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Lefschetz fibrations and pseudoholomorphic curves

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Abstract. This survey of the main results of [13] and [14] discusses how, using constructions of S. Donaldson and I. Smith [3], one can exploit the existence of Lefschetz fibrations on symplectic 4-manifolds to deduce facts about the Gromov–Taubes invariants of such manifolds.

1 Gromov–Witten and Gromov–Taubes invariants

Let (X, ω) be a symplectic 4-manifold, $\alpha \in H^2(X, \mathbb{Z})$, and $A = PD(\alpha)$. For some time, it has been realized that counting pseudoholomorphic curves representing the class A should give rise to interesting symplectic invariants of (X, ω). When one sets about carefully defining such invariants, though, one inevitably runs into certain technical difficulties, the most serious of which arises from the fact that the moduli space of pseudoholomorphic curves in class A can have an undesirably-large boundary component consisting of multiple covers.

If one attempts to restrict to *embedded* curves representing A, all such curves will (by the adjunction formula) have genus $g(\alpha) = 1 + \frac{1}{2}(\alpha^2 + \kappa_X \cdot \alpha)$, and the expected complex dimension of the space of such curves is $d(\alpha) = \frac{1}{2}(\alpha^2 - \kappa_X \cdot \alpha)$. (Here κ_X denotes the canonical class of (X, ω), *i.e.*, the first Chern class of T^*X considered as a complex vector bundle using any almost complex structure j on X tamed by ω.) From these formulas, one can verify that, for generic j, the only source of noncompactness of the moduli spaces arises from the fact that, for some $T \in H_2(X, \mathbb{Z})$ and $m > 1$, a sequence of embedded square-zero tori representing a class mT might converge to a double cover of a torus in class T. One would like to define the invariant by choosing some generic ω-tame almost complex structure j and counting embedded j-holomorphic curves with sign, but the scenario described above and pictured in Figure 1 (which depicts a possible parametrized moduli space of embedded curves in classes T and $2T$ for a family of almost complex structures varying from j_0 to j_1) shows that this cannot be expected to work: one cannot

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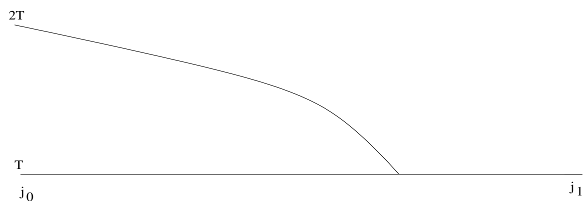

Figure 1

assign weights ± 1 to all the curves representing $2T$ in Figure 1 in a way that yields the same result for j_0 as for j_1.

Two approaches which emerged in the mid-1990s to surmounting this issue are the following:

Approach 1.1 (Ruan-Tian [7]**)** *Define the Gromov–Witten invariant*

$$\Psi_{\alpha,g(\alpha),d(\alpha)}([\overline{M}_{g(\alpha),d(\alpha)}];pt,\ldots,pt)$$

to be the number of maps u, counted with sign, from curves Σ of genus $g(\alpha)$ to X which satisfy $u_*[\Sigma] = A$, $\Omega \subset Im(u)$, and $(\bar{\partial}_j u)(x) = \nu(x, u(x))$, where Ω is a generic set of $d(\alpha)$ points, j is a generic ω-tame almost complex structure on X, and ν is a generic inhomogeneous term (see [7] *for the precise definition).*

Approach 1.2 (Taubes [12]**)** *For C a pseudoholomorphic square-zero torus, find general weights $r(C,m)$, depending only on the spectral flows of the operators D_ι obtained by twisting the linearization of the $\bar{\partial}$ operator at C by the four real line bundles over C, with the property that if m-fold covers of C are counted as contributing to the invariant for the class $m[C]$ with multiplicity $r(C,m)$, the total count does not change under the wall crossing scenario pictured in Figure 1. Count pseudoholomorphic curves which are not square-zero tori with signs according to the usual prescription. Define the Gromov invariant $Gr(\alpha)$ by, for generic j and Ω, counting each j-holomorphic curve (connected or not) with total homology class A which passes through Ω with a weight equal to the product of the weights of its components.*

Formulas for the $r(C,m)$ are given on p. 832 of [12]; they are determined uniquely by the requirements that the quantity $Gr(\alpha)$ defined using them be independent of the choice of almost complex structure and that, if j is integrable on a neighborhood of C and the four operators D_ι are all surjective, then $r(C,m) = 1$ for every m.

Note that, in Approach 1.1, the multiple-cover problem is eliminated by the fact that, if the inhomogeneous term ν depends nontrivially on x, then multiple covers of the "(j,ν)-holomorphic curves" being counted will not themselves be (j,ν)-holomorphic curves. From the standpoint of producing invariants, the Ruan–Tian approach is the more powerful one, since it extends without difficulty to give invariants which count curves of arbitrary genus in any symplectic manifold which satisfies a semipositivity condition which prevents bubble trees containing multiply covered spheres of negative Chern number from contributing large boundary components to the moduli spaces (semipositive manifolds include all symplectic 4-manifolds, as well as Calabi–Yau manifolds and many others in arbitrary dimension). Meanwhile, the methods involved in the definition of the Gromov–Taubes invariant seem to be uniquely adapted to counting curves in 4-manifolds with genus given by the adjunction formula (indeed, E. Ionel's lecture at this conference addressed some of the very considerable difficulties involved in trying to construct an analogue of the Gromov–Taubes invariant for Calabi–Yau 3-folds).

On the other hand, in certain ways the Gromov–Taubes invariant seems more natural, since the objects it counts are genuine pseudoholomorphic curves, rather than maps which satisfy a perturbed Cauchy–Riemann equation. Taubes was led to his invariant while studying the Seiberg–Witten equations on symplectic 4-manifolds; he discovered that when the perturbation term in those equations is sent to infinity in a certain direction, the vanishing loci of the sections of a line bundle which occur as one of the components of solutions to the equations converge in the sense of currents to a pseudoholomorphic curve. These vanishing loci of course are only sets and in particular do not come equipped with maps into the manifold, so an invariant counting pseudoholomorphic *sets* (as Gr does), rather than (approximately) pseudoholomorphic *maps* (as does the invariant of Ruan-Tian) was better-suited to Taubes' work.

