Twisted geodesic flows and symplectic topology

Mike Usher

UGA

September 12, 2008/ UGA Geometry Seminar

Outline

1. Classical mechanics of a particle in a potential on a manifold
    - Lagrangian formulation
    - Hamiltonian formulation
2. Magnetic flows
    - Lagrangian formulation
    - The twisted cotangent bundle
3. Closed orbits
    - Existence theorems
    - Floer homology

The dynamics of a classical particle moving on a Riemannian manifold (M,g) are dictated by a *Lagrangian*

$$L : TM \to \mathbb{R}.$$

If the particle has mass 1 and is subjected to a conservative force $F = -\nabla U$ where $U : M \to \mathbb{R}$, we'll have

$$L(q,v) = \frac{1}{2}|v|^2 - U(q).$$

The dynamics of a classical particle moving on a Riemannian manifold (M, g) are dictated by a *Lagrangian*

$$L : TM \to \mathbb{R}.$$

If the particle has mass 1 and is subjected to a conservative force $F = -\nabla U$ where $U : M \to \mathbb{R}$, we'll have

$$L(q, v) = \frac{1}{2}|v|^2 - U(q).$$

The principle of least (really, stationary) action says that the trajectory $q : [0, T] \to M$ of the particle will be a critical point of the *action*

$$S[q] = \int_0^T L(q(t), \dot{q}(t))dt,$$

where q varies among paths with $q(0), q(T)$ fixed.

The Euler-Lagrange equation states that these critical points q are solutions to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}(q(t), \dot{q}(t))\right) = \frac{\partial L}{\partial q_i}(q(t), \dot{q}(t)),$$

where $(q_1, \ldots, q_n, v_1, \ldots, v_n)$ is a standard coordinate chart on TM.

The principle of least (really, stationary) action says that the trajectory $q : [0,T] \to M$ of the particle will be a critical point of the *action*

$$S[q] = \int_0^T L(q(t), \dot{q}(t))dt,$$

where q varies among paths with $q(0), q(T)$ fixed.
The Euler-Lagrange equation states that these critical points q are solutions to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}(q(t), \dot{q}(t))\right) = \frac{\partial L}{\partial q_i}(q(t), \dot{q}(t)),$$

where $(q_1, \ldots, q_n, v_1, \ldots, v_n)$ is a standard coordinate chart on TM.

If

$$L(q,v) = \frac{1}{2}|v|^2 - U(q),$$

by taking the q_i to form a normal coordinate chart around $q(t)$ with respect to g the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}(q(t),\dot{q}(t))\right) = \frac{\partial L}{\partial q_i}(q(t),\dot{q}(t))$$

is easily seen to take the form

$$\frac{D\dot{q}}{dt} = -\nabla U(q(t)).$$

This is consistent with Newton's second law, and in particular when the potential energy U is identically zero we find that the possible trajectories are precisely geodesics.

If

$$L(q,v) = \frac{1}{2}|v|^2 - U(q),$$

by taking the q_i to form a normal coordinate chart around $q(t)$ with respect to g the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}(q(t),\dot{q}(t))\right) = \frac{\partial L}{\partial q_i}(q(t),\dot{q}(t))$$

is easily seen to take the form

$$\frac{D\dot{q}}{dt} = -\nabla U(q(t)).$$

This is consistent with Newton's second law, and in particular when the potential energy U is identically zero we find that the possible trajectories are precisely geodesics.

If

$$L(q,v) = \frac{1}{2}|v|^2 - U(q),$$

by taking the q_i to form a normal coordinate chart around $q(t)$ with respect to g the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}(q(t),\dot{q}(t))\right) = \frac{\partial L}{\partial q_i}(q(t),\dot{q}(t))$$

is easily seen to take the form

$$\frac{D\dot{q}}{dt} = -\nabla U(q(t)).$$

This is consistent with Newton's second law, and in particular when the potential energy U is identically zero we find that the possible trajectories are precisely geodesics.

