alpha)$. Given arbitrary generic fibration-compatible j, however, as in the proof of Theorem 2.11, the arguments of Sections 4 and 5 of [U1] go through as long as we can find an isotopy (C_t, j_t) of pairs consisting of almost complex structures j_t compatible with the fibration $f\colon X \to S^2$ and j_t-holomorphic curves C_t such that $(C_0, j_0) = (C, j)$ and j_1 is integrable on a neighborhood of C_1. This is indeed possible; if near a critical point of $f|_C$ C has the form $\{z = w^n + O(n+1)\}$, we can take C_t such that C_t agrees with C away from a neighborhood of $Crit(f|_C)$ and C_1 has the form $\{z = w^n\}$ on a smaller neighborhood of the critical point, and then we can choose j_t to make C_t holomorphic. (The easiest approach to this seems to be to have C_t be constant for $t \leq 1/2$ and arrange the function $b_{1/2}(z, w)$ in the notation (2.2) to depend only on w near the critical points; then for $t > 1/2$, the form of C_t determines uniquely a z-independent function b_t which causes C_t to be j_t-holomorphic, and we will have $b_1(z, w) = 0$ near the critical point. Details are left to the reader.)

3. The family standard surface count

While much is known about the structure the Gromov–Taubes invariants, which count embedded holomorphic curves in symplectic 4-manifolds, we know comparatively little about invariants counting singular curves. We explain here an approach to nodal curves using Donaldson and Smith's constructions.

We should mention first of all that whereas Taubes' work gives us a natural invariant $Gr(\alpha)$ counting all embedded curves (regardless of their connected-component decomposition) Poincaré dual to some class α, if we instead wish to assemble all of the possibly-reducible curves Poincaré dual to α and having some number $n > 0$ of ordinary double points into an invariant $Gr_n(\alpha)$, it is somewhat unclear how we should proceed in many cases. Just as with the difficulties surrounding the Gromov–Taubes invariant, this stems from the multiple-cover problem: if for some class $\beta \in H^2(X, \mathbb{Z})$ and $m > 1$ we have $d(\beta) \geq \max\{0, d(m\beta) - n\}$, then for generic almost complex structures j there will arise the possibility of a sequence of curves

Poincaré dual to $m\beta$ which have n double points converging to an m-fold cover of a curve Poincaré dual to β. When $n = 0$, as was noted in the previous section the formula for $d(\beta)$ and the adjunction formula imply that this only arises when β is Poincaré dual to a square-zero torus, and Taubes' work shows how to incorporate multiple covers into the definition of Gr in the correct way. When $n > 0$, the equation $d(\beta) \geq d(m\beta) - n$ becomes easier to satisfy and it is less clear how multiple covers should be dealt with, especially in the case of a strict inequality $d(\beta) > d(m\beta) - n$, when the multiple covers form a space of larger dimension than that of the space we are interested in.

Of course, there will typically be at least some classes for which this issue does not arise:

Definition 3.1. *A class $\alpha \in H^2(X, \mathbb{Z})$ is called* strongly n-semisimple *if there exist* no *decompositions $\alpha = \alpha_1 + \cdots + \alpha_l$ into nonnegatively-intersecting classes α_i such that each α_i has $d(\alpha_i) \geq 0$ and is Poincaré dual to the image of a symplectic immersion, and α_1 is equal to $m\beta$ ($m > 1$) where β satisfies $d(\beta) \geq \max\{0, d(\alpha_1) - n + \alpha_1 \cdot (\alpha - \alpha_1)\}$. α is called* weakly n-semisimple *if the only decompositions $\alpha = \alpha_1 + \cdots + \alpha_n$ as above which exist have $\alpha_1^2 = \kappa_X \cdot \alpha_1 = 0$.*

For instance, every class is weakly 0-semisimple, while the only classes which are not weakly 1-semisimple are those classes α such that there exists a class $\beta \in H^2(X;\mathbb{Z})$ such that $\beta \cdot (\alpha - 2\beta) = 0$ and β is Poincaré dual either to a symplectic sphere of square 0 or a symplectic genus-two curve of square 1, while $\alpha - 2\beta$ is Poincaré dual to some embedded (and possibly disconnected) symplectic submanifold. For strong semisimplicity, one needs to add the assumption that α is not Poincaré dual to a symplectic immersion having a component which is a square-zero torus in a non-primitive homology class.

For a weakly- or strongly-n-semisimple classes α, there is an obvious analogue of the Gromov–Taubes invariant $Gr_n(\alpha)$, defined by counting j-holomorphic curves C which are unions of curves C_i Poincaré classes α_i carrying multiplicities m_i which are equal to 1 unless C_i is a square-zero torus with $\sum m_i\alpha_i = \alpha$, such that C has n transverse double points and passes through a generic set of $d(\alpha) - n$ points of X; each such C contributes the product of the Taubes weights $r(C_i, m_i)$ to the count $Gr_n(\alpha)$. Since the condition of n-semisimplicity is engineered to rule out the only additional possible source of noncompactness of the relevant moduli spaces, the proof that $Gr(\alpha)$ is independent of the choice of almost complex structure used to define it goes through to show the same result for $Gr_n(\alpha)$.

For that matter, if α is weakly n-semisimple and we have $n_i \geq 0$ and α_i with $\sum \alpha_i = \alpha$ and $\sum n_i = n - \sum_{i<j} \alpha_i \cdot \alpha_j$, we can form a refinement $Gr_{(n_1,\ldots,n_k)}(\alpha; \alpha_1, \ldots, \alpha_k)$ along the lines of Definition 1.1 which counts (modulo the usual square-zero torus issues) curves with reducible components which are Poincaré dual to the α_i and have n_i transverse self-intersections. In this case, under Assumption 2.7 it is also straightforward to modify the constructions of the previous section to produce an invariant $\widetilde{\mathcal{DS}}_{(n_1,\ldots,n_k)}(\alpha; \alpha_1, \ldots, \alpha_k)$ which counts holomorphic sections s of $X_r(f)$ in the homotopy class c_α which decompose into a sum of C^1 sections $s_i \in c_{\alpha_i}$ such that each s_i has n_i tangencies to the diagonal stratum of $X_{r_i}(f)$ and does not itself decompose as a nontrivial sum of C^1 sections. Furthermore, the proof of Theorem 2.11 goes through unchanged to show that

$$Gr_{(n_1,\ldots,n_k)}(\alpha; \alpha_1, \ldots, \alpha_k) = \widetilde{\mathcal{DS}}_{(n_1,\ldots,n_k)}(\alpha; \alpha_1, \ldots, \alpha_k).$$

Instead, though, we aim to produce an invariant similar to $Gr_n(\alpha)$ which does not require α to be n-semisimple. For general α, the multiple cover problem discussed above has its mirror on the side of $\widetilde{\mathcal{DS}}$ in the fact that the moduli spaces for the latter will tend to have undesirably-large components consisting of sections which are mapped entirely into the diagonal stratum, so $\widetilde{\mathcal{DS}}$ will not be much help toward this goal. Instead, we take a hint from the approach used by A.K. Liu in [L1] and construct family versions of the standard surface count. These new invariants will use almost complex structures which generally do not make the diagonal stratum pseudoholomorphic, and so we will not encounter moduli spaces with unexpectedly large components consisting of sections mapped into Δ.

Be given a symplectic Lefschetz fibration $f\colon X \to S^2$. Write $f_0 = f$, $X_0 = \{pt\}$, $X_1 = X$, and let $g_0\colon X_1 \to X_0$ be the map of X to a point. As in [L1], for $n \geq 1$ form $X^0_{n+1} = X_n \times_{g_{n-1}} X_n$, and let X_{n+1} be the blowup of the relative diagonal in X^0_{n+1}. Let $g_n\colon X_{n+1} \to X_n$ be the projection onto the first factor. Each $X^b := g_n^{-1}(b)$ ($b \in X_n$) is then an n-fold blowup of X, with the parameter b indicating which points have been blown up. Composing the maps g_n gives a map $X_{n+1} \to X_1 = X$; let $f_n\colon X_{n+1} \to S^2$ be the composition of this map with the Lefschetz fibration f. (Equivalently, on each n-fold blowup $X^b = g_n^{-1}(b)$, $f_n|_{X^b}$ is the composition of the blowdown map with the Lefschetz fibration f.)

Write $f^b = f_n|_{X^b}$. $f^b\colon X\# n\overline{\mathbb{C}P^2} \to S^2$ then has the same structure as f, except that if k points on some fiber (in class $[\Phi]$) are among the blown up points, that (initially irreducible) fiber has been replaced by a reducible curve with components in classes $[\Phi] - E_1 - \cdots - E_k$, $E_1, \ldots, E_k$, where the E_i are classes of exceptional spheres. Straightforward local coordinate calculations show that, if none of the blown-up points are critical points of any of the f_i ($i < n$), then the only intersection points between components are ordinary double points, and that near the double points f^b has form $(z, w) \mapsto zw$. In particular, each $f^b = f_n|_{X^b}$ is still a Lefschetz fibration provided that no critical points of any of the intermediate fibrations are blown up in forming X^b.

Notation 3.2. *Denote a point $b \in X_n$ by $(p_1, \ldots, p_n)$, where each $p_{j+1} \in X^{(p_1,\ldots,p_j)}$. Let:*

(i) *X'_n be the set of $(p_1, \ldots, p_n) \in X_n$ such that no p_{j+1} is a critical point of $f^{(p_1,\ldots,p_j)}\colon X^{(p_1,\ldots,p_j)} \to S^2$.*
(ii) *X''_n be the set of $(p_1, \ldots, p_n) \in X_n$ such that no p_{j+1} lies in a singular fiber of $f^{(p_1,\ldots,p_j)}\colon X^{(p_1,\ldots,p_j)} \to S^2$.*

If $b \in X'_n$, then, our above remarks show that $f^b\colon X^b \to S^2$ is a Lefschetz fibration; if moreover $b \in X''_n$, then no fiber of f^b will contain more than one critical point (and also none of the n blowups involved in the creation of X^b will be at a point on an exceptional divisor of a previous blowup).

Notation 3.3. (i) *For any $b \in X'_n$, $F^b\colon X^b_r(f^b) \to S^2$ shall denote the relative Hilbert scheme constructed from f^b as in the Appendix of* [DS] *and Section 3 of* [Sm2].
(ii) $\mathcal{X}^n_r(f) = \{(D, b) : b \in X'_n, D \in X^b_r(f^b)\}$. *In particular we have a map $\mathcal{F}^n\colon \mathcal{X}^n_r(f) \to S^2 \times X'_n$.*

For $b \in X''_n$, X^b contains disjoint exceptional divisors $E_1, \ldots, E_n$, and our intention is to define an invariant counting sections of the various $X^b_r(f^b)$ which descend to curves Poincaré dual to $\alpha - 2\sum PD(E_i)$, as b ranges over X_n''. We have to be somewhat careful in the definition of this invariant, though, since our parameter space X_n'' is noncompact.

Lemma 3.4. *For $b \in X_n''$, $X_r^b(f^b)$ is a smooth symplectic manifold, as is the total space of $\mathcal{X}_r^n(f) \to S^2 \times X_n'$.*

Proof. That the relative Hilbert scheme constructed from any Lefschetz fibration (such as f^b when $b \in X_n''$) in which there is at most one critical point per fiber is smooth is shown in Theorem 3.4 of [Sm2] (as noted in Remark 3.5 of [Sm2], Smith's provision of a local coordinate description for the relative Hilbert scheme makes irrelevant his assumption that all of the fibers of the Lefschetz fibration are irreducible). When $b \in X_n' \setminus X_n''$, so that f^b, while still a Lefschetz fibration, may have more than one critical point per fiber, the individual $X_r^b(f^b)$ will tend not to be smooth near points on the Hilbert scheme of the singular fibers Σ_0 which are sent by the map $Hilb^{[r]}\Sigma_0 \to S^r\Sigma_0$ to divisors which contain more than one of the nodes of Σ_0. We will show presently, though, that the freedom to vary $b \in X_n'$ results in the total space $\mathcal{X}_r^n(f)$ still being smooth at these points.