Notwithstanding the above distinctions, it should be mentioned that a theorem of E. Ionel and T. Parker [4] shows that Gr may be expressed as a (somewhat complicated) combination of the Ruan-Tian invariants, reflecting the fact that, in spirit, these invariants are counting the same things.

From early on in Gromov–Witten theory it has been understood that, in order to get the appropriate invariants, certain perturbations would have to be made: for instance, when the manifold is Kähler and so has a natural integrable complex structure, in formulating the definition of Gromov–Witten invariants it is useful to perturb that complex structure to a nearby nonintegrable almost complex structure (as was noted as early as [15, p. 134], for instance). One reason that the Ruan–Tian count of pseudoholomorphic maps is comparatively easy to define is that one may perturb the notion of what it means to be a pseudoholomorphic map in such a way that multiple covers of perturbed-pseudoholomorphic maps are no longer perturbed-pseudoholomorphic. Meanwhile, there does not seem to be an obvious way of perturbing what it means for a *set* to be pseudoholomorphic in such a way that multiple covers of perturbed-pseudoholomorphic sets are no longer perturbed-pseudoholomorphic. Hence, if one wants to define invariants counting pseudoholomorphic sets, one will be obliged to explicitly include the effects of multiple covers, which Taubes discovered in [12] to be a rather complicated affair for embedded curves in dimension four, and which at this writing has not even been shown to be possible in the symplectic category in other contexts.

The present article surveys work relating to a construction of Donaldson and Smith which, in dimension four, uses a Lefschetz fibration structure on a blowup of X to effectively provide a perturbation of what it means to be a pseudoholomorphic set in a way that breaks the multiple cover symmetry. This enables one to define Gromov-type invariants more easily, although such invariants might *a priori* depend on the choice of Lefschetz fibration. A result of the author discussed below shows that they do not; more specifically, they agree with the Gromov–Taubes invariant. Along with results of [3] and [10], this yields new proofs of various results concerning pseudoholomorphic curves in symplectic 4-manifolds which had previously only been known as consequences of Seiberg–Witten theory and of Taubes' famous theorem [11] relating the Seiberg–Witten and Gromov invariants. Moreover, the Donaldson–Smith invariant may be refined to invariants which at least in simple cases agree with an invariant counting nodal curves in the 4-manifold, and the methods of [3] may be extended to prove vanishing results for these new invariants.

2 Symplectic Lefschetz fibrations and the standard surface count

Let us assume that $[\omega] \in H^2(X, \mathbb{Z})$; note that by slightly perturbing ω to make it rational and then scaling, we can always deform ω through symplectic forms to arrange that this be the case, and that doing so will not alter the Gromov invariants since these are unchanged under symplectic deformation. According to [2], then, some blowup X' of (X, ω) admits the structure of a symplectic Lefschetz fibration, *i.e.*, a map $f\colon X' \to S^2$ whose generic fiber is a smooth symplectic 2-submanifold and which is modelled near each of its finitely many critical points by the function $(z, w) \mapsto zw$ in complex coordinates compatible with the orientations of X and S^2. To briefly recall how such a structure is obtained,[1] note first that the integrality of ω allows us to find a complex line bundle $L \to X$ admitting a connection with curvature $-2\pi i\omega$; this should be viewed as analagous to a positive line bundle in Kähler geometry. A reference almost complex structure on X then gives rise to a Dolbeault operator $\bar{\partial}$ on sections of L. In Kähler geometry, high tensor powers of a positive line bundle become very ample and thus admit many holomorphic sections. In the almost Kähler setting we cannot hope for this, but results of [1] show that, for large k, $L^{\otimes k}$ will admit many sections s satisfying an "approximate holomorphicity" condition that may loosely be summed up as $|\bar{\partial} s| \ll |\partial s|$. Taking two of these sections s_0 and s_1 then gives rise to a map

$$\begin{aligned} f\colon X \setminus B &\to \mathbb{C}P^1 \\ x &\mapsto [s_0(x) : s_1(x)], \end{aligned}$$

where B is the common vanishing locus of s_0 and s_1 and so (under favorable circumstances) will be a finite set. Moreover, at least if s_0 and s_1 satisfy certain transversality properties, f will lift to a map $f\colon X' \to \mathbb{C}P^1$ defined on all of the blowup X' of X at B, with the exceptional divisors of the blowup appearing as sections of f. The fibers of f can be arranged to be symplectic (essentially as a result of the approximate holomorphicity of s_0 and s_1), and the behavior near critical points may be taken to be such that f defines a symplectic Lefschetz fibration. Note that the fibers of f are Poincaré dual to the pullback of the class $k[\omega]$, so that by taking k large we can obtain symplectic Lefschetz fibrations on blowups of X whose fibers have arbitrarily large symplectic area. Results of [9] allow one to assume that the singular fibers contain only one node, which is nonseparating.

Since if $\pi\colon X' \to X$ is a blowup we have $Gr(\alpha) = Gr(\pi^*\alpha)$, it follows that to understand the Gromov invariants of symplectic 4-manifolds it suffices to understand the Gromov invariants of symplectic Lefschetz fibrations. Accordingly, let $f\colon X \to S^2$ be a symplectic Lefschetz fibration on a 4-manifold X. In order to exploit the Lefschetz fibration structure to study pseudoholomorphic curves, we shall restrict attention to almost complex structures j on X with respect to which f is a pseudoholomorphic map. Such almost complex structures may be constructed by pushing forward the standard complex structure from the complex coordinate charts near the critical points and then extending to the rest of X (on which f

[1] This sketch omits several highly nontrivial technical points, for which the reader is referred to [1] and [2]

is a genuine fibration) by means of a connection on the fibration (as is seen for instance in Section 4 of [3]). Moreover, Lemma 2.1 of [13] shows that there are sufficiently many such structures that for a generic choice of one of them (say j), all moduli spaces of j-holomorphic curves representing classes of smaller symplectic area than the fibers of f will be of the correct dimension, and, if the moduli spaces are cut down by incidence conditions to be zero-dimensional, they will consist only of curves which miss the critical points of f.