The Legendre transform

Given the Lagrangian $L : TM \to \mathbb{R}$ (which should satisfy a convexity hypothesis, *e.g.* it should grow quadratically in sufficiently large $|v|$), its Legendre transform is the "Hamiltonian"

$$H : T^*M \to \mathbb{R}$$

defined by

$$H(q,p) = \sup_{v \in T_qM} \left(\langle p,v \rangle - L(q,v) \right).$$

At least for the Lagrangians that we'll consider, the supremum will be attained at the unique $v_p \in T_qM$ satisfying

$$(d(L|_{T_qM}))_{v_p} = p.$$

If $L(q,v) = \frac{1}{2}|p|^2 - U(q)$ then v_p is just the metric dual to p, and

$$H(q,p) = \frac{1}{2}|p|^2 + U(q).$$

The Legendre transform

Given the Lagrangian $L : TM \to \mathbb{R}$ (which should satisfy a convexity hypothesis, *e.g.* it should grow quadratically in sufficiently large $|v|$), its Legendre transform is the "Hamiltonian"

$$H : T^*M \to \mathbb{R}$$

defined by

$$H(q,p) = \sup_{v \in T_qM} \left(\langle p,v \rangle - L(q,v) \right).$$

At least for the Lagrangians that we'll consider, the supremum will be attained at the unique $v_p \in T_qM$ satisfying

$$(d(L|_{T_qM}))_{v_p} = p.$$

If $L(q,v) = \frac{1}{2}|p|^2 - U(q)$ then v_p is just the metric dual to p, and

$$H(q,p) = \frac{1}{2}|p|^2 + U(q).$$

T^*M carries a canonical 1-form $\lambda_{can} = \sum p_i dq_i$, given by

$$(\lambda_{can})_{(q,p)}(V) = p(\pi_* V).$$

Then

$$\omega = -d\lambda_{can} = \sum_i dq_i \wedge dp_i \in \Omega^2(T^*M)$$

is symplectic (i.e. closed and nondegenerate).Via the Legendre transform, the Euler-Lagrange equation translates to the statement that the trajectories $(q(t),p(t))$ in T^*M are given by

$$(\dot{q}(t),\dot{p}(t)) = X_H(q,p).$$

Here the Hamiltonian vector field X_H is given by

$$\omega(X_H,\cdot) = dH.$$

One has $L_{X_H}\omega = di_{X_H}\omega = ddH = 0$, and $dH(X_H) = 0$, so the flow of X_H preserves both the symplectic structure and the function H.

T^*M carries a canonical 1-form $\lambda_{can} = \sum p_i dq_i$, given by

$$(\lambda_{can})_{(q,p)}(V) = p(\pi_* V).$$

Then

$$\omega = -d\lambda_{can} = \sum_i dq_i \wedge dp_i \in \Omega^2(T^*M)$$

is symplectic (i.e. closed and nondegenerate). Via the Legendre transform, the Euler-Lagrange equation translates to the statement that the trajectories $(q(t), p(t))$ in T^*M are given by

$$(\dot{q}(t), \dot{p}(t)) = X_H(q,p).$$

Here the Hamiltonian vector field X_H is given by

$$\omega(X_H, \cdot) = dH.$$

One has $L_{X_H}\omega = di_{X_H}\omega = ddH = 0$, and $dH(X_H) = 0$, so the flow of X_H preserves both the symplectic structure and the function H.

T^*M carries a canonical 1-form $\lambda_{can} = \sum p_i dq_i$, given by

$$(\lambda_{can})_{(q,p)}(V) = p(\pi_* V).$$

Then

$$\omega = -d\lambda_{can} = \sum_i dq_i \wedge dp_i \in \Omega^2(T^*M)$$

is symplectic (i.e. closed and nondegenerate).Via the Legendre transform, the Euler-Lagrange equation translates to the statement that the trajectories $(q(t),p(t))$ in T^*M are given by

$$(\dot{q}(t),\dot{p}(t)) = X_H(q,p).$$

Here the Hamiltonian vector field X_H is given by

$$\omega(X_H,\cdot) = dH.$$

One has $L_{X_H}\omega = di_{X_H}\omega = ddH = 0$, and $dH(X_H) = 0$, so the flow of X_H preserves both the symplectic structure and the function H.

T^*M carries a canonical 1-form $\lambda_{can} = \sum p_i dq_i$, given by

$$(\lambda_{can})_{(q,p)}(V) = p(\pi_* V).$$

Then

$$\omega = -d\lambda_{can} = \sum_i dq_i \wedge dp_i \in \Omega^2(T^*M)$$

is symplectic (i.e. closed and nondegenerate).Via the Legendre transform, the Euler-Lagrange equation translates to the statement that the trajectories $(q(t), p(t))$ in T^*M are given by

$$(\dot{q}(t), \dot{p}(t)) = X_H(q,p).$$

Here the Hamiltonian vector field X_H is given by

$$\omega(X_H, \cdot) = dH.$$

One has $L_{X_H}\omega = di_{X_H}\omega = ddH = 0$, and $dH(X_H) = 0$, so the flow of X_H preserves both the symplectic structure and the function H.