To see this, note that Donaldson and Smith show (c.f. the proof of Proposition A.8 of [DS]) that when f only has one node per fiber, at a singular point of a fiber of $X_s(f)$ (corresponding to a divisor with points near the node of a fiber) the behavior of $F\colon X_s(f) \to S^2$ is modeled by $(z_1, \dots, z_{s+1}) \mapsto z_1 z_2$. When there are multiple nodes in a fiber, then, the relative Hilbert scheme will be modeled near a point corresponding to a divisor containing s_i copies of the nodes p_i $(i = 1, \dots, l)$ by the fiber product of the various maps $(z_1^{(i)}, \dots, z_{s_i+1}^{(i)}) \mapsto z_1^{(i)} z_2^{(i)}$. This fiber product is the common vanishing locus of the various $z_1^{(i)} z_2^{(i)} - z_1^{(j)} z_2^{(j)}$ (which is of course singular where $z_1^{(i)} = z_2^{(i)} = 0$ for all i).

More generally, though, if p_i is a node lying near the fiber over zero, $X_s(f) \to S^2$ is modeled near points corresponding to divisors with points near p_i by $(z_1^{(i)}, \dots, z_s^{(i)}) \mapsto z_1^{(i)} z_2^{(i)} + f(p_i)$. In our present context the fibration map is f^b; say for notational simplicity that $b = (p_1, \dots, p_n)$ gives rise to an n-fold blowup with all exceptional divisors in the same fiber (of course if some exceptional divisors are in different fibers we can work fiber-by-fiber and reduce to this case). The space $\mathcal{X}_r^n(f)$ is then, at worst, modeled locally by

$$\{(\vec{z}^{(0)}, \vec{z}^{(1)}, \dots, \vec{z}^{(n)}, q_1, \dots, q_n) : z_1^{(0)} z_2^{(0)} = z_1^{(i)} z_2^{(i)} + f^{(p_1,\dots,p_{i-1})}(q_i)\}. \tag{3.1}$$

Here $\vec{z}^{(0)}$ are the coordinates on the relative Hilbert scheme corresponding to divisors which contain any nodes that may have existed in our fiber before blowing up (and we are of course assuming throughout that the original f was chosen so that there is at most one such). The q_i are elements of a coordinate chart centered on $p_i \in X^{(p_1,\dots,p_{i-1})}$. But (3.1) defines a smooth manifold at any point with $q_i = p_i$ as long as none of the p_i are critical points for $f^{(p_1,\dots,p_{i-1})}$, and this latter condition is precisely ensured by the fact that $b \in X_n'$.

This shows that $\mathcal{X}_r^n(f)$ is smooth; the existence of a symplectic structure on it then follows exactly as in the proof of the existence of a symplectic structure on $X_r(f)$ in [DS]: where $\mathcal{X}_r^n(f)$ fails to be a fibration we have a local Kähler model for it, and we can extend the resulting form to the entire manifold by the usual methods of Gompf and Thurston. □

We consider almost complex structures J on the $X_r^b(f^b)$ which make the fibration maps $F^b\colon X_r^b(f^b) \to S^2$ pseudoholomorphic and have the following special type: for each reducible fiber of X^b, letting E denote the union of the spherical components of that fiber, we require that there exist neighborhoods $U \supset V$ of E with $f^b(U) = f^b(V) = W \subset S^2$ and almost complex structures J_1^q and J_2^q on the restricted relative Hilbert schemes $X_q(f^b|_U)$ and $X_{r-q}(f^b|_{(f^b)^{-1}(W)-V})$ such that the natural "addition map" $X_q^b(f^b|_U) \times_{F^b} X_{r-q}^b(f^b|_{(f^b)^{-1}(W)-V}) \to X_r^b(f^b)$ is $(J_1^q \times_{F^b} J_2^q, J)$-holomorphic; moreover, we require that J_1^q agree with the complex structure induced (via the algebro-geometric description for the relative Hilbert scheme given in Section 3 of [Sm2]) by an integrable complex structure on $U \supset E$ with respect to which f^b is holomorphic. Note that one way of forming such a J is by taking any almost complex structure on $X_r^b(f^b)$ which agrees near the singular fibers with the almost complex structure $\mathbb{J}_j$ tautologically corresponding to a structure j on X^b which is integrable near the singular fibers of X^b. If j is instead integrable only on the neighborhood U of the exceptional spheres, we still obtain a Hölder almost complex structure satisfying the requirement, which may then be Hölder-approximated by smooth almost complex structures also satisfying the requirement by smoothing the almost complex structures J_2^q in a coherent way at points of the $X_{r-q}(f^b|_{f^{-1}(W)-V})$ corresponding to divisors having points missing U.

Let $\mathcal{J}$ denote the space of smooth tame almost complex structures on $\mathcal{X}_r^n(f)$ which restrict to each $X_r^b(f^b) = (\mathcal{F}^n)^{-1}(S^2 \times \{b\})$ as a J of the above form. For each b, the blowdown map $\pi^b\colon X^b \to X$ naturally induces a generically injective map $\Pi^b\colon X_r^b(f^b) \to X_r(f)$ on relative Hilbert schemes. For $J \in \mathcal{J}$ we obtain commutative diagrams

$$\begin{array}{ccc} X_q^b(f^b|_U) \times_{F^b} X_{r-q}^b(f^b|_{(f^b)^{-1}(W)-V}) & \longrightarrow & X_r^b(f^b) \\ \downarrow & & \downarrow{\scriptstyle \Pi^b} \\ X_q(f|_{\pi^b(U)}) \times_F X_{r-q}(f^b|_{f^{-1}(W)-\pi^b(V)}) & \longrightarrow & X_r(f) \end{array}$$

in which Π^b pushes J forward to a smooth almost complex structure J_b on $X_r(f)$. The J_b vary smoothly in b, and indeed extend by continuity to a smoothly X_n-parametrized family of almost complex structures on $X_r(f)$ (rather than just an X_n'-parametrized family). Since our sections of the $F^b\colon X_r^b(f^b) \to S^2$ pass through all of the fibers of F^b, restricting our almost complex structures to behave in this way near the special fibers of F^b will not prevent moduli spaces of J-holomorphic sections of the $X_r^b(f^b)$ from being of the expected dimension for generic $J \in \mathcal{J}$.

For $\alpha \in H^2(X;\mathbb{Z})$, $b \in X_n''$, and e_i $(i = 1, \dots, n)$ the Poincaré duals to the exceptional divisors of the blowups which form X^b, note that the expected complex dimension of the space of curves Poincaré dual to $\alpha - 2\sum e_i$ is $d(\alpha - 2\sum e_i) = d(\alpha) - 3n$, so since the the *real* dimension of X_n'' is $4n$ we would expect the space of such curves appearing in any X^b as b ranges over X_n'' to have complex dimension $d(\alpha) - n$.

Lemma 3.5. *Let $\alpha \in H^2(X;\mathbb{Z})$, and choose a generic set Ω of $d(\alpha) - n$ points in X. For generic $J \in \mathcal{J}$, and also for generic paths J_t in $\mathcal{J}$ connecting two such generic J, the spaces*

$$\mathcal{M}_{J,\Omega}^n(\alpha - 2\sum e_i) = \{(s,b) : b \in X_n'',\ s \in c_{\alpha-2\sum e_i} \subset \Gamma(X_r^b(f^b)),\ \bar{\partial}_J s = 0,\ \Omega \subset C_s\}$$

and

$$\mathcal{PM}^n_{(J_t),\Omega}(\alpha-2\sum e_i) = \{(s,b,t) : b \in X_n'',\ s \in c_{\alpha-2\sum e_i} \subset \Gamma(X_r^b(f^b)),\ \overline{\partial}_{J_t}s = 0,\ \Omega \subset C_s\}$$

are compact manifolds of real dimensions zero and one, respectively, provided that $r = \langle \alpha, [\Phi]\rangle \geq g + 3n$ *where* g *is the genus of the generic fiber of* $f\colon X \to S^2$.

Proof. That the dimensions will generically be as expected is a standard result (for the general theory of "parametrized Gromov–Witten invariants" of the sort that we are in the process of defining see [Ru], though the compactness result proved presently makes much of Ruan's machinery unnecessary for our purposes), so we only concern ourselves with compactness.

Let (s^m, b^m) be a sequence of J-holomorphic sections (or J_{t_m}-holomorphic sections with $J_{t_m} \to J$) from either of the sets at issue. *A priori*, there are two possible sources of noncompactness: the b^m might have a limit in $X_n \setminus X_n''$, or the b^m might converge to $b \in X_n''$ with the s^m converging to a bubble tree. As usual for section-counting invariants, we can eliminate the second possibility: because $J|_{X_r^b(f)}$ makes $X_r^b(f) \to S^2$ holomorphic, any bubbles must be contained in the fibers, and so the section component of the resulting bubble tree would descend to a set Poincaré dual to $\alpha - 2\sum e_i - PD(i_*B)$, where B is some class in one of the fibers $(f^b)^{-1}(t)$ of the fibration $f^b : X^b \to S^2$. If $(f^b)^{-1}(t)$ is irreducible, B will necessarily be a positive multiple of the fundamental class of the fiber, and just as in Section 4 of [Sm2] we will have $d(\alpha - 2\sum e_i - PD(i_*B)) \leq d(\alpha - 2\sum e_i) - (r - g + 1)$, which rules such bubble trees out for generic one-parameter families of J. If $(f^b)^{-1}(t)$ is reducible, with components in classes $[\Phi] - E$ and E, then B will have form $m([\Phi] - E) + pE$ where $m, p \geq 0$ and at least one is positive, and a routine computation then yields that

$$d(\alpha-2\sum e_i - PD(i_*B)) - d(\alpha-2\sum e_i) = -m(r-g+1) - \frac{5}{2}(p-m) - \frac{1}{2}(p-m)^2,$$

which, since we have assumed that $r \geq g+3$, will always be negative when $m, p \geq 0$ and are not both zero. Thus for generic J or J_t, none of the possible bubble trees appear.

There remains the issue that the b^m might converge to some $b \notin X_n''$. We rule this out in two steps: first, we prove:

Sublemma 3.6. *If* $b \in X_n \setminus X_n'$ *then* b^m *cannot converge to* b.

Proof of the sublemma. Let $\pi^{b^m}\colon X^{b^m} \to X$ be the blowdown map, and $\Pi^{b^m}\colon X_r^{b^m}(f^{b^m}) \to X_r(f)$ the map that it induces on relative Hilbert schemes. By the definition of our space $\mathcal{J}$ of almost complex structures, the $\Pi^{b^m} \circ s^m$ are J_{b^m}-holomorphic sections of $X_r(f)$ in the class c_α, and so converge modulo bubbling to a J^b-holomorphic section $\bar{s}$ of $X_r(f)$. In fact, we can rule out bubbling, since we can assume that the family J_b is regular as a $4n$-real-dimensional family of almost complex structures on $X_r(f)$, and so as above no bubbles can form in the limit thanks to the fact that all fibers of f are irreducible and

$$\begin{aligned} 2n + d(\alpha - mPD[\Phi]) &= d(\alpha) + 2n - m(r-g+1) \\ &\leq d(\alpha) - n - (r - g + 1 - 3n) < d(\alpha) - n \end{aligned}$$

by the hypothesis of the lemma.

Since $b \notin X_n'$, where $b = (p_1, \dots, p_n)$ there will be some minimal l such that p_{l+1} is a critical point of $f^{(p_1,\dots,p_l)}\colon X^{(p_1,\dots,p_l)} \to S^2$. Suppose first that p_{l+1} lies on just one irreducible component of its fiber (so that it is a double point of that component). Write $t^m = f^{(p_1^m,\dots,p_l^m)}(p_{l+1}^m)$ and $T = f^{(p_1,\dots,p_l)}(p_{l+1})$. Now since $C_{s^m} \subset X^b$ meets the exceptional divisor formed by blowing up p_{l+1}^m transversely exactly twice, we deduce that $\Pi \circ s^m \in \Gamma(X_r(f))$ acquires a tangency to the diagonal at a divisor containing two copies of $\pi^{b^m}(p_{l+1}^m)$; more specifically, assuming that $\bar{s}(T)$ corresponds to a divisor containing p_{l+1} with multiplicity q, for large m in a neighborhood U around $T, t^m \in S^2$ we have a decomposition $\Pi \circ s^m|_U = +(s_1^m, s_2^m)$ into disjoint summands $s_1^m \in \Gamma(X_q(f)|_U)$ and $s_2^m \in \Gamma(X_{r-q}(f)|_U)$, with s_1^m tangent to the diagonal at a point of form $\{p_{l+1}^m, p_{l+1}^m, x_3, \dots, x_q\}$. Since the divisors $s_1^m(t)$ and $s_2^m(t)$ are disjoint for $t \in U$, the smoothness of the $\Pi \circ s^m$ implies the smoothness of s_1^m and s_2^m over U. Similarly, where V is a neighborhood of p_{l+1} with $f(V) \subset U$ $\bar{s}$ splits near T into disjoint sections $\bar{s}_1$ of $\mathcal{H}_q \cong X_q(f|_V)$ and $\bar{s}_2$ of $X_{r-q}(f|_{f^{-1}(f(V))-V})$; here $\mathcal{H}_q$ is the q-fold relative Hilbert scheme of the map $(z, w) \mapsto zw$. Moreover, we have $s_1^m \to \bar{s}_1$. But then since $p_{l+1}^m \to p_{l+1}$, $\bar{s}_1$ must then be tangent to the diagonal in $\mathcal{H}_q$ at a point corresponding to $\{(0,0), \dots, (0,0)\} \in Sym^q\{zw = 0\}$. This, however, is impossible, since $\bar{s}_1$ is a *section* of $\mathcal{H}_q$, so that $Im(d\bar{s}_1)_T$ cannot be tangent to the fiber, whereas according to Theorem 6.5 in Section 6.2 the tangent cone to $\Delta \subset \mathcal{H}_q$ is contained in the tangent space to the fiber at $\bar{s}_1(T)$.