Let C be a j-holomorphic curve in X (possibly reducible, and with its components allowed to have multiplicity larger than 1) with no fiber components, as is of course assured to be the case if $\langle [\omega], [C] \rangle < \langle [\omega], [fiber] \rangle$. Now our assumption on j ensures that the fibers themselves are j-holomorphic, so all intersections between C and any fiber $f^{-1}(t)$ will contribute positively to the intersection number $r := [C] \cdot [fiber]$. As such, $C \cap f^{-1}(t)$ is a subset of $f^{-1}(t)$ consisting of points with positive multiplicities which add up to the number r; putting it less clumsily, $C \cap f^{-1}(t)$ is naturally viewed as an element of the symmetric product $S^r f^{-1}(t)$. Thus, giving the curve C is equivalent to giving a family of elements of the symmetric products $S^r f^{-1}(t)$ as t ranges over S^2.

In [3], Donaldson and Smith construct the *relative Hilbert scheme* associated to a Lefschetz fibration f and an integer $r > 0$ as a space $X_r(f)$ with a map $F\colon X_r(f) \to S^2$ whose fiber over a regular value t of F is naturally identified with $S^r f^{-1}(t)$, and moreover show that $X_r(f)$ is a smooth symplectic manifold as long as no fiber of f contains more than one node. Thus, the curve C of the previous paragraph, by determining elements of the various spaces $S^r f^{-1}(t)$, gives rise to a *section* s_C of $X_r(f)$. Conversely, a continuous section s of $X_r(f)$ naturally "sweeps out" a cycle C_s in X determined as the union of all the points occurring in the various divisors $s(t)$.

From j, we may form an almost complex structure $\mathbb{J}_j$ on $X_r(f)$ as follows. Note first that since a generic point of $X_r(f)$ is a set $\{p_1, \ldots, p_r\}$ of points in X such that each $f(p_i) = f(p_j)$, a tangent vector at $\{p_1, \ldots, p_r\}$ is obtained by taking tangent vectors $v_i \in T_{p_i}X$ with the property that the "horizontal parts" $f_* v_i \in T_{f(p_i)}S^2$ are all equal. We then simply define

$$\mathbb{J}_j\{v_1, \ldots, v_r\} = \{jv_1, \ldots, jv_r\};$$

observe that the horizontal parts $f_* j v_i$ are all equal by virtue of the fact that f is a j-holomorphic map. $\mathbb{J}_j$ may then be extended to nongeneric points by continuity.

With $\mathbb{J}_j$ understood, it is then easy to see that, in the above notation

$$C \text{ is } j\text{-holomorphic } \Leftrightarrow \bar{\partial}_{\mathbb{J}_j} s_C = 0. \tag{2.1}$$

This indicates that an alternate approach to counting j-holomorphic curves in X might be to count $\mathbb{J}_j$-holomorphic sections in $X_r(f)$; perhaps we should use some Ruan–Tian-type invariant for $X_r(f)$. This is indeed what we will do, but it should be mentioned that using $\mathbb{J}_j$ to directly evaluate such an invariant is not an option, by virtue of the fact that, as originally observed in [8], $\mathbb{J}_j$ is typically only Hölder-continuous at the *diagonal stratum* in $X_r(f)$ consisting of divisors with one or more points repeated. (Non-differentiable almost complex structures cannot be used in the evaluation of Gromov–Witten invariants because the construction of such invariants invokes the implicit function theorem for a map which is only as smooth as the almost complex structures being used.) The right approach is suggested by the following:

Theorem 2.1 *([10], Section 4) Given $\alpha \in H^2(X,\mathbb{Z})$, there is at most one homotopy class c_α of sections of $X_r(f)$ with the property that sections s in the class c_α descend to sets $C_s \subset X$ which are Poincaré dual to α. Furthermore, the complex index of the $\bar\partial$ operator acting on sections in the class c_α for* any *almost complex structure on $X_r(f)$ is $d(\alpha) = \frac{1}{2}(\alpha^2 - \kappa_X \cdot \alpha)$*

Recall that $d(\alpha)$ is none other than the expected dimension of the space of pseudoholomorphic submanifolds of X Poincaré dual to α. From this we see that, for a generic set Ω of $d(\alpha)$ points in X, both the moduli space

$$\mathcal{M}^j_X(\alpha)$$

consisting of j-holomorphic subvarieties of X Poincaré dual to α and passing through Ω where j is a generic almost complex structure on X and the moduli space

$$\mathcal{M}^J_{X_r(f)}(c_\alpha)$$

consisting of J-holomorphic sections of $X_r(f)$ in the class c_α with $\Omega \subset C_s$ where J is a *generic, smooth* almost complex structure on $X_r(f)$ (which will typically have nothing to do with j or with any other almost complex structure on X) will be finite sets.

Definition 2.2 *([3] for $\alpha = \kappa_X$, [10] in general) The standard surface count $\mathcal{DS}_f(\alpha)$ is defined as the Gromov–Witten invariant which counts the elements of $\mathcal{M}^J_{X_r(f)}(c_\alpha)$ for generic J and Ω with sign according to the spectral flow.*

Quite standard and elementary arguments in Gromov–Witten theory show that $\mathcal{DS}_f(\alpha)$ is independent of the choice of J and Ω in the definition. Its dependence on f is a rather subtler question, though we shall see in the coming section that it is in fact independent of f as long as the fibers have large enough area (which, we recall, may always be arranged using Donaldson's construction).

Returning to the point of view advocated at the end of the introduction, we note that by the correspondence (2.1) we have

$$\mathcal{M}^j_X(\alpha) = \mathcal{M}^{\mathbb{J}_j}_{X_r(f)}(c_\alpha),$$

so that by perturbing the almost complex structure $\mathbb{J}_j$ to some generic, smooth almost complex structure J on $X_r(f)$ and then counting the elements of $\mathcal{M}^J_{X_r(f)}(c_\alpha)$ we are effectively perturbing the notion of what it means to be a pseudoholomorphic subset of X Poincaré dual to α, and using this perturbed notion to define an invariant. Noting that sections can never be multiply covered, we see that using this perturbed notion of pseudoholomorphicity evades the multiple cover issue. Happily, although these constructions are rather difficult to carry through in detail in any specific cases, we shall see below that it is possible to prove general theorems about the behavior of $\mathcal{DS}$ which yield interesting information about pseudoholomorphic curves in symplectic 4-manifolds.

3 Relation to the Gromov–Taubes invariant

That the information contained by $\mathcal{DS}_f$ ultimately relates to the four-manifold and not just to the Lefschetz fibration f is ensured by the following.

Theorem 3.1 *([13]) Let $f\colon (X,\omega) \to S^2$ be a symplectic Lefschetz fibration and $\alpha \in H^2(X,\mathbb{Z})$ any class such that $\omega\cdot\alpha < \omega\cdot(fiber)$. Then $\mathcal{DS}_{(X,f)}(\alpha) = Gr(\alpha)$.*

Proof The proof of Theorem 3.1 occupies the bulk of [13]; we shall only briefly summarize it here. One first needs to find a rather special type of almost complex structure j on X with which to evaluate $Gr(\alpha)$. j should make f a pseudoholomorphic map, and the curves C in the zero-dimensional moduli space $\mathcal{M}_X^j(\alpha)$ which contribute to $Gr(\alpha)$ should obey a number of special properties with respect to f, the most subtle of which is that on some neighborhood of C there should exist some other *integrable* complex structure $\tilde{j}$ which both preserves TC and makes $f\colon X \to S^2$ pseudoholomorphic. Generic almost complex structures j making f pseudoholomorphic will *not* have moduli spaces whose curves all satisfy this property, since one usually encounters obstructions to constructing such a $\tilde{j}$ near the critical points of the restrictions $f|_C$. However, it is shown in Section 3 of [13] that one may delicately perturb a generic initial choice j' to a j which does satisfy this property. Further, there will exist a path j_t of almost complex structures defined near C which connects $j_0 = \tilde{j}$ to $j_1 = j$, with each j_t preserving TC and making f pseudoholomorphic.

For each t, we can use Taubes' definition from [12] to determine the numbers $r_t(C)$ with which C would contribute to $Gr(\alpha)$ if we were using j_t to evaluate Gr. Now, via the correspondence (2.1), we can define numbers $r'_t(C)$ which may be viewed as the contribution of s_C to $\mathcal{DS}$ "using $\mathbb{J}_{j_t}$ to evaluate $\mathcal{DS}$" as follows: take a generic smooth almost complex structure J_t on $X_r(f)$ which is Hölder-close to our (only Hölder-continuous) almost complex structure $\mathbb{J}_{j_t}$. By Gromov compactness, all J-holomorphic sections of $X_r(f)$ will be close to one and only one $\mathbb{J}_{j_t}$-holomorphic section, and we take $r'_t(C)$ to be the signed count of those J-holomorphic sections which are close to the section s_C, with the signs obtained according to the spectral flow in the usual way.

For $t = 0$ the fact that $\tilde{j} = j_0$ is integrable near C is easily seen to imply that $\mathbb{J}_{j_0}$ is integrable (and so smooth) near s_C, and so we may use $\mathbb{J}_{j_0}$ directly to evaluate $r'_0(C)$; no perturbation is needed. Indeed, the integrability of $\mathbb{J}_{j_0}$ implies that $r'_0(C) = 1$, which fortuitously agrees with the Taubes weight $r_0(C) = 1$ since j_0 is integrable. We now consider the effect of varying t up to 1. In Section 5 of [12], Taubes carefully analyzed the changes that the spaces $\mathcal{M}_X^{j_t}(\alpha)$ can undergo under a generic variation in the almost complex structure; in order to obtain an invariant (*i.e.*, a total count which does not depend on the almost complex structure), he was obliged to have the $r_t(C)$ remain constant for all but finitely many values of t, at which they had to obey certain specific wall-crossing formulas. Borrowing his analysis of the moduli spaces, one can likewise see (as is shown in Section 5 of [13]) that the fact that $\mathcal{DS}$ is already known to be independent of the almost complex structure J used to define it implies that the numbers $r'_t(C)$ must obey identical wall-crossing formulas to those of $r_t(C)$. The matching initial conditions $r_0(C) = r'_0(C) = 1$ combined with the fact that r_t and r'_t change in the same way as t varies then proves that $r_1(C) = r'_1(C)$. In other words, for each C which contributes to $Gr(\alpha)$, the corresponding section s_C contributes to $\mathcal{DS}_f(\alpha)$ with the same weight; thanks to the correspondence (2.1), this implies that the invariants agree. $\square$

Theorem 3.1 had been conjectured by Smith in [10] based partly on the following result, which bore a striking resemblance to the duality $Gr(\alpha) = \pm Gr(\kappa_X - \alpha)$ which arises from the charge-conjugation symmetry in Seiberg–Witten theory combined with Taubes' equivalence [11] between SW and Gr (note that Theorems

3.1 and 3.2 provide a new proof of this duality for symplectic 4-manifolds with $b^+ > b_1 + 1$):

Theorem 3.2 *([10]) If $b^+(X) > b_1(X) + 1$ and if $f\colon X' \to S^2$ is a symplectic Lefschetz fibration on a blowup $\pi\colon X' \to X$ of X obtained by Donaldson's construction with k sufficiently large, then*

$$\mathcal{DS}_f(\pi^*\alpha) = \pm\mathcal{DS}_f(\pi^*(\kappa_X - \alpha)). \tag{3.1}$$

We shall first outline the proof of this theorem in the comparatively easy case when

$$\omega \cdot \alpha > \omega \cdot \kappa_X. \tag{3.2}$$

In this case, since holomorphic curves cannot have negative symplectic area, the right hand side of Equation 3.1 is clearly zero. As for the left hand side, one can check using the adjunction formula and the fact that the fibers of f are Poincaré dual to $k[\omega]$ that if $\alpha' \in H^2(X', \mathbb{Z})$ is obtained from $\pi^*\alpha$ by adding the Poincaré dual of each exceptional divisor of the blowup, we have $\mathcal{DS}_f(\alpha') = \mathcal{DS}_f(\alpha)$ and

$$r := \langle \alpha', fiber\rangle > 2g - 2,$$

where g is the genus of the fibers of the fibration f.