The other possibility is that p_{l+1} is an intersection point between two irreducible components of its fiber, in which case one of those components is the exceptional sphere E formed by a previous blowup (say at p_a). Where again $t^m = f^{(p_1^m,\dots,p_l^m)}(p_{l+1}^m)$, in local coordinate systems U^m around t^m (which may be shrinking but are scale-invariant) we have

$$\Pi \circ s^m = \{c_m z, d_m z\} + s_2^m(z)$$

where s_2^m is a local section of $X_{r-2}(f)$ which does not meet $z \mapsto \{c_m z, d_m z\}$. Now the fact that $p_m^{l+1} \to p_{l+1}$ which is an intersection point between the fiber containing $p_{l+1} \in X^{(p_1,\dots,p_l)}$ and the exceptional sphere of one of the blowups implies that, in X (where the blowup has not yet taken place), the two branches $c_m z$ and $d_m z$ of $\Pi \circ s^m$ near $\pi^{b^m}(p^{l+1})$ both tend toward the vertical, so that $c_m, d_m \to \infty$. But then this implies that $|d(\Pi \circ s^m)_{t^m}| \to \infty$, which is impossible by elliptic regularity since $\Pi \circ s^m \to \bar{s}$. $\square$

Finally we show that, generically, if $b^m \to b \in X'_n$ then in fact $b \in X''_n$. Indeed, since $b \in X'_n$, so that $X_r^b(f^b) \subset \mathcal{X}_r^n(f)$, Gromov compactness on the symplectic manifold $\mathcal{X}_r^n(f)$ implies that after passing to a subsequence the sections s^m will converge to some smooth section $\bar{s}$ of $X_r^b(f^b)$. Just as above, the fact that $\bar{s}$ is a smooth section implies that it misses the critical locus of $F^b\colon X_r^b(f^b) \to S^2$; in particular, if $b \in X'_n \setminus X''_n$, $Im(\bar{s})$ is contained in the smooth part of the relative Hilbert scheme $X_r^b(f^b)$. But then a neighborhood of $Im(\bar{s})$ in $X_r^b(f^b)$ will be diffeomorphic to a neighborhood of $Im(s^m)$ in $X_r^{b^m}(f^{b_m})$ for large m, and so the index of the Cauchy-Riemann operator acting on perturbations of the former will be the same as the index of the Cauchy-Riemann operator acting on perturbations of the latter, namely $d(\alpha) - 3n$. Hence since the real dimension of $X'_n \setminus X''_n$ is $4n-2$, the expected complex dimension of the space of possible limits $\bar{s}$ with $b \in X'_n \setminus X''_n$ is $d(\alpha) - n - 1$, so for generic J, and also for generic one-real-parameter families J_t, on $\mathcal{X}_r^n(f)$, no such limits $\bar{s}$ with $C_{\bar{s}}$ satisfying our $d(\alpha) - n$ incidence conditions will exist. $\square$

Given this compactness result, the standard cobordism argument permits us to make the following definition.

Definition 3.7. *Let α be as in Lemma 3.5. $\mathcal{FDS}_f^n(\alpha - 2\sum e_i)$ is then defined as the number of elements, counted with sign according to the spectral flow, in the moduli space $\mathcal{M}^n_{J,\Omega}(\alpha - 2\sum e_i)$ for generic J and Ω as in Lemma 3.5.*

Theorem 3.8. *Suppose that α is as in Lemma 3.5 and is strongly n-semisimple. Then*

$$n!Gr_n(\alpha) = \mathcal{FDS}_f^n\left(\alpha - 2\sum e_i\right),$$

provided that $\langle \omega_X, [\Phi]\rangle > \omega_X \cdot \alpha \geq g(\Phi) + 3n$.

Proof. As in the proof of Theorem 2.11, we may evaluate $Gr_n(\alpha)$ using an almost complex structure j which makes the Lefschetz fibration f pseudoholomorphic and which has the property that, for any of the curves $C = \bigcup_i C^i$ being counted by $Gr_n(\alpha)$, j is integrable on a neighborhood of $\bigcup_i Crit(f|_{C^i})$; each intersection point between the C^i occurs away from $\bigcup_i Crit(f|_{C^i})$; and C misses the critical locus of the fibration f. For each $b \in X_n$, let j_b be pullback of j via the blowup $\pi^b\colon X^b \to X$ (see Section 6.1 for the proof that j_b exists and is Lipschitz), so that $X^b \to X$ is (j_b, j)-holomorphic. Then, for any of the $n!$ elements b of X'_n corresponding to the $n!$ different orders in which the nodes of C may be blown up, the proper transform $\tilde{C}$ of C will be a curve in X^b (with $b \in X'_n$ as a result of the fact that C misses the critical points of f) Poincaré dual to $\alpha - 2\sum e_i$. In fact, we claim that for a generic initial choice of j these proper transforms $\tilde{C}$ are guaranteed to be the only j_b-holomorphic curves Poincaré dual to $\alpha - 2\sum e_i$ in any X^b which have no components contained in the fibers of $f^b\colon X^b \to S^2$.

Indeed, suppose that $\tilde{C} = \cup_i \tilde{C}_i$ is a j_b-holomorphic curve in one of the X^b Poincaré dual to $\alpha - 2\sum e_i$, with the (possibly-multiply-covered) components $\tilde{C}_i$ Poincaré dual to $\beta_i - \sum c_{ik} e_k$. We need to show that, where $\pi^b\colon X^b \to X$ is the blowup, $\pi^b(\tilde{C})$ has n nodes, located at the points $p_i, \dots, p_n$ which were blown up to form X^b (as $\pi^b(\tilde{C})$ is obviously a j-holomorphic curve Poincaré dual to α). Now for each k, $\sum_i c_{ik} = -2$, while by positivity of intersections in X^b, we have each $c_{ik} \leq 0$. If k is such that there are distinct q and s with $c_{qk} = c_{sk} = -1$, then the curves $\pi(\tilde{C}_q)$ and $\pi(\tilde{C}_s)$ intersect transversely at the point p_k, contributing the desired node. On the other hand, if k is such that the only nonzero c_{ik} is some $c_{qk} = -2$, then $\pi^b(C_q)$ might *a priori* be either a singly-covered curve Poincaré dual to β_q which has a self-intersection at p_k, or a double cover of a curve in class $\beta_q/2$ which passes through p_k. However, the n-semisimplicity condition rules the second possibility out for generic choices of j, since we will have either $d(\beta_q/2) < 0$ or $d(\beta_q/2) < d(\beta_q) - n \leq d(\alpha) - n$, and so no such curves satisfying our incidence conditions will exist.

We conclude, then, that the only j_b-holomorphic curves $\tilde{C}$ in any X^b Poincaré dual to $\alpha - 2\sum e_i$ are proper transforms of j-holomorphic curves which contribute to $Gr_n(\alpha)$. With this established, the proof of the theorem becomes almost just an application of our usual methods. Since the restriction of j_b to the exceptional spheres is standard, we can choose smooth almost complex structures j'_b which are integrable near the exceptional spheres and are C^0-close to the j^b. By Gromov compactness for C^0 convergence of almost complex structures [IS] and the fact that $d(\alpha - 2\sum e_i) = -n$, we deduce as usual that for generic choices of these perturbed j'_b each $\tilde{C}$ will have finitely many j'_{b_i}-holomorphic curves $\tilde{C}_1, \dots, \tilde{C}_N$ near it (for various b_i near b). On the relative Hilbert schemes we have almost complex structures $\mathbb{J}_{j'_b}$. If $\tilde{C}_i$ is one of the curves above with the intersections of its components resolved by the blowup $X^{b_i} \to X$, we define $r''(\tilde{C}_i)$ as the signed count of $J_{b'}$ holomorphic sections of $X_r^{b'}(f^{b'})$ near $s_{\tilde{C}_i}$ for b' near b_i and $J_{b'}$ a generic family of smooth almost complex structures Hölder-close to the $\mathbb{J}_{j_{b_i}}$.

For C a curve contributing to the Gromov invariant with nodes resolved by $X^b \to X$ and proper transform $\tilde{C}$, we define the contribution $r'(C)$ of C to $\mathcal{FDS}$ as $\sum_{i=1}^n r''(\tilde{C}_i)$ where the $\tilde{C}_i$ are obtained as above. When j is integrable near C, each $j_{b'}$ will be integrable near $\tilde{C}$ and near the exceptional spheres of $X^{b'}$ for b' near b,, so that the first perturbation of the $j_{b'}$ to $j'_{b'}$ is not necessary and the only $\tilde{C}_i$ is $\tilde{C}$ itself. Moreover, each $\mathbb{J}_{j_{b'}}$ will be integrable near $s_{\tilde{C}}$ for b' near b, and so (under suitable nondegeneracy assumptions) both contributions will be 1. Further, exactly as in the proof of Theorem 2.11, the contributions transform under variations in j in the same way by virtue of the fact that $\mathcal{FDS}$ is independent of the almost complex structure used to define it. The agreement of the invariants then follows. □

If α is only weakly n-semisimple, then if $C \in PD(\alpha)$ is the disjoint union of a double cover of a square-zero torus with a curve having $n-1$ nodes, then the proper transform of C under blowup at the nodes of C and at any point on the torus gives a curve in some X^b Poincaré dual to $\alpha - 2\sum e_i$, even though C does not contribute to $Gr_n(\alpha)$. On perturbing the family $(\mathbb{J}_{j_b})$ on $\mathcal{X}_r^n(f)$ to a generic family (J_b), we might find that the sections corresponding to these curves contribute to $\mathcal{FDS}_f^n(\alpha - 2\sum e_i)$. It seems reasonable, though, to believe that these additional contributions could be expressed in terms of the various other Gromov invariants of X, consistently with Conjecture 1.3.

4. A review of Smith's constructions

Our vanishing theorem for $\mathcal{FDS}$ will follow by adapting the constructions found in Section 6 of [Sm2] to the family context. Let us review these.