From f, the *relative Picard scheme* is constructed in [3] as a smooth symplectic manifold $P_r(f)$ with a map $G\colon P_r(f) \to S^2$ whose fiber $G^{-1}(t)$ over a regular value t of f is naturally identified with the Picard variety $Pic^r f^{-1}(t)$ of degree-r holomorphic line bundles on $f^{-1}(t)$. We then have an Abel-Jacobi map

$$AJ\colon X_r(f) \to P_r(f)$$

which takes an element of $X_r(f)$ (*i.e.*, an effective divisor of degree r on some $f^{-1}(t)$) to its associated line bundle on $f^{-1}(t)$. Now for $L \in Pic^r f^{-1}(t) \subset P_r(f)$, the preimage $AJ^{-1}(L)$ consists of the vanishing loci of holomorphic sections of L, so that $AJ^{-1}(L) = \mathbb{P}H^0(L)$. We have $h^1(L) = h^0(\kappa_{f^{-1}(t)} \otimes L^*) = 0$ since $\kappa_{f^{-1}(t)} \otimes L^*$ has degree $2g - 2 - r < 0$, and so Riemann-Roch tells us that $h^0(L) = r - g + 1$, independently of L. Examining AJ slightly more carefully, one can see that it is a submersion, so that (thanks to the fact that $r > 2g-2$) AJ defines a $\mathbb{C}P^{r-g}$-bundle $X_r(f) \to P_r(f)$.

Now a computation of Section 5 of [3] shows that the real index of the $\bar{\partial}$ operator acting on sections of $P_r(f)$ is $1 + b_1(X) - b^+(X)$, which is assumed to be negative in the statement of the theorem. Thus generic almost complex structures J' on $P_r(f)$ will admit no holomorphic sections. Since $AJ\colon X_r(f) \to P_r(f)$ is a fibration over $P_r(f)$ whose fibers $\mathbb{C}P^{r-g}$ admit natural complex structures, it is straightforward to find almost complex structures J on $X_r(f)$ such that AJ is a (J, J')-holomorphic map. But then J is an almost complex structure on $X_r(f)$ which admits no pseudoholomorphic sections at all, since any such section, when composed with AJ, would give a J'-holomorphic section of $P_r(f)$. Using this J to evaluate the invariant, we immediately conclude that $\mathcal{DS}_f(\alpha') = 0$, proving the theorem in the case of present concern.

When $r \leq 2g - 2$, the structure of the Abel–Jacobi map is more complicated. Recall that in Donaldson's construction the number of exceptional sections is $k^2[\omega]^2$, so $r = \langle \alpha', fiber\rangle = k^2[\omega]^2 + k\alpha \cdot [\omega]$. Meanwhile by the adjunction formula $2g-2 = k^2[\omega]^2 + k\kappa_X \cdot [\omega]$, so if the degree k of the pencil is sufficiently high, then $r/(g-1)$ will be close to two. In this case, Serre duality gives rise to an embedding of $X_{2g-2-r}(f)$ in $P_r(f)$. The Abel–Jacobi map may then be seen to generically restrict over $X_{2g-2-r}(f)$ as a $\mathbb{C}P^{r-g+1}$-bundle, while over the rest of $P_r(f)$ it is a $\mathbb{C}P^{r-g}$-bundle. The almost complex structures on $X_{2g-2-r}(f)$, $P_r(f)$, and $X_r(f)$ may then be chosen so that any holomorphic section of $X_r(f)$ in the class $c_{\alpha'}$ is sent by AJ to a holomorphic section of $X_{2g-2-r}(f)$ in class $c_{\kappa_{X'}-\alpha'}$. A detailed analysis of this correspondence, performed in Section 6 of [10], then yields the duality theorem.

4 Refinements

Taubes' discovery that, when $b^+(X) > 1$, $Gr(\alpha) = 0$ whenever $\omega\cdot\alpha > \omega\cdot\kappa_X$ long seemed to be one of the many rather mysterious consequences of Seiberg–Witten theory; however, Theorems 3.1 and 3.2 hopefully shed somewhat more light on it, at least when $b^+ > b_1 + 1$. Theorem 3.1 shows that, in counting j-holomorphic curves in X (or, equivalently, $\mathbb{J}_j$-holomorphic sections of $X_r(f)$) according to Taubes' prescription, we obtain the same information if we use the perturbed notion of pseudoholomorphicity provided by the Donaldson–Smith approach of counting J-holomorphic sections of $X_r(f)$ for arbitrary J. The proof of Theorem 3.2 then shows that, when $\omega \cdot \alpha > \omega \cdot \kappa_X$, there is a particular choice of this perturbed J for which the moduli space of sections counted by the invariant is empty, so that obviously the invariant must vanish. Now, our identification of an empty moduli space contains strictly more information than the vanishing result, since in principle an invariant like $Gr(\alpha)$ could vanish while receiving contributions from a wide variety of sources which happen to cancel each other out. One might hope to deduce further results from this.

In this direction, recall that $Gr(\alpha)$ counts all of the j-holomorphic submanifolds of X, including disconnected ones and even ones with multiply-covered square-zero toroidal components, which are Poincaré dual to α. It is natural to want an invariant which, say, only counts connected curves, or more generally counts possibly-reducible curves while keeping track of their decomposition into reducible components. The scenario pictured in Figure 1 prevents us from doing this in full, but we can at least keep track of the non-toroidal components and make the following definition.