In addition to the relative Hilbert scheme, Donaldson and Smith constructed from the Lefschetz fibration $f\colon X \to S^2$ a *relative Picard scheme* $P_r(f)$ whose fiber over a regular value $t \in S^2$ is naturally identified with the Picard variety $Pic^r\Sigma_t$ of degree-r line bundles on Σ_t. Over each Σ_t, we have an Abel–Jacobi map $S^r\Sigma_t \to Pic^r\Sigma_t$ mapping a divisor D to its associated line bundle $\mathcal{O}(D)$; letting t vary over S^2, we then get a map

$$AJ\colon X_r(f) \to P_r(f)$$

(that all of these constructions extend smoothly over the critical values of $f\colon X \to S^2$ is seen in the Appendix of [DS]). Meanwhile, by composing the Abel–Jacobi map for effective divisors of degree $2g-2-r$ with the Serre duality map $\mathcal{L} \mapsto \kappa_{\Sigma_t} \otimes \mathcal{L}^\vee$, we obtain a map

$$\begin{aligned} i\colon X_{2g-2-r}(f) &\to P_r(f) \\ D &\mapsto \mathcal{O}(\kappa - D). \end{aligned} \tag{4.1}$$

Moreover, using a result from Brill-Noether theory due to Eisenbud and Harris [EH], Smith obtains that (cf. Theorem 6.1 and Proposition 6.2 of [Sm2]):

Lemma 4.1. *(*[Sm2]*) For a generic choice of fiberwise complex structures on X, if $3r > 4g-11$ where g is the genus of the fibers of $f\colon X \to S^2$, then $i\colon X_{2g-2-r}(f) \to P_r(f)$ is an embedding. Further, $AJ\colon X_r(f) \to P_r(f)$ restricts to $AJ^{-1}(i(X_{2g-2-r}(f)))$ as a $\mathbb{P}^{r-g+1}$-bundle, and is a $\mathbb{P}^{r-g}$-bundle over the complement of $i(X_{2g-2-r}(f))$.*

The reason for this is that in general $AJ^{-1}(\mathcal{L}) = \mathbb{P}H^0(\mathcal{L})$, which by Riemann-Roch is a projective space of dimension $r-g+h^1(\mathcal{L})$. The result of [EH] ensures that for $r > (4g-11)/3$ and for generic families of complex structures on the Σ_t, none of the fibers of f admit any line bundles $\mathcal{L}$ with degree r and $h^1(\mathcal{L}) > 1$; then $Im(i) \subset P_r(f)$ consists of those bundles for which $h^1(\mathcal{L}) = h^0(\kappa \otimes \mathcal{L}^\vee) = 1$. To see the bundle structure, rather than just set-theoretically identifying the fibers, note that on any Σ_t, when we identify the tangent space to $Pic^r\Sigma_t$ with $H^0(\kappa_{\Sigma_t})$, the orthogonal complement of the linearization $(AJ_*)_D$ at $D \in S^r\Sigma_t$ consists of those elements of $H^0(\kappa_{\Sigma_t})$ which vanish along D (this follows immediately from the fact that, after choosing a basepoint $p_0 \in \Sigma_t$ and a basis $\{\phi_1, \dots, \phi_g\}$ for $H^0(\kappa_{\Sigma_t})$ in order to identify $Pic^r(\Sigma_t)$ with $\mathbb{C}^g/H^1(\Sigma_t, \mathbb{Z})$, AJ is given by $AJ(\sum p_i) = \left(\sum \int_{p_0}^{p_i} \phi_1, \dots, \sum \int_{p_0}^{p_i} \phi_g\right)$). If $AJ(D) \notin Im(i)$, so that $H^0(\kappa - D) = 0$, this shows that $(AJ_*)_D$ is surjective, so that AJ is indeed a submersion away from $AJ^{-1}(Im\, i)$. Meanwhile, if $\mathcal{L} = i(D') \in Im(i)$, the above description shows that the only directions in the orthogonal complement of any $Im(AJ_*)_D$ with $AJ(D) = \mathcal{L}$ are those 1-forms which vanish at D, but since $AJ(D) = i(D')$ such 1-forms also vanish at D' and so are also orthogonal to $Im(i_*)_{D'}$. So if $AJ(D) = i(D')$, $Im(AJ_*)_D$ contains $T_{i(D')}(Im\, i)$, implying that AJ does in fact restrict to $AJ^{-1}(Im\, i)$ as a submersion and hence as a $\mathbb{P}^{r-g+1}$ bundle.

Smith's duality theorem, and also the vanishing result in this paper, depend on the construction of almost complex structures which are especially well-behaved with respect to the Abel-Jacobi map. From now on, we will fix complex structures on the fibers of X satisfying the conditions of Lemma 4.1; these induce complex structures on the fibers of the $X_r(f)$ and $P_r(f)$, but on all of our spaces (including X) we still have the freedom to vary the "horizontal-to-vertical" parts of the almost complex structures. Almost complex structures agreeing with these fixed structures on the fibers will be called "compatible."

The following is established in the discussion leading to Definition 6.4 of [Sm2].

Lemma 4.2. *(*[Sm2]*) In the situation of Lemma 4.1, for any compatible almost complex structure J_1 on $X_{2g-2-r}(f)$ and any compatible J_2 on $P_r(f)$ such that $J_2|_{T(Im\, i)} = i_*J_1$, there exist compatible almost complex structures J on $X_r(f)$ with respect to which $AJ\colon X_r(f) \to P_r(f)$ is (J, J_2)-holomorphic.*

We outline the construction of J: Since $AJ\colon AJ^{-1}(Im\, i) \to X_{2g-2-r}(f)$ is a $\mathbb{P}^{r-g+1}$-bundle, given the natural complex structure on $\mathbb{P}^{r-g+1}$ and the structure J_1, the structures on $AJ^{-1}(Im\, i)$ making this fibration pseudoholomorphic correspond precisely to connections on the bundle; since this bundle is the projectivization of the vector bundle with fiber $H^0(\kappa - D)$ over D, a suitable connection on the latter gives rise to a connection on our projective-space bundle and thence to an almost complex structure J on $AJ^{-1}(Im\, i)$ making the restriction of AJ pseudoholomorphic.

To extend J to all of $X_r(f)$, we first use the fact that, as in Lemma 3.4 of [DS],

$$AJ_*\colon \left(N_{AJ^{-1}(Im\, i)}X_r(f)\right)|_{AJ^{-1}(i(D))} \to (N_{Im\, i}P_r(f))_{i(D)}$$

is modeled by the map

$$\{(\theta,[x]) \in V^* \times \mathbb{P}(V) | \theta(x) = 0\} \to V^*$$
$$(\theta,[x]) \mapsto \theta,$$

where $V = H^0(\kappa_{\Sigma_t} - D)$, so that the construction of Lemma 5.4 of [DS] lets us extend J to the closure of some open neighborhood U of $AJ^{-1}(Im\, i)$. But then since AJ is a $\mathbb{P}^{r-g}$-bundle over the complement of $AJ^{-1}(Im\, i)$, the problem of extending J suitably to all of $X_r(f)$ amounts to the problem of extending the connection induced by J from ∂U to the entire bundle, which is possible because, again, our bundle is the projectivization of a vector bundle and connections on vector bundles can always be extended from closed subsets.

Our vanishing results are consequences of the following:

Lemma 4.3. *([Sm2],p.965) Assume that $b^+(X) > b_1(X)+1$. For any fixed compatible smooth almost complex structure J_1 on $X_{2g-2-r}(f)$ and for generic smooth compatible almost complex structures J_2 such that $J_2|_{Im\, i} = i_* J_1$, all J_1-holomorphic sections of $P_r(f)$ are contained in $i(X_{2g-2-r}(f))$.*

This follows from the fact that, as Smith has shown, the index of the $\bar{\partial}$-operator on sections of $P_r(f)$ is $1 + b_1 - b^+$, which under our assumption is negative, and so since J_2 may be modified as we please away from $Im\, i$, standard arguments show that for generic J_2 as in the statement of the lemma all sections will be contained in $Im\, i$.

5. Proof of Theorem 1.5

Lemma 5.1. *If $b^+(X) > b_1(X) + 4n + 1$, then $\mathcal{FDS}^n_f(\alpha - 2\sum e_i) = 0$ for all $\alpha \in H^2(X;\mathbb{Z})$ such that $r = \langle \alpha, [\Phi]\rangle$ satisfies $r > \max\{g(\Phi) + 3n, 2g(\Phi) - 2\}$.*

Proof. Let $(J'_b)_{b\in X_n}$ be a smooth family of almost complex structures on the relative Picard scheme $P_r(f)$ such that

(i) For each b, the map $G\colon\ P_r(f) \to S^2$ is pseudoholomorphic with respect to J'_b, and for all critical values t of f J agrees near $G^{-1}(t)$ with the standard complex structure on the relative Picard scheme induced by an integrable complex structure near $f^{-1}(t)$;

(ii) For each $b = (p_1, \dots, p_n)$, where $t_i = f \circ \pi^{(p_1,\dots,p_{i-1})}(p_i)$, J'_b also agrees near each $G^{-1}(t_i)$ with the standard complex structure induced by an integrable complex structure near $f^{-1}(t_i)$.

Thanks to the assumption that $b^+(X) > b_1(X) + 4n + 1$ and the fact that the index of the $\bar{\partial}$-operator on sections of $P_r(f)$ is $1 + b_1 - b^+$, for a generic such family $(J'_b)_{b\in X_n}$ there will be no J'_b holomorphic sections of $P_r(f)$ for any b. Now, as in Section 4, since $r > 2g - 2$, so that $AJ\colon\ X_r(f) \to P_r(f)$ is a projective-space bundle, we can construct a family J_b of almost complex structures on $X_r(f)$ such that $AJ\colon\ X_r(f) \to P_r(f)$ is (J_b, J'_b)-holomorphic for each b. By construction, for each b J_b agrees with the standard complex structure on the relative Hilbert scheme $F\colon\ X_r(f) \to S^2$ near each singular fiber and also near each $F^{-1}(t_i)$ where the t_i are as above. Since X^b is formed from X by performing blowups at points in $f^{-1}(t_i)$, for $b \in X'_n$ J_b lifts to an almost complex structure $\tilde{J}_b$ on $X^b_r(f^b)$ such that the map $\Pi^b\colon\ X^b_r(f^b) \to X_r(f)$ induced by blowup is $(\tilde{J}_b, J_b)$-holomorphic.

Let J^m be almost complex structures on the $\mathcal{X}_r^n(f)$ from the Baire set in the definition of $\mathcal{FDS}$ which converge to an almost complex structure that agrees on each $X_r^b(f^b)$ with $\tilde{J}_b$. If the invariant were nonzero, we would obtain J^m-holomorphic sections s^m of some $X_r^{b_m}(f^{b_m})$ ($b_m \in X_n'$); after passing to a subsequence we assume $b_m \to \bar{b} \in X_n$ (since X_n, though not X_n', is compact). By the definition of our class of almost complex structures (see the text before Lemma 3.5) there are compatible almost complex structures $J_{b_m}^m$ on $X_r(f)$ such that $\Pi^{b_m}\colon X_r^{b_m}(f^{b_m}) \to X_r(f)$ is $(J^m, J_{b_m}^m)$-holomorphic; further, we will have $J_{b_m}^m \to J_{\bar{b}}$. So the $\Pi^{b_m} \circ s^m$ are $J_{b_m}^m$-holomorphic sections of $X_r(f)$, whence after passing to a subsequence they converge modulo bubbling to a $J_{\bar{b}}$-holomorphic section $\bar{s}$. (As usual, even if bubbling occurs, the bubble tree will contain a component which is a $J_{\bar{b}}$-holomorphic section by virtue of the fact that all bubbles will be contained in the fibers.) But then $AJ \circ \bar{s}$ would be a $J_{\bar{b}}'$-holomorphic section, contradicting the fact that no J_b'-holomorphic sections exist for any $b \in X_n$. □

The intermediate case where $\max\{g(\Phi) + 3n + d(\alpha), (4g(\Phi) - 11)/3\} < r \leq 2g(\Phi) - 2$ takes slightly more work. In this case, as in Section 4 we use the fact that combining the Abel-Jacobi map with Serre duality gives a map

$$i\colon X_{2g-2-r}(f) \to P_r(f);$$

as before since $3r > 4g - 11$ generic choices of the complex structures on the fibers of f result in this map being an embedding. Similarly to the proof of Lemma 5.1, consider families of almost complex structures J_b'' ($b \in X_n$) on $X_{2g-2-r}(f)$ which make $X_{2g-2-r}(f) \to S^2$ holomorphic and are standard near the singular fibers and near the fibers containing the points which are blown up to form X^b. Form almost complex structures J_b' on $P_r(f)$ restricting to $i(X_{2g-2-r}(f))$ as $i_* J_b''$ and which are also standard near the singular fibers and near the fibers containing the points which are blown up to form X^b. The fact that $b^+ > b_1 + 1 + 4n$ implies that if the family J_b' is chosen generically among almost complex structures with this property, then any J_b' holomorphic sections of $P_r(f)$ for any b must be contained in $i(X_{2g-2-r}(f))$.

We then form almost complex structures J_b on $X_r(f)$ such that $AJ\colon X_r(f) \to P_r(f)$ is (J_b, J_b')-holomorphic. As in the proof of Lemma 5.1, a nonvanishing invariant $\mathcal{FDS}_f^n(\alpha - 2\sum e_i)$ would give rise to a sequence of sections of $X_r(f)$ in the homotopy class c_α which converge modulo bubbling to a $J_{\bar{b}}$-holomorphic section $\bar{s}$ of $X_r(f)$. Since all fibers of f are irreducible, any bubbles that arise will descend to a multiple covering of one of the fibers of f, and so for some $m \geq 0$ we will have $\bar{s} \in c_{\alpha - mPD[\Phi]}$ where as usual $[\Phi]$ is the class of the fiber.

$AJ \circ \bar{s}$ will then be a $J_{\bar{b}}'$-holomorphic section of $P_r(f)$, and so must be contained in $i(X_{2g-2-r}(f))$. By the construction of i, then, $i^{-1} \circ AJ \circ \bar{s}$ is a $J_{\bar{b}}''$-holomorphic section of $X_{2g-2-r}(f)$ in the homotopy class $c_{\kappa_X - \alpha + mPD[\Phi]}$.

Now one computes using the adjunction formula for the fiber Φ that

$$\begin{aligned} d(\kappa_{X^b} - \alpha + mPD[\Phi]) &= d(\kappa_X - \alpha) + d(m\Phi) + m\langle \kappa_X - \alpha, [\phi]\rangle \\ &= d(\alpha) - \frac{m}{2}\langle \kappa_X, [\Phi]\rangle + m\langle \kappa_X - \alpha, [\Phi]\rangle \\ &= d(\alpha) - m(r - g(\Phi) + 1). \end{aligned}$$

Thus by choosing the $4n$-real-dimensional family J''_b generically we ensure that $m = 0$ thanks to the assumption that $r > d(\alpha) + g(\Phi) + 3n$ in the statement of the theorem.

Now take a family of almost complex structures j_b on X which are standard near the singular fibers of the fibrations f and also near the fibers containing the points blown up to form X^b; these induce tautological almost complex structures $\mathbb{J}_{j_b}$ on $X_{2g-2-r}(f)$. Let J''^m_b be families of smooth almost complex structures on $X_{2g-2-r}(f)$ which are generic in the sense of the previous paragraph and which converge in Hölder norm to the $\mathbb{J}_{j_b}$. For each m there is some b_m such that $J''^m_{b_m}$ admits a holomorphic section in the class $c_{\kappa_X-\alpha}$, so Gromov compactness guarantees the existence of a $\mathbb{J}_{j_{b_0}}$-holomorphic section of some $X_{2g-2-r}(f)$ in $c_{\kappa_X-\alpha}$ for some b_0; this section then tautologcally corresponds to a j_{b_0}-holomorphic curve C Poincaré dual to $\kappa_X - \alpha$; setting $j = j_{b_0}$, this is the curve that we desire.

To get the j-holomorphic curve Poincaré dual to α, we simply consider the almost complex structures j_b on the members X^b of the family blowup induced in the almost complex category by j. Let j^m_b be a sequence of almost complex structures C^0-approximating the j_b which are integrable near the exceptional spheres, and apply Gromov compactness to a sequences of almost complex structures on $\mathcal{X}^n_r(f)$ whose restrictions to $X^b_r(f^b)$ Hölder-approximate the family $\mathbb{J}_{j^m_b}$; in this way our nonvanishing invariant guarantees the existence of a $\mathbb{J}_{j^m_{b_m}}$-holomorphic section of some $X^{b_m}_r(f^{b_m})$ in the class $c_{\alpha-2\sum e_i}$ and so of a $j^m_{b_m}$-holomorphic curve Poincaré dual to $\alpha - 2\sum e_i$. Appealing to Gromov compactness for these curves then gives a j_b-holomorphic curve, and this latter is sent by the blowdown map to the j-holomorphic curve which we desire. Theorem 1.5 is thus proven.

If X admits an integrable complex structure j making the fibration holomorphic, then for our original family of almost complex structures j_b we can take the constant family j, justifying a statement made near the end of the introduction. For arbitrary j, though, this argument does not work, because it was crucial in the construction of the curve Poincaré dual to $\kappa_X - \alpha$ that each of the j_b was integrable near the fibers containing the points blown up in forming X^b.

6. Two technical matters

6.1. **Blowing up a point in an almost complex manifold.** In the proof of Theorem 3.8 we have used the fact that, if $\pi\colon X' \to X$ is the blowup of a 4-manifold at a point and J is an almost complex structure on X, then there is a Lipschitz almost complex structure J' on X' such that π is (J', J)-holomorphic. Since we have not found a proof of this fact in the literature, we present one here. As the dimension of X does not affect the argument, we prove the result for almost complex manifolds of arbitrary complex dimension n. The blowup, of course, has the effect of replacing the point p being blown up with an exceptional divisor $E \cong \mathbb{C}P^{n-1}$; we note that, as will be seen in the proof, $J'|_{TE}$ agrees with the standard complex structure on $\mathbb{C}P^{n-1}$. If (X, ω) is symplectic, recall from, *e.g.*, Chapter 7 of [MS2] that X' can be endowed with symplectic forms ω_ϵ for small $\epsilon > 0$, with the parameter ϵ reflecting the size of the exceptional divisor E in the symplectic manifold (X', ω_ϵ). One can easily check that if the almost complex structure J on X is ω-tame, then J' will be ω_ϵ-tame for small enough ϵ.

Our method only proves Lipschitz regularity for J'; it is unclear whether J' is differentiable in directions normal to E. In principle, one would also like to be able to blow up almost complex submanifolds $V \subset (X, J)$ of arbitrary dimension in the almost complex category. Our method does not readily extend to show that the pullback of J under the blowup extends even continuously over the exceptional divisor of the blowup when $\dim V > 0$. Nonetheless, the case of blowing up a point suffices for our application.

We begin with the following lemma, which will later be used to construct coordinate charts on the blowup.

Lemma 6.1. *Let J be an almost complex structure on $\mathbb{C}^n$ agreeing at the origin with the standard complex structure J_0. Given $\kappa_0 \in \mathbb{C}P^{n-1}$ there exists a constant ρ_0 with the following property. Let $\rho < \rho_0$ and let U_ρ be the ball of radius ρ around κ_0 in $\mathbb{C}P^{n-1}$ and D_ρ the disc of radius ρ in $\mathbb{C}$. There is a smooth map*

$$\Theta\colon\ D_\rho \times U_\rho \to \mathbb{C}^n$$

such that each $\Theta|_{D_\rho \times \{\kappa\}}$ ($\kappa \in U_\rho \subset \mathbb{C}P^{n-1}$) is an embedding whose image is a J-holomorphic disc which is tangent at the origin to the line $l_\kappa \subset \mathbb{C}^n$ determined by κ.

Proof. The proof quite closely parallels some of the arguments in Section 5 of [T1]; we outline it for completeness. By a complex linear change of coordinates we may assume that $\kappa_0 = [1 : 0 : \cdots : 0]$. Where $c = (c_1, \ldots, c_{n-1}) \in (D_\rho)^{n-1}$ and $\kappa = [1 : \kappa_1 : \cdots : \kappa_{n-1}]$ is close to $[1 : 0 : \cdots : 0]$, we search for a J-holomorphic disc

$$q_{c,\kappa}(z) = (z, c_1 + \kappa_1 z + u_1(c, \kappa, z), \ldots, c_{n-1} + \kappa_{n-1} z + u_{n-1}(c, \kappa, z))$$

defined for $z \in D_\rho$. As in [T1], this is equivalent to a system of equations

$$\frac{\partial u_i}{\partial \bar{z}} = Q_i\left(c, \kappa, u_1(c, \kappa, z), \ldots, u_{n-1}(c, \kappa, z)\right)$$

such that for certain constants γ_k we have

$$\|Q_i\|_{C^k} \leq \gamma_k \|J - J_0\|_{C^k(D^n_{2\rho})}. \tag{6.1}$$

Note that by decreasing ρ and rescaling the coordinates we can make the right hand side of (6.1) as small as we like.

Now introduce a cutoff function $\chi_\rho\colon \mathbb{C} \to [0, 1]$ which equals 1 for $|z| < \rho$ and 0 for $|z| > 3\rho/2$, and search for a solution to

$$\frac{\partial u_i}{\partial \bar{z}} = \chi_\rho Q_i \quad (i = 1, \ldots, n-1)$$

by, on the class of $(n-1)$-tuples of $C^{2,1/2}$ functions u_i restricting to the circle of radius 4ρ around zero in the span of $\{e^{ik\theta} | k < 0\}$, searching for a tuple $(u_1, \ldots, u_{n-1})$ obeying

$$\left(u_i(z) = \frac{1}{\pi} \int \frac{\chi_\rho Q_i(c, \kappa, u_i(c, z))}{z - w} d^2 w \right)_{i=1,\ldots,n-1} \tag{6.2}$$

Applying the contractive mapping theorem on this class of functions (viewed as a Banach space using the $(n-1)$-fold direct sum of the norm used on p. 886 of [T1]), thanks to the smallness of the Q_i we can find a unique small solution of (6.2). Furthermore as in Lemma 5.5 of [T1] the solution varies smoothly in each of z, c, and κ, and satisfies bounds

$$\left|\frac{\partial u}{\partial c_i}\right| < C\rho, \quad \left|\frac{\partial u}{\partial \kappa_i}\right| < C\rho^2, \quad \|u\|_{C^0} < C(\rho^2 + \rho(|c| + |\kappa|)), \quad \|u\|_{C^1} < C(\rho + (|c| + |\kappa|)).$$

Letting σ denote the map which assigns to (c, κ) the pair consisting of $q_{c,\kappa}(0)$ and the tangent space to $Im\, q_{c,\kappa}$ at $q_{c,\kappa}(0)$, the implicit function theorem then allows us to solve the equation $\sigma(c, \tilde{\kappa}) = ((0, \ldots, 0), \kappa)$ for c and $\tilde{\kappa}$ in terms of κ. The desired map Θ is then

$$\begin{aligned}\Theta\colon\ D_\rho \times U_\rho &\to \mathbb{C}^n \\ (z, \kappa) &\mapsto q_{c(\kappa),\tilde{\kappa}(\kappa)}(z).\end{aligned}$$

□

For any even-dimensional manifold X with $p \in X$, we form the blowup X' of X at p as a topological manifold by removing a ball B^{2n} around p, embedding B^{2n} in $\mathbb{C}^n$ in standard fashion, and replacing B^{2n} in X by $B' = \{(l, e) \in \mathbb{C}P^{n-1} \times \mathbb{C}^n | e \in l \cap B^{2n}\}$. The blowdown map $\pi\colon\ X' \to X$ is of course just the identity outside B' and the map $(l, e) \to e$ inside B'. The exceptional divisor is $E = \{(l, e) \in B' | e = 0\} \subset X'$.

If $\kappa_0 = [1 : 0 : \cdots : 0]$ in Lemma 6.1 and we write κ near κ_0 as $[1 : \kappa_1 \cdots : \kappa_{n-1}]$, the map Θ has the form

$$(z, \kappa) \mapsto (z, \kappa_1 z + \tilde{u}_1(\kappa, z), \ldots, \kappa_{n-1} z + \tilde{u}_{n-1}(\kappa, z))$$

where the $\tilde{u}_i$ are smooth functions satisfying $|\tilde{u}_i(\kappa, z)| < C|z|^2$ for an appropriate constant C. (In the notation of the proof of Lemma 6.1, $\tilde{u}_i(\kappa, z) = u_i(c(\kappa), \tilde{\kappa}(\kappa), z) + (\tilde{\kappa}_i(\kappa) - \kappa_i)z$.)

We hence obtain a local homeomorphism $\tilde{\Theta} = \tilde{\Theta}_{\kappa_0}\colon\ D^2 \times D^2 \to \tilde{\mathbb{C}}^n$ such that, where $\pi\colon\ \tilde{\mathbb{C}}^n \to \mathbb{C}^n$ is the blowdown, $\pi \circ \tilde{\Theta} = \Theta$. We use the $\tilde{\Theta}_{\kappa_0}$ as κ_0 varies over $\mathbb{C}P^{n-1}$ as an atlas for $\tilde{\mathbb{C}}^n$ near the exceptional divisor E (away from E we of course just use charts pulled back by π from charts on $\mathbb{C}^n$ not containing the origin). From the definition of the $\tilde{\Theta}_{\kappa_0}$ and the fact that tangencies of J-holomorphic curves in $\mathbb{C}^n$ are C^1-diffeomorphic to tangencies between J_0-holomorphic curves [Si], one can see that the transition functions have the form

$$\tilde{\Theta}^{-1}_{\kappa_0} \circ \tilde{\Theta}_{\kappa_0'}(z, \kappa_1, \ldots, \kappa_{n-1}) =$$

$$\left(z, \kappa_1 + z^{-1}(f_1(\kappa)z^2 + O(|z|^3)), \ldots, \kappa_{n-1} + z^{-1}(f_{n-1}(\kappa)z^2 + O(|z|^3))\right),$$

and in particular are C^1. We have thus provided an atlas for $\tilde{\mathbb{C}}^n$ as a C^1 manifold.