Definition 4.1 *Let $\alpha \in H^2(X;\mathbb{Z})$. Let*

$$\alpha = \beta_1 + \cdots + \beta_m + c_1\tau_1 + \cdots + c_n\tau_n$$

be a decomposition of α into distinct summands, where none of the β_i satisfies $\beta_i^2 = \kappa_X \cdot \beta_i = 0$, while the τ_i are distinct classes which are primitive in the lattice $H^2(X;\mathbb{Z})$ and all satisfy $\tau_i^2 = \kappa_X \cdot \tau_i = 0$. Then

$$Gr(\alpha; \beta_1, \ldots, \beta_m, c_1\tau_1, \cdots, c_n\tau_n)$$

is the invariant counting ordered $(m+n)$-tuples $(C_1, \ldots, C_{m+n})$ of transversely intersecting smooth pseudoholomorphic curves in X, where

(i) *for $1 \le i \le m$, C_i is a connected curve Poincaré dual to β_i which passes through some prescribed generic set of $d(\beta_i)$ points;*

(ii) *for $m+1 \le k \le m+n$, C_k is a union of connected curves Poincaré dual to classes $l_{k,1}\tau_k, \cdots, l_{k,p}\tau_k$ decorated with positive integer multiplicities $m_{k,q}$ with the property that*

$$\sum_q m_{k,q} l_{k,q} = c_k.$$

The weight of each component of each such curve is to be determined according to the prescription given in the definition of the Gromov invariant in [12] *(in particular, the components $C_{k,q}$ in class $l_{k,q}\tau_k$ are given the weight $r(C_{k,q}, m_{k,q})$ specified in Section 3 of* [12]*), and the contribution of the entire curve is the product of the weights of its components. As notation, we set $Gr(0;0) = 1$.*

The objects counted by $Gr(\alpha; \alpha_1, \ldots, \alpha_n)$ will then be reducible curves with smooth irreducible components and a total of $\sum \alpha_i \cdot \alpha_j$ nodes arising from intersections between these components. $Gr(\alpha)$ is the sum over all decompositions of α into classes which are pairwise orthogonal under the cup product of the

$$\frac{d(\alpha)!}{\prod(d(\alpha_i)!)} Gr(\alpha; \alpha_1, \ldots, \alpha_n);$$

in turn, one has

$$Gr(\alpha; \alpha_1, \ldots, \alpha_n) = \prod_{i=1}^{n} Gr(\alpha_i; \alpha_i).$$

It is natural to attempt to make similar refinements on the Donaldson–Smith side. Now Theorem 2.1 tells us that all of the sections s of $X_r(f)$ with C_s Poincaré dual to α are homotopic regardless of how C_s itself decomposes into components. Thus, if J_t is a one-parameter family of almost complex structures on $X_r(f)$, in the parametrized moduli space $\mathcal{PM}^{(J_t)}_{X_r(f)}(c_\alpha) = \{(s,t) | s \in \mathcal{M}^{J_t}_{X_r(f)}(c_\alpha)\}$ we might in principle have an interval whose left endpoint corresponds to a curve which decomposes into reducible components in a different fashion than does the curve corresponding to its right endpoint. If this were to happen, the signed count of J-holomorphic sections with a given decomposition type could not be expected to be independent of the choice of almost complex structure J.

Let Δ be the diagonal stratum in $X_r(f)$ consisting of divisors with one or more points repeated. A more careful analysis reveals that if s_t is an interval of J_t-holomorphic sections, then the decomposition type of the s_t has the potential to change at precisely those t for which the number of nontransverse intersections of s_t with Δ jumps. For a generic family of almost complex structures J_t, assuming the moduli spaces $\mathcal{M}^{J_t}_{X_r(f)}(c_\alpha)$ have been cut down by incidence conditions to be zero-dimensional, one expects finitely many t for which some $s \in \mathcal{M}^{J_t}_{X_r(f)}(c_\alpha)$ has a one-real dimensional tangency to Δ. This prevents such families of almost complex structures from being used to define refinements of $\mathcal{DS}$ which keep track of how the descendant curves of the sections being counted decompose into reducible components.

On the other hand, if we require the members of the family J_t to preserve the diagonal stratum Δ, then all nontransverse intersections of a J_t-holomorphic section with Δ will be two-real dimensional. As such, for a generic one-parameter family of such sections, no nontransverse intersections will arise. More generally, if we impose $d(\alpha) - n$ incidence conditions, so that the expected dimension of the space of J-holomorphic sections in class c_α with n tangencies to the diagonal is zero, the parametrized moduli spaces for one-parameter families of almost complex structures will not contain any sections with more than n tangencies to the diagonal. Thus, the decomposition types of the sections with n tangencies to the diagonal will remain constant on the intervals within such a parametrized moduli space.

Requiring Δ to be pseudoholomorphic does introduce an additional possible source of noncompactness in our moduli spaces: sequences of the sections with isolated intersections with the diagonal that we wish to count might converge to sections entirely contained in Δ. By restricting attention to only certain decomposition types, we may rule this out, and obtain:

Theorem 4.2 *Let $\alpha = \alpha_1 + \cdots + \alpha_n$, with none of the α_i equal to $m\beta$ for a class β with $\beta^2 = \kappa_X \cdot \beta = 0$, and with $\alpha_i \cdot e \geq -1$ with equality only if $\alpha_i = e$ whenever e is the class of a (-1)-sphere. Then the number $\widetilde{\mathcal{DS}}_f(\alpha; \alpha_1, \ldots, \alpha_n)$ which counts with sign the J-holomorphic sections s of $X_r(f)$ with $\Omega \subset C_s$ and C_s equal to a union of surfaces Poincaré dual to the α_i for an almost complex structure J generic among those which preserve Δ and a generic set Ω of $d(\alpha) - \sum_{i<j} \alpha_i \cdot \alpha_j$ points is independent of the generic pair (J, Ω) used to define it.*

The sections s contributing to $\widetilde{\mathcal{DS}}_f(\alpha; \alpha_1, \ldots, \alpha_n)$ will each have $\sum_{i<j} \alpha_i \cdot \alpha_j$ tangencies to Δ, one for each intersection between the reducible components of the descendant curve C_s. In fact, it can be shown that (because J preserves $T\Delta$), if s is any J-holomorphic section, then all intersection points between reducible components of C_s will contribute positively to the intersection number of those components (similarly to the situation with genuine pseudoholomorphic curves).

Once we know we have an invariant $\widetilde{\mathcal{DS}}_f(\alpha; \alpha_1, \ldots, \alpha_n)$, we can evaluate it by taking J to be close to some $\mathbb{J}_j$, and the proof of Theorem 3.1 then goes through with only minor technical changes to show that:

Theorem 4.3 *If $\alpha = \alpha_1 + \cdots + \alpha_n$ is a decomposition as in Theorem 4.2, then*

$$\frac{(\sum d(\alpha_i))!}{\prod(d(\alpha_i)!)} Gr(\alpha; \alpha_1, \ldots, \alpha_n) = \widetilde{\mathcal{DS}}_f(\alpha; \alpha_1, \ldots, \alpha_n)$$

provided that the degree of the fibration is large enough that $\langle [\omega_{X'}], fiber \rangle > [\omega_{X'}] \cdot \alpha$.

The complete proofs of Theorems 4.2 and 4.3 will appear in [14].

One might then hope that the proof of Theorem 3.2 could be used to glean information about $\widetilde{\mathcal{DS}}_f$ and hence about $Gr(\alpha; \alpha_1, \ldots, \alpha_n)$. Unfortunately, this is not the case, as a result of the fact that the special almost complex structures considered by Smith cannot be taken to preserve the diagonal. If such an argument could be used, we would be able to conclude that each invariant $Gr(\alpha; \alpha_1, \ldots, \alpha_n)$ vanishes whenever α has larger symplectic area than the canonical class and $\widetilde{\mathcal{DS}}_f(\alpha; \alpha_1, \ldots, \alpha_n)$ is defined. However, the manifold considered in [6] admits a symplectic form such that, for certain primitive classes α, β, γ, and δ each with positive symplectic area, the canonical class is $2(\alpha + \beta + \gamma)$ but the invariant $Gr(2(\alpha + \beta + \gamma) + \delta; \alpha, \beta, \gamma, \alpha + \beta + \gamma + \delta)$ is nonzero. (This provides a counterexample to a statement of which the author had previously mistakenly announced a proof, which had been based on an attempt to refine $\mathcal{DS}$ using almost complex structures which do not preserve Δ.)

Another way in which Gr may sometimes be refined is by counting pseudoholomorphic curves in a given homology class which, instead of being embedded, have a prescribed number of self-intersections. In general, it is somewhat unclear what the proper analogue of the Gromov–Taubes invariant in this context should be, since the multiple cover problem is more serious than in the embedded case. The expected complex dimension of the space of pseudoholomorphic curves Poincaré dual to the class α having n self-intersections is $d(\alpha) - n$, and if some for some

$m > 1$ α/m is an integral class with $d(\alpha) - n \leq d(\alpha/m)$, any attempts to define an invariant will be complicated by the possibility of a sequence of the curves we wish to count converging to an m-fold cover of a curve Poincaré dual to α/m. (In the embedded case where $n = 0$, one can see from the formula for $d(\alpha)$ and the adjunction formula that the only classes α with $d(\alpha) \geq 0$ for which this issue can arise are classes of square-zero tori, which can be handled by Taubes' prescription. It is not clear how to generalize Taubes' prescription to the various cases which can arise when $n > 0$, especially those for which $d(\alpha/m)$ is strictly larger than $d(\alpha) - n$.)

With this caveat in place, we make the following:

Definition 4.4 *A class $\alpha \in H^2(X, \mathbb{Z})$ is called* strongly n-semisimple *if there exist* no *decompositions $\alpha = \alpha_1 + \cdots + \alpha_l$ such that each α_i has $d(\alpha_i) \geq 0$ and is Poincaré dual to the image of a symplectic immersion, and α_1 is equal to $m\beta$ ($m > 1$) where β satisfies $d(\beta) \geq \max\{0, d(\alpha_1) - n + \alpha_1 \cdot (\alpha - \alpha_1)\}$. α is called* weakly n-semisimple *if the only decompositions $\alpha = \alpha_1 + \cdots + \alpha_n$ as above which exist have $\alpha_1^2 = \kappa_X \cdot \alpha_1 = 0$.*

For instance, every class is weakly 0-semisimple, while the only classes which are not weakly 1-semisimple are those classes α such that there exists a class $\beta \in H^2(X; \mathbb{Z})$ such that $\beta \cdot (\alpha - 2\beta) = 0$ and β is Poincaré dual either to a symplectic sphere of square 0 or a symplectic genus-two curve of square 1, while $\alpha - 2\beta$ is Poincaré dual to some embedded (and possibly disconnected) symplectic submanifold. For strong semisimplicity, one needs to add the assumption that α is not Poincaré dual to a symplectic immersion having a component which is a square-zero torus in a non-primitive homology class.

For weakly- or strongly n-semisimple classes α, there is an obvious analogue of the Gromov–Taubes invariant $Gr_n(\alpha)$, defined by counting j-holomorphic curves C which are unions of curves C_i Poincaré classes α_i carrying multiplicities m_i which are equal to 1 unless C_i is a square-zero torus with $\sum m_i\alpha_i = \alpha$, such that C has n transverse double points and passes through a generic set of $d(\alpha) - n$ points of X; each such C contributes the product of the Taubes weights $r(C_i, m_i)$ to the count $Gr_n(\alpha)$. Here a double point arising as an intersection of the components C_i and C_j is counted as contributing a multiplicity $m_i m_j$ toward the total n. Since the condition of n-semisimplicity is engineered to rule out the only additional possible source of noncompactness of the relevant moduli spaces, the proof that $Gr(\alpha)$ is independent of the choice of almost complex structure used to define it goes through to show the same result for $Gr_n(\alpha)$.

While in ordinary Gromov–Taubes theory it is difficult to find an analogue of Gr for nodal curves in most homology classes, it does turn out to be possible to build a more general candidate for such an invariant by mixing the Donaldson–Smith approach with a family blowup construction along the lines of that used in [5]. To motivate this, consider a curve C Poincaré dual to α having just one double point p (which might be either a transverse intersection of two of its reducible components or a self-intersection of one component), which is to be counted by a putative invariant $Gr_1(\alpha)$. Let X_2 denote the blowup of the diagonal in $X \times X$; note that X_2 fibers naturally over X, the fiber X^q over $q \in X$ being the blowup of X at q. Now where e is the Poincaré dual of the exceptional divisor, for $q \neq p$ the proper transform of C in X^q still has a double point and will be Poincaré dual to either α or $\alpha - e$ depending on whether $q \in C$. For $q = p$, however, the proper transform of C will be an embedded curve Poincaré dual to $\alpha - 2e$. Thus, the curves in which we are interested correspond to those curves contributing to one of the $Gr_{X^p}(\alpha - 2e)$ as p ranges over X.