This atlas depends on the almost complex structure J, and it is worth noting that the charts corresponding to different J might not be C^1-related. For example, for a particular J $\Theta_{[1:0:\cdots:0]}$ could conceivably have the form

$$\Theta_{[1:0:\cdots:0]}(z, \kappa_1, \ldots, \kappa_n) = (z, \kappa_1 z + \bar{z}^2, \kappa_2 z, \ldots, \kappa_{n-1} z).$$

In this case, in terms of the *standard* smooth coordinates on $\tilde{\mathbb{C}}^n$ (equivalently, those induced by the above construction using the standard complex structure J_0),

$$\tilde{\Theta}_{[1:0:\cdots:0]}(z, \kappa_1, \ldots, \kappa_{n-1}) = (z, \kappa_1 + \bar{z}^2/z, \kappa_2, \ldots, \kappa_{n-1}),$$

which is Lipschitz but not C^1 along the exceptional divisor $\{z = 0\}$. Of course, these resulting manifolds are still abstractly C^1-diffeomorphic; this is somewhat reminiscent of the fact that distinct complex structures on a Riemann surface Σ induce smooth charts on the symmetric products $S^d\Sigma$ which are related by transition maps that are only Lipschitz, as noted for instance in Remark 4.4 of [Sa].

Proposition 6.2. *Let $\pi\colon \tilde{\mathbb{C}}^n \to \mathbb{C}^n$ denote the blowup of $\mathbb{C}^n$ at the origin, and let J be an almost complex structure on $\mathbb{C}^n$ agreeing with the standard almost complex structure J_0 at the origin. Then there is a unique Lipschitz continuous almost complex structure $\tilde{J}$ on $\tilde{\mathbb{C}}^n$ such that π is a $(\tilde{J}, J)$ holomorphic map.*

Proof. Let $E \cong \mathbb{C}P^{n-1}$ denote the exceptional divisor of the blowup π. Of course, π restricts to a diffeomorphism $\tilde{\mathbb{C}^n} \setminus E \to \mathbb{C}^n \setminus (0, \ldots, 0)$, so our $\tilde{J}$ must agree away from E with $\pi^* J = \pi_*^{-1} \circ J \circ \pi_*$ away from E and uniqueness even of a continuous almost complex structure $\tilde{J}$ is clear from the fact that $\tilde{\mathbb{C}}^n \setminus E$ is dense in $\tilde{\mathbb{C}}^n$. We show now that $\pi^* J$ extends over E in Lipschitz fashion by exhibiting a Lipschitz continuous basis of vector fields for its antiholomorphic tangent space $T^{0,1} \subset T\tilde{\mathbb{C}}^n \otimes \mathbb{C}$ near any given point $x \in E$.

Lemma 6.1 and the remarks thereafter provide us with one element of this basis: the maps Θ_{κ_0} map each $D_\rho \times \{\kappa\}$ diffeomorphically to a J-holomorphic disc Δ_κ in $\mathbb{C}^n$ in a way that varies smoothly in κ. We then obtain a (complexified) vector field $\tilde{\alpha}_\kappa$ along each $D_\rho \times \{\kappa\}$ defined by the property that $\alpha_\kappa = (\Theta_{\kappa_0})_* \tilde{\alpha}$ generates the J-antiholomorphic tangent space to Δ_κ. Choosing the α_κ to depend smoothly on κ causes the $\tilde{\alpha}_\kappa$ to do so as well, and so to give a vector field α on a neighborhood of our basepoint x which is transverse to E and which is antiholomorphic for the pulled back almost complex structure $\pi^* J$ where the latter is defined.

After a complex linear change of coordinates on $\mathbb{C}^n$ we may assume that $x = ([1 : 0 : \cdots : 0], (0, \ldots, 0))$ and $\pi(x) = (0, \ldots, 0)$. In terms of the coordinate chart given by $\Theta_{[1:0\cdots:0]}$, the blowdown map π has the form

$$(s, t_1, \ldots, t_{n-1}) \mapsto (s, st_1 + u_1(s, t_1, \ldots, t_{n-1}), \ldots, st_{n-1} + u_{n-1}(s, t_1, \ldots, t_{n-1})),$$

where $|u_i(s, t_1, \ldots, t_{n-1})| < C|s|^2$. Away from the exceptional sphere $s = 0$, this is a diffeomorphism whose complexified linearization with respect to the coordinates $(s, \bar{s}, t_1, \bar{t}_1, \ldots, t_{n-1}\bar{t}_{n-1})$ has inverse of the form

$$((\pi_*)^{-1})_{\pi(s,t_1,\ldots,t_{n-1})} = \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ -t_1/s & 0 & 1/s & 0 & \cdot & \cdot & 0 \\ 0 & -\bar{t}_1/\bar{s} & 0 & 1/\bar{s} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -t_{n-1}/s & 0 & \cdot & \cdots & 0 & 1/s & 0 \\ 0 & -\bar{t}_{n-1}/s & 0 & \cdot & \cdots & 0 & 1/\bar{s} \end{pmatrix} + B(s, t_1, \ldots, t_n),$$

where B is smooth away from $s = 0$ and bounded (but not necessarily continuous) as $s \to 0$.

Write the coordinates on $\mathbb{C}^n$ as $(w, z_1, \ldots, z_{n-1})$. Since J agrees with J_0 at the origin, for $i = 1, \ldots, n-1$ there are J-antiholomorphic vector fields

$$\beta_i = \partial_{\bar{z}_i} + \sum_j a_{ij}(z_1, \ldots, z_n)\partial_{z_j} + \sum_{j \neq i} b_{ij}(z_0, \ldots, z_n)\partial_{\bar{z}_j} + c_i(z_0, \ldots, z_n)\partial_w,$$

where $a_{ij}(0,\dots,0)=b_{ij}(0,\dots,0)=c_i(0,\dots,0)=0$. Away from E we then have

$$\begin{aligned}(\pi_*^{-1}\beta_i)_{(u,v_1,\dots,v_{n-1})}&=\frac{1}{\bar{u}}\partial_{\bar{v}_i}+\sum_j a_{ij}(\pi(u,v_1,\dots,v_{n-1}))\left(\frac{1}{u}\partial_{v_j}\right)\\&+\sum_{j\neq i} b_{ij}(\pi(u,v_1,\dots,v_{n-1}))\left(\frac{1}{\bar{u}}\partial_{\bar{v}_j}\right)\\&+c_i(\pi(u,v_1,\dots,v_{n-1}))\left(\partial_u-\sum_j\frac{v_j}{u}\partial_{v_j}\right)+\tilde{\gamma_i}\end{aligned}$$

where $\tilde{\gamma_i}=B\beta_i$ has bounded coeffecients. So

$$\begin{aligned}\tilde{\beta}_i:=\bar{u}\pi_*^{-1}\beta_i=\partial_{\bar{v}_i}&+\sum\frac{\bar{u}}{u}\left(a_{ij}(u,uv_1,\dots,uv_{n-1})-v_jc_{ij}(u,uv_1,\dots,uv_{n-1})\right)\partial_{v_j}\\+\sum_{j\neq i}b_{ij}(u,uv_1,\dots,uv_{n-1})\partial_{\bar{v}_j}&+\bar{u}c_i(u,uv_1,\dots,uv_{n-1})\partial_w+\bar{u}\tilde{\gamma_i}\end{aligned}$$

is an antiholomorphic tangent vector for π^*J away from $E=\{u=0\}$. Further, we note that since a_{ij},b_{ij}, and c_i are differentiable and vanish at the origin while $\tilde{\gamma_i}$ is L^∞, so that $|a_{ij}(u,uv_1,\dots,uv_{n-1})|$, $|b_{ij}(u,uv_1,\dots,uv_{n-1})|$, $|c_i(u,uv_1,\dots,uv_{n-1})|$, and $\|\bar{u}\gamma_i\|$ are all bounded by a constant times $|u|$, $\tilde{\beta}_i$ extends over E in Lipschitz fashion, agreeing with $\partial_{\bar{v}_i}$ at E.

Hence, defining $\tilde{J}$ near x by

$$T^{0,1}_{\tilde{J}}=\langle\tilde{\alpha},\tilde{\beta}_1,\dots,\tilde{\beta}_{n-1}\rangle,$$

we see that $\tilde{J}$ is Lipschitz and agrees with π^*J where the latter is defined. So since $\tilde{J}$ preserves TE and since at each point of E there is a $\tilde{J}$-holomorphic disc transverse to E mapped holomorphically to a J-holomorphic disc by π, we conclude that $\pi\colon\tilde{\mathbb{C}}^n\to\mathbb{C}^n$ is (J',J)-holomorphic. □

Corollary 6.3. *Let (X,J) be an almost complex manifold with $p\in X$, and let X' denote the blowup of X at p. Then there is a unique almost complex structure J' on X' which is Lipschitz continuous such that $\pi\colon X'\to X$ is (J',J)-holomorphic. Further J' restricts to E as the standard complex structure on $\mathbb{C}P^{n-1}$.*

Proof. Since π is a diffeomorphism away from $E=\pi^{-1}(p)$ (which thus determines J' on $X'\setminus E$ as the smooth almost complex structure π^*J), this follows from the proposition and its proof by choosing a chart around p which sends $(p,J|_{T_pX})$ to $(0,J_0|_{T_0\mathbb{C}^n})$ in $\mathbb{C}^n$ (as may easily be done by modifying any chart around p by an appropriate real linear map). □

6.2. **The diagonal in the relative Hilbert scheme.** Let $F\colon\mathcal{H}_r\to D^2$ denote the r-fold relative Hilbert scheme of the map $f\colon(z,w)\mapsto zw$; the spaces $\mathcal{H}_r\times\mathbb{C}^{s-r}$ form the local models for the relative Hilbert scheme $X_s(g)$ of a Lefschetz fibration g near points of $X_s(g)$ which correspond to divisors containing r copies of a critical point of g. In this subsection we prove the fact, used in the proof of the compactness result underlying the construction of $\mathcal{FDS}$, that at a point in the diagonal Δ of the relative Hilbert scheme $\mathcal{H}_r$ corresponding to the divisor in the nodal fiber $f^{-1}(0)$ consisting of r copies of $(0,0)$, the tangent cone to the diagonal is contained in the tangent cone to the fiber $F^{-1}(0)\subset\mathcal{H}_r$. (Note that since the natural map $F^{-1}(t)\to S^rf^{-1}(t)$ is an isomorphism if and only if t is a regular value of f, there are many points in $F^{-1}(0)$ corresponding to $\{(0,0),\ldots,(0,0)\}$, as will be seen later on when we review the definition of $\mathcal{H}_r$.) Our proof of this fact uses the description of the relative Hilbert scheme in terms of linear algebra provided in Section 3 of [Sm2] based on work of Nakajima [N], and boils down to a rather arcane fact about the discriminants of the characteristic polynomials of certain matrices. It would certainly not surprise us if there exists a more elegant way of proving this result via algebraic geometry, but the argument we give presently is the only one we have at the moment. As will be seen later on, the relevant characteristic polynomials have the form considered in the following lemma.

Lemma 6.4. *There is a universal, nonzero polynomial $P(c_{k+1},\ldots,c_{k+l+1})$ with $P(0,\ldots,0)=0$ such that, given a degree $r=k+l+1$ polynomial*

$$f(x)=x^r+\sum_{a=1}^{k}\epsilon(c_a+O(\epsilon))x^{r-a}+\sum_{b=1}^{l+1}\epsilon^b(c_{k+b}+O(\epsilon))x^{l+1-b}, \tag{6.3}$$

the discriminant $\delta(f)$ of f has the form

$$\delta(f)=P(c_{k+1},\ldots,c_{k+l+1})\,\epsilon^{r+l^2-1}+O(\epsilon^{r+l^2}). \tag{6.4}$$

Proof. For $i=0,\ldots,r=k+l+1$, let a_i be the coefficient of x^{r-i} in f (so in particular $a_0=1$). Recall that $\delta(f)=(-1)^{r(r-1)/2}a_0^{-1}Res(f,f')$ ("Res" denoting the resultant; see, *e.g.*, Section V.10 of [La]), so it suffices to prove the expansion (6.4) for $Res(f,f')$. $Res(f,f')$ is given as the determinant

(6.5)

$$\begin{vmatrix}
a_0 & a_1 & a_2 & \cdots & \cdots & a_r & & & \\
 & a_0 & a_1 & a_2 & \cdots & \cdots & a_r & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & & a_0 & a_1 & a_2 & \cdots & \cdot & a_r \\
ra_0 & (r-1)a_1 & (r-2)a_2 & \cdots & a_{r-1} & & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & & ra_0 & (r-1)a_1 & (r-2)a_2 & \cdots & a_{r-1} & \\
 & & & & ra_0 & (r-1)a_1 & (r-2)a_2 & \cdots & a_{r-1}
\end{vmatrix}$$

Each term in the expansion of this determinant will be a constant times $\prod_{j=0}^{r}a_j^{i_j}$ for some natural numbers i_j satisfying

$$\sum i_j=2r-1$$

(since this is a $(2r-1)\times(2r-1)$ matrix) and

$$\sum ji_j=r(r-1)$$

(since if the roots of f are $\alpha_1,\ldots,\alpha_r$, the discriminant $\prod_{a<b}(\alpha_a-\alpha_b)^2$ has degree $r(r-1)$ in the α_b, while the coefficient a_j has degree j in the α_b). Let

$$e(i_0,\ldots,i_r)=\max\{e\in\mathbb{N}|a_0^{i_0}\cdots a_r^{i_r}=O(\epsilon^e)\}.$$

To prove the lemma we need to show that:

(i) For each $\prod a_j^{i_j}$ appearing in the expansion of the resultant (6.5), $e(i_0,\ldots,i_r)\geq r+l^2-1$, with equality implying that $i_1=\cdots=i_k=0$ (the latter condition being needed to show that our polynomial P depends only on $c_{k+1},\ldots,c_r$ and vanishes when all of these c_j are 0); and

(ii) There are particular values of the c_j for which $Res(f,f') \neq O(\epsilon^{r+l^2})$.

Point (ii) above is easy: in the statement of the lemma, let

$$c_j = \begin{cases} 1 & i = k+1, n \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$f(x) = x^r + (\epsilon + O(\epsilon^2))x^l + (\epsilon^{l+1} + O(\epsilon^{l+2})).$$

We then see that the unique lowest-order term in the expansion of the determinant 6.5 is obtained by choosing $a_0 = 1$ from the first $k+1$ columns, $(r-k-1)a_{k+1} = l\,\epsilon + O(\epsilon^2)$ from the next r columns, and $a_n = \epsilon^{l+1} + O(\epsilon^{l+2})$ from the last $l-1$ columns, so that

$$Res(f,f') = \pm(l\,\epsilon)^r(\epsilon^{l+1})^{l-1} + \text{higher order terms} = \pm l^n\,\epsilon^{r+l^2-1} + O(\epsilon^{r+l^2}).$$

We now set about the proof of point (i). Assume that $\prod a_j^{i_j}$ is a term appearing in the expansion of the determinant (6.5). Let q be the quotient and p be the remainder when $\sum_{m=0}^{l} m i_{r-m}$ is divided by l, and set $s = \sum_{m=0}^{l} i_{r-m} - q$ (note that the above sums only go up to $l = r-k-1$). We then have

$$\sum_{j=r-l}^{r} j i_j = \sum_{j=0}^{l}(r-l+j)i_{r-l+j} = (r-l)q + rs - p.$$

Now since $\sum_{j=0}^{r} i_j = \sum_{j=0}^{k} i_j + q + s = 2r-1$ and since $2r-1 = r+k+l$, we see

$$\begin{aligned} s &= 2r - 1 - q - \sum_{j=0}^{k}(i_j - 1) - (k+1) \\ &= r + l - 1 - q - \sum_{j=0}^{k}(i_j - 1). \end{aligned}$$

Hence

$$\begin{aligned} r^2 - r = \sum_{j=0}^{r} j i_j &= \sum_{j=0}^{k} j i_j + q(r-l) + rs - p \\ &= \sum_{j=0}^{k} j i_j + q(r-l) - p + r(r + l - 1 - q - \sum_{j=0}^{k}(i_j - 1)) \\ &= r^2 - r + l(r-q) - p + \sum_{j=0}^{k}(j i_j - r(i_j - 1)), \end{aligned}$$

i.e.,

$$l(r-q) = p + \sum_{j=0}^{k}(r(i_j - 1) - j i_j). \tag{6.6}$$

Meanwhile

$$\begin{aligned}
e(i_0,\ldots,i_r) &= \sum_{j=1}^{k} i_j + \sum_{j=0}^{l}(1+j)i_{r-l+j} \\
&= \sum_{j=1}^{k} i_j + q + s(l+1) - p \\
&= \sum_{j=1}^{k} i_j + q + \left(r + l - 1 - q - \sum_{j=0}^{k}(i_j - 1)\right)(l+1) - p \\
&= l(r-q) + r + l^2 - 1 - (l+1)\sum_{j=0}^{k}(i_j-1) + \sum_{j=1}^{k} i_j - p \\
&= r + l^2 - 1 + \sum_{j=0}^{k}\left(r(i_j-1) - ji_j\right) - (l+1)\sum_{j=0}^{k}(i_j-1) + \sum_{j=1}^{k} i_j \\
&= r + l^2 - 1 + k\sum_{j=0}^{k}(i_j-1) + \sum_{j=1}^{k}(1-j)i_j, \qquad (6.7)
\end{aligned}$$

where in the penultimate equality we have used (6.6) and in the last we have used the fact that $r-(l+1)=k$.

In our term $\prod a_j^{i_j}$ in the expansion of the determinant (6.5), each of those a_j which are chosen from the first $(k+1)$ columns necessarily has $j \le k$. For each j write $i_j = w_j + z_j = w_j + x_j + y_j$ where w_j denotes the number of a_j's chosen from the first $(k+1)$ columns and x_j denotes the number of a_j's chosen from columns $k+2$ through $2k+1$; evidently $w_j = 0$ for $j > k$ while $\sum_{j=0}^{k} w_j = k+1$, *i.e.*,

$$\sum_{j=0}^{k}(w_j - 1) = 0. \qquad (6.8)$$

Rearrange our term $\prod_{j=0} a_j^{i_j}$ as

$$a_{p_1}\cdots a_{p_{2r-1}},$$

where the entry a_{p_n} is culled from the *nth* column in the matrix in (6.5); label the row from which a_{p_n} is taken as m_n. Denoting

$$\bar{m} = \begin{cases} m & m \le r-1 \\ m+1-r & m \ge r \end{cases},$$

we see from the form of the resultant matrix that

$$\bar{m}_n = n - p_n.$$

Consider the quantity

$$\sum_{n=1}^{2k+1} \bar{m}_n.$$

Obviously, the way to minimize this quantity is by using rows $1,2,\ldots,k,r,r+1,\ldots,r+k$ (or, just as well, rows $1,\ldots,k+1,r,\ldots,r+k-1$) when we pick the

$a_{p_1}, \ldots, a_{p_{2k+1}}$; such a choice then yields $\{\bar{m}_n | n \leq 2k+1\} = \{1, 1, \ldots, k, k, k+1\}$ and

$$\sum_{n=1}^{2k+1} \bar{m}_n = \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} = (k+1)^2.$$

If $x_0 \neq 0$, we have some $n \in [k+2, 2k+1]$ with $p_n = 0$ and so $\bar{m}_n = n > k+1$; in this vein, one may easily check that

$$\sum_{n=1}^{2k+1} \bar{m}_n \geq (k+1)^2 + \frac{x_0(x_0+1)}{2};$$

in particular

$$\sum_{n=1}^{2k+1} \bar{m}_n \geq (k+1)^2 + x_0,$$

with equality requiring that either $x_0 = 0$ and $\{\bar{m}_n | n \leq 2k+1\} = \{1, 1, \ldots, k, k, k+1\}$ or $x_0 = 1$ and $\{\bar{m}_n | n \leq 2k+1\} = \{1, 1, \ldots, k, k, k+2\}$.

Thus,

$$\begin{aligned}
(k+1)^2 + x_0 \leq \sum_{n=1}^{2k+1} \bar{m}_n &= \sum_{n=1}^{2k+1} (n - p_n) \\
&= (k+1)(2k+1) - \sum_{n=1}^{k+1} p_n - \sum_{n=k+2}^{2k+1} p_n \\
&= (k+1)^2 - \sum_{j=1}^{k} j w_j + \sum_{n=k+2}^{2k+1} (k+1-p_n) \\
&\leq (k+1)^2 - \sum_{j=1}^{k} j w_j + \sum_{n=k+2, p_n \leq k}^{2k+1} (k+1-p_n) \\
&= (k+1)^2 - \sum_{j=1}^{k} j w_j + \sum_{j=0}^{k} (k+1-j) x_j \qquad (6.9)
\end{aligned}$$

So

$$kz_0 + \sum_{j=1}^{k}(k+1-j)z_j \geq kx_0 + \sum_{j=1}^{k}(k+1-j)x_j \geq \sum_{j=1}^{k} j w_j \geq \sum_{j=1}^{k}(j-1)w_j, \qquad (6.10)$$

i.e., $k\sum_{j=0}^{k} z_j + \sum_{j=1}^{k}(1-j)(w_j+z_j) \geq 0$, so that since $\sum_{j=0}^{k}(w_j - 1) = 0$ and $i_j = w_j + z_j$, we at last conclude that

$$k\sum_{j=0}^{k}(i_j - 1) + \sum_{j=1}^{k}(1-j)i_j \geq 0. \qquad (6.11)$$

In light of Equation 6.7, this shows that $e(i_0, \ldots, i_r) \geq r + l^2 - 1$ with equality if and only if equality holds in (6.11); equality in (6.11) requires among other things that

(i) either $x_0 = 0$ and $\{\bar{m}_n | n \leq 2k+1\} = \{1, 1, \ldots, k, k, k+1\}$ or $x_1 = 1$ and $\{\bar{m}_n | n \leq 2k+1\} = \{1, 1, \ldots, k, k, k+2\}$; and

(ii) due to (6.10), $z_j = x_j$ for $j \le k$ (so that for $j \le k$ all of the a_j in our term $\prod_{j=0}^{r} a_j^{i_j}$ come from the first $2k+1$ columns of the resultant matrix).

For $n = 1,2,3,4$ let M_n denote the $(2k+1) \times (2k+1)$ matrix constructed from the resultant matrix (6.5) by taking columns 1 through $2k+1$ and rows $1, \ldots, k, r, \ldots, r+k$ (for $n=1$), rows $1, \ldots, k+1, r, \ldots, r+k-1$ (for $n=2$), rows $1, \ldots, k, r, r+k-1, \ldots, r+k+1$ (for $n=3$), or rows $1, \ldots, k, k+2, r, \ldots, r+k-1$ (for $n=4$). Let M'_n be the $(2r-2k-2) \times (2r-2k-2)$ constructed from the other rows and columns. Assume that our term $\prod_{j=0}^{r} a_j^{i_j}$ in the resultant gives rise to the lowest possible value of $e(i_0, \ldots, i_j)$. (i) above then ensures that $\prod_{j=0}^{r} a_j^{i_j}$ is constructed by multiplying a term in the determinant of one of the M_n by a term in the determinant of the corresponding M'_n. In searching for the optimal such monomial, we may then vary the contributions from M_n and M'_n separately. But on examining the form of the M_n, one sees immediately that the term in $\det(M_n)$ giving rise to the *strictly* lowest possible power of ϵ is obtained by a product of $k+1$ a_0's (from columns 1 through $k+1$ for $n=1,2$ and columns $1, \ldots, k, k+2$ for $n=3,4$) and k a_{k+1}'s (and in particular contains no a_j for $1 \le j \le k$). By (ii), any optimal monomial from M'_n can't contain any a_j with $j \le k$. Thus any $\prod_{j=1}^{r} a_j^{i_j}$ with $(i_1, \ldots, i_k) \neq (0, \ldots, 0)$ must have $e(i_1, \ldots, i_r)$ strictly greater than the lowest possible value (which has been shown above to be $n + l^2 - 1$). This proves the lemma. □

We now recall the linear algebra definition of the relative Hilbert scheme from [Sm2]. Let

$$\tilde{\mathcal{H}}_r = \{(A, B, t, v) \in M_r(\mathbb{C})^2 \times D^2 \times \mathbb{C}^r | AB = BA = tId, (*)\}, \tag{6.12}$$

where the stability condition (*) states that the matrices A and B share no proper invariant subspaces containing the vector v. The relative Hilbert scheme of the map $(z, w) \mapsto zw$ is then

$$\mathcal{H}_r = \tilde{\mathcal{H}}_r / GL_r(\mathbb{C}),$$

where $GL_r(\mathbb{C})$ acts by

$$g \cdot (A, B, t, v) = (gAg^{-1}, gBg^{-1}, t, gv).$$

The projection map $F\colon \mathcal{H}_r \to D^2$ is just $[A, B, t, v] \mapsto t$. To briefly motivate this, remark that a point of the r-fold relative Hilbert scheme of f is naturally viewed from an algebro-geometric standpoint as an ideal $I \le \mathbb{C}[z, w]$ with the property that $V = \mathbb{C}[z, w]/I$ is an r-dimensional vector space and, for some t, I is supported on $f^{-1}(t)$ (*i.e.*, $\langle zw - t\rangle < I$). To go from such an ideal to an element of $\mathcal{H}_r$, let $v \in V$ be the image of $1 \in \mathbb{C}[z, w]$ under the projection, and let A and B be the operators on V defined by multiplication by the polynomials z and w respectively. For more details see [N] and [Sm2].