Iterating this construction, one obtains fibrations $X_{n+1} \to X_n$ whose fiber X^b over $b \in X_n$ is an n-fold blowup of X, with b parameterizing the set of points which are blown up. This converts the problem of counting curves Poincaré dual to α with n nodes to the problem of enumerating embedded curves Poincaré dual to $\alpha - 2\sum_{l=1}^{n} e_l$ where the e_i are the exceptional divisors of the n blowups, as b ranges over X_n. By itself, this doesn't significantly simplify the problem of defining the invariants $Gr_n(\alpha)$ for α which are not n-semisimple, since over various substrata of the parameter space X_n one will find undesirably-large boundary components of the moduli spaces that we are concerned with (though we should mention that in the algebraic category Liu has developed in [5] a method for analyzing these strata in order to obtain an invariant). In [14], though, it is shown that from a symplectic Lefschetz fibration $f\colon X \to S^2$, by lifting f to maps f^b on the various blowups X^b we can obtain a family of relative Hilbert schemes $\mathcal{X}_r^n(f) \to S^2 \times X_n'$ over a subset $X_n' \subset X_n$ whose complement has codimension 4 such that the total space $\mathcal{X}_r^n(f)$ is smooth and symplectic and each restriction over $S^2 \times \{b\}$ is the relative Hilbert scheme $X_r^b(f^b)$ constructed from $f^b\colon X^b \to S^2$. This allows one to define a family standard surface count $\mathcal{FDS}_f^n$ which enumerates holomorphic sections of the various $X_r^b(f^b)$ as b ranges over X_n'; in spite of the noncompactness of the parameter space X_n', it is shown in Lemma 3.5 of [14] that if the almost complex structure on $\mathcal{X}_r^n(f)$ is chosen generically from an appropriate family, the relevant moduli spaces will be compact and in particular will not contain sections of $X_r^{b_m}(f^{b_m})$ for a sequence b_m converging to an element of the codimension-four set $X_n \setminus X_n'$. Since working in families has allowed us to eliminate the nodes from our curves, $\mathcal{FDS}_f^n$ can be evaluated using a generic almost complex structure on $\mathcal{X}_r^n(f)$; no special behavior with respect to the diagonal stratum is necessary. When we perturb the tautological almost complex structure $\mathbb{J}_j$ to a generic one, the sections corresponding the multiple covers which stand in the way of defining $Gr_n(\alpha)$ are perturbed to non-multiple sections on the same footing as all of the others.

Using an Abel–Jacobi map from $\mathcal{X}_r^n(f)$ to a family of relative Picard schemes $\mathcal{P}_r^n(f)$, one then constructs an almost complex structure J like that in the proof of Theorem 3.2; if the class α has larger area than the canonical class, the relevant moduli space for this almost complex structure is then guaranteed to be empty as long as $b^+(X) > b_1(X) + 1 + 4n$. (The extra $4n$ term is the dimension of the family X_n over which we are working, the point being that we need to ensure that there are no holomorphic sections of any of the relative Picard schemes $P_r(f^b)$ as b ranges over X_n, so that we require $\mathrm{ind}_{P_r(f)}\,\bar{\partial} + \dim X_n < 0$.) More generally, still assuming that $b^+(X) > b_1(X) + 1 + 4n$, we find that regardless of the area of α, as in the proof of Theorem 3.2 a nonvanishing invariant $\mathcal{FDS}_f^n(\alpha)$ would give rise to J-holomorphic sections of one of the members of the family $\mathcal{X}_{2g-2-r}^n(f)$ in the class $c_{\kappa_{X'}-\alpha'}$ for generic J on $\mathcal{X}_{2g-2-r}^n(f)$. Appealing to Gromov compactness and the correspondence between $\mathbb{J}_j$-holomorphic sections and j-holomorphic curves, one then finds that if $\mathcal{FDS}_f^n(\alpha) \neq 0$ there is a particular j for which there exist j-holomorphic curves Poincaré dual to both classes α and $\kappa_X - \alpha$.

This outline should make plausible the following theorem; the proofs (and the somewhat-cumbersome precise statements) of the various lemmas indicated above appear in [14]:

Theorem 4.5 (i) *If $b^+(X) > b_1(X) + 1 + 4n$, then for all $\alpha \in H^2(X;\mathbb{Z})$ either $\mathcal{FDS}_f^n(\alpha) = 0$ or there exists an almost complex structure j on X simultaneously admitting j-holomorphic curves Poincaré dual to α and $\kappa_X - \alpha$.*

(ii) *If α is strongly n-semisimple, then $\mathcal{FDS}_f^n(\alpha) = n!Gr_n(\alpha)$.*

As is explained in the introduction to [14], the second alternative in part (i) above can be ruled out for generic almost complex structures j; however, the j in the statement of the theorem cannot be taken generically.

We note that unless α is strongly n-semisimple, so that there exists an invariant $Gr_n(\alpha)$ which only depends on the four-manifold with which it is possible to equate $\mathcal{FDS}_f^n(\alpha)$, it is unclear whether or not $\mathcal{FDS}_f^n(\alpha)$ depends on the choice of Lefschetz fibration f. In light of the situation with the ordinary invariant $\mathcal{DS}_f$, it is natural to conjecture that $\mathcal{FDS}_f^n$ is in fact independent of f and agrees with some combination of Ruan–Tian invariants. If this is the case, $\mathcal{FDS}$ would seem to be an appropriate candidate for a generalization of the Gromov–Taubes invariant to nodal curves. It would also be interesting to know whether, in the case where X is Kähler, $\mathcal{FDS}$ might be related to the invariants constructed by Liu in [5] using family Seiberg–Witten theory and algebro-geometric methods.

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