Given $[A, B, t, v] \in \mathcal{H}_r$, the fact that A and B commute implies that they can be simultaneously conjugated to be upper triangular; assuming that this has been done, the natural map $\phi_t\colon F^{-1}(t) \to Sym^r f^{-1}(t)$ takes $[A, B, t, v]$ to $\{(A_{11}, B_{11}), \ldots, (A_{rr}, B_{rr})\}$. For $t \neq 0$, according to (6.12), A is invertible and $B = tA^{-1}$, so ϕ_t is an isomorphism; ϕ_0, meanwhile, is a nontrivial partial resolution. On the diagonal $\Delta \subset \mathcal{H}_r$, A and B will both have repeated eigenvalues, occurring in corresponding Jordan blocks.

The main result of this section is:

Theorem 6.5. *Let $F\colon \mathcal{H}_r \to D^2$ denote the r-fold relative Hilbert scheme of the map $(z,w) \mapsto zw$, ϕ_0 the partial resolution map $F^{-1}(0) \to Sym^r\{zw = 0\}$, and $\Delta \subset \mathcal{H}_r$ the diagonal stratum. At any point $p \in \Delta \cap F^{-1}(0)$ with $\phi_0(p) = \{(0,0),\dots,(0,0)\}$, where $T_p\Delta$ is the tangent cone to Δ at p, we have $T_p\Delta \subset T_pF^{-1}(0)$.*

Proof. According to the above description, the points p under concern are of the form $[A,B,0,v]$ with A and B both nilpotent matrices such that $AB = BA = 0$ Further, letting k be such that $A^kv \neq 0$ but $A^{k+1}v = 0$, the stability condition $(*)$ in (6.12) ensures that, where $r = k+l+1$,

$$\{A^kv,\dots,Av,B^lv,\dots,Bv,v\}$$

is a basis for $\mathbb{C}^r$. All operators on $V \cong \mathbb{C}^r$ appearing in the rest of the proof will be written as matrices in terms of this basis.

Since $AB = 0$ we can write

$$B^{l+1}v = aA^kv + \sum_{i=1}^{l} b_{l-i}B^iv.$$

With respect to our above basis, we have

$$A = \left(\begin{array}{cccc|cccc|c}
0 & 1 & & & 0 & \cdot & \cdot & 0 & 0 \\
 & \ddots & \ddots & & \cdot & \cdot & & \cdot & \vdots \\
 & & 0 & 1 & \cdot & & \cdot & \cdot & 0 \\
 & & & 0 & 0 & \cdot & \cdot & 0 & 1 \\
\hline
0 & & \cdots & 0 & 0 & \cdots & & & 0 \\
\vdots & & & & & & & & \vdots \\
0 & & \cdots & 0 & 0 & \cdots & & & 0
\end{array}\right),$$

$$B = \left(\begin{array}{cccc|c|cccc}
0 & \cdot & \cdot & 0 & a & 0 & \cdot & \cdot & 0 \\
\cdot & \cdot & & \cdot & 0 & \cdot & & & \cdot \\
\cdot & & \cdot & \cdot & \vdots & \cdot & & \ddots & \cdot \\
\cdot & & & \cdot & 0 & 0 & \cdot & \cdot & 0 \\
\hline
0 & \cdot & \cdot & 0 & b_1 & 1 & 0 & \cdots & 0 \\
\hline
\cdot & \cdot & & \cdot & \vdots & 0 & 1 & \cdots & 0 \\
\cdot & & & \cdot & b_{l-1} & \cdot & \ddots & \ddots & \vdots \\
\cdot & & \cdot & \cdot & b_l & \cdot & & \cdot & 1 \\
0 & \cdot & \cdot & 0 & 0 & 0 & \cdot & \cdot & 0
\end{array}\right),$$

and $v = e_r = (0,\dots,0,1)$ (in both of the above matrices, the upper left block is of size $k \times k$). Let

$$(C,D,\mu,w) \in T_{(A,B,0,e_r)}\tilde{\mathcal{H}}_r.$$

Letting $\pi\colon \tilde{\mathcal{H}}_r \to \mathcal{H}_r$ be the projection, we have $\mu = F_*(\pi_*)_{(A,B,0,e_r)}(C,D,\mu,w)$, so our goal is to show that if (C,D,μ,w) is tangent to $\pi^{-1}\Delta$ then $\mu = 0$. Linearizing the defining equations for $\tilde{H}_r$ gives

$$CB + AD = BC + DA = \mu Id,$$

which implies, among other things,

$$\text{For } i > k, \ \begin{cases} aC_{i1} + \sum b_m C_{i,k+m} = \mu\delta_{i,j} \\ C_{i,j-1} = \mu\delta_{i,j} \text{ if } j \geq k+2 \end{cases}$$

$$\text{For } j = 1 \text{ or } k+1 \leq j \leq r-1, \ \begin{cases} aC_{k+1,j} = \mu\delta_{1,j} \\ b_{i-k}C_{k+1,j} + C_{i+1,j} = \mu\delta_{i,j} \text{ if } k+1 \leq i \leq r-1 \end{cases}$$

If $a = 0$, we have $\mu = \mu\delta_{1,1} = aC_{k+1,1} = 0$ and we are done.

If $a \neq 0$, we find from the above equations that

$$C = \left(\begin{array}{cccc|cccc|c} * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ \hline \mu/a & * & * & * & 0 & 0 & \cdot & 0 & * \\ \hline -b_1\mu/a & * & * & * & \mu & 0 & \cdots & 0 & * \\ -b_2\mu/a & * & * & * & 0 & \mu & \cdot & 0 & * \\ \vdots & * & * & * & \cdot & \cdot & \ddots & \cdot & * \\ -b_l\mu/a & * & * & * & 0 & \cdot & \cdot & \mu & * \end{array} \right),$$

where again the upper left block is size $k \times k$ and all asterisks denote undetermined entries.

We consider now the characteristic polynomials of the matrices $A + \epsilon\, C$ for small ϵ. The matrix $A + \epsilon\, C - \lambda Id$ is

$$\left(\begin{array}{cccc|cccc|c} -\lambda + \epsilon\, C_{11} & 1 + \epsilon\, C_{12} & * & * & * & * & * & * & * \\ * & -\lambda + \epsilon\, C_{22} & \ddots & * & * & * & * & * & * \\ * & * & \ddots & 1 + \epsilon\, C_{k-1,k} & * & * & * & * & * \\ * & * & * & -\lambda + \epsilon\, C_{kk} & * & * & * & * & 1 + \epsilon\, C_{rk} \\ \hline \epsilon\, \mu/a & * & * & * & -\lambda & 0 & \cdot & 0 & * \\ \hline -\epsilon\, b_1\mu/a & * & * & * & \epsilon\, \mu & -\lambda & \ddots & 0 & * \\ -\epsilon\, b_2\mu/a & * & * & * & 0 & \epsilon\, \mu & \cdot & 0 & * \\ \vdots & * & * & * & \vdots & \cdot & \ddots & -\lambda & * \\ -\epsilon\, b_l\mu/a & * & * & * & 0 & \cdot & \cdot & \epsilon\, \mu & -\lambda + \epsilon\, C_{rr} \end{array} \right)$$

where an asterisk in the (i,j)th entry signifies $\epsilon\, C_{ij}$. When we expand the determinant of this matrix, among the terms that we obtain are

$$(-\lambda)^r \text{ and } \ \pm(-\epsilon\, b_m\mu/a) \cdot 1^{k-1}(-\lambda)^{m-1}(\epsilon\, \mu)^{l-m+1} = \pm\frac{\mu^{l-m+2}b_m}{a}\, \epsilon^{l-m+2}\, \lambda^{m-1};$$

note that these latter have combined degree exactly $l+1$ in ϵ and λ. Any other term in the expansion of the determinant will have degree at least 1 in ϵ and at least $l+2$ in ϵ and λ combined, the reason being that each of the entries denoted with an asterisk above lies in either the same row or the same column as an entry of form $1 + \epsilon\, C_{ij}$, so a term in the determinant containing one of the asterisked entries can contain at most $k-1$ of the k $(1 + \epsilon\, C_{ij})$'s and hence must contain at least $r - (k-1) = l+2$ other terms, each of which is of combined order at least 1 in ϵ and λ. In other words, for constants $c_1, \dots, c_{k+l+1}$ where

$$c_{k+m} = \pm\frac{b_{l+1-m}\mu^m}{a} \text{ for } 1 \le m \le l \text{ and } c_{k+l+1} = \frac{\mu^{l+1}}{a},$$

the characteristic polynomial of $A + \epsilon C$ has form

$$p_{A+\epsilon C}(x) = (-x)^r + \epsilon\sum_{a=1}^{k}(c_a + O(\epsilon))(-x)^{r-a} + \sum_{b=1}^{l+1}\epsilon^b(c_{k+b} + O(\epsilon))(-x)^{l+1-b},$$

which, since $r = k + l + 1$, is precisely the sort of polynomial considered in Lemma 6.4. By replacing ϵ with $\nu\epsilon$ in the statement of that lemma, we see that the polynomial $P(a_{r-l}, \dots, a_r)$ provided by its conclusion scales as

$$P(\nu a_{r-l}, \nu^2 a_{r-l+1}, \dots, \nu^{l+1}a_r) = \nu^{r+l^2-1}P(a_{r-l}, \dots, a_r),$$

so that

$$\begin{aligned} P(c_{k+1}, \dots, c_r) &= P\left(\pm\frac{b_l\mu}{a}, \pm\frac{b_{l-1}\mu^2}{a}, \dots, \pm\frac{b_1\mu^l}{a}, \frac{\mu^{l+1}}{a}\right) \\ &= \mu^{r+l^2-1}P(\pm b_l/a, \pm b_{l-1}/a, \dots, \pm b_1/a, 1/a). \end{aligned}$$

So since P is not the zero polynomial, at least for a generic initial choice of our base point $[A, B, 0, e_r]$ (equivalently, for generic $a, b_1, \dots, b_l$) we conclude that if $(C, D, \mu, w) \in T_{(A,B,0,e_r)}\tilde{\mathcal{H}}_r$, we have

$$\delta(p_{A+\epsilon C}) = \mu^{r+l^2-1}M\,\epsilon^{n+l^2-1} + O(\epsilon^{n+l^2}), \tag{6.13}$$

where M is a nonzero constant depending only on A. Let

$$\begin{aligned} \tilde{\Delta}_1 &= \{(A', B', t, v') \in \tilde{\mathcal{H}}_r | A' \text{ has a repeated eigenvalue}\} \\ &= \{(A', B', t, v') \in \tilde{\mathcal{H}}_r | \delta(p_A) = 0\} \end{aligned}$$

Equation 6.13 then shows that, for $(C, D, \mu, w) \in T_{(A,B,0,e_r)}\tilde{\mathcal{H}}_r$,

$$(C, D, \mu, w) \in T_{(A,B,0,e_r)}\tilde{\Delta}_1 \Leftrightarrow \mu = 0$$

($T_{(A,B,0,e_r)}\tilde{\Delta}_1$ denoting the tangent cone at $(A, B, 0, e_r)$ if $\tilde{\Delta}_1$ is singular there). Where again $\pi\colon \tilde{\mathcal{H}}_r \to \mathcal{H}_r$ is the projection, we have $T\Delta \subset \pi_* T\tilde{\Delta}_1$, so if $\alpha \in T_{[A,B,0,e_r]}\Delta$, writing $\alpha = \pi_*(C, D, \mu, w)$, we have that $F_*\alpha = \mu = 0$. This conclusion initially only applies at those $[A, B, 0, e_r] \in \Delta$ which are generic in the sense that $P(\pm b_l/a, \pm b_{l-1}/a, \dots, \pm b_1/a, 1/a) \neq 0$, but then since the conclusion is a closed condition it in fact applies to all $[A, B, 0, e_r]$ lying on the diagonal Δ. □

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