T^*M carries a canonical 1-form $\lambda_{can} = \sum p_i dq_i$, given by

$$(\lambda_{can})_{(q,p)}(V) = p(\pi_* V).$$

Then

$$\omega = -d\lambda_{can} = \sum_i dq_i \wedge dp_i \in \Omega^2(T^*M)$$

is symplectic (i.e. closed and nondegenerate).Via the Legendre transform, the Euler-Lagrange equation translates to the statement that the trajectories $(q(t), p(t))$ in T^*M are given by

$$(\dot{q}(t), \dot{p}(t)) = X_H(q,p).$$

Here the Hamiltonian vector field X_H is given by

$$\omega(X_H, \cdot) = dH.$$

One has $L_{X_H}\omega = di_{X_H}\omega = ddH = 0$, and $dH(X_H) = 0$, so the flow of X_H preserves both the symplectic structure and the function H.

The simplest way to incorporate a velocity dependence in the Lagrangian $L : TM \to \mathbb{R}$ aside from the kinetic energy is to choose a 1-form $\alpha \in \Omega^1(M)$ and put

$$L(q,v) = \frac{1}{2}|v|^2 - \alpha(v).$$

The Euler-Lagrange equation in this case works out (in normal coordinates, where $\alpha = \sum_j \alpha_j dq_j$) to

$$\ddot{q}_i - \sum_j \frac{\partial \alpha_i}{\partial q_j}\dot{q}_j = -\sum_j \frac{\partial \alpha_j}{\partial q_i}\dot{q}_j,$$

i.e., for all $w \in TM$,

$$\langle \frac{D\dot{q}}{dt}, w\rangle = d\alpha(\dot{q}, w).$$

The simplest way to incorporate a velocity dependence in the Lagrangian $L : TM \to \mathbb{R}$ aside from the kinetic energy is to choose a 1-form $\alpha \in \Omega^1(M)$ and put

$$L(q,v) = \frac{1}{2}|v|^2 - \alpha(v).$$

The Euler-Lagrange equation in this case works out (in normal coordinates, where $\alpha = \sum_j \alpha_j dq_j$) to

$$\ddot{q}_i - \sum_j \frac{\partial \alpha_i}{\partial q_j}\dot{q}_j = -\sum_j \frac{\partial \alpha_j}{\partial q_i}\dot{q}_j,$$

i.e., for all $w \in TM$,

$$\langle \frac{D\dot{q}}{dt}, w \rangle = d\alpha(\dot{q}, w).$$

Thus where $B : TM \to TM$ is the skew-symmetric endomorphism given by $\langle Bv, w\rangle = d\alpha(v, w)$ the equations of motion are

$$\frac{D\dot{q}}{dt} = B\dot{q}.$$

In the case $M \subset \mathbb{R}^3$, any skew-symmetric endomorphism is cross product with a vector; say $Bv = v \times \vec{\beta}$. So the equation of motion is $\ddot{q} = \dot{q} \times \vec{\beta}$, which empirically describes the motion of a particle of unit mass and charge in the magnetic field $\vec{\beta}$.

The statement that $d\alpha$ is exact translates to the statement that the magnetic field $\vec{\beta}$ has a global vector potential: $\vec{\beta} = \nabla \times \vec{\gamma}$.
One of Maxwell's equations is $\nabla \cdot \vec{\beta} = 0$; this corresponds to the statement that the form $d\alpha$ is closed. Thus our Lagrangian formulation accomodates magnetic fields which have global vector potentials, but not a general magnetic field $\vec{\beta}$, for which we'd just know that the 2-form σ given by

$$\sigma(v,w) = \langle v \times \vec{\beta}, w \rangle$$

is closed.
However, passing to the Hamiltonian picture we'll be able to remedy this.

The statement that $d\alpha$ is exact translates to the statement that the magnetic field $\vec{\beta}$ has a global vector potential: $\vec{\beta} = \nabla \times \vec{\gamma}$. One of Maxwell's equations is $\nabla \cdot \vec{\beta} = 0$; this corresponds to the statement that the form $d\alpha$ is closed. Thus our Lagrangian formulation accomodates magnetic fields which have global vector potentials, but not a general magnetic field $\vec{\beta}$, for which we'd just know that the 2-form σ given by

$$\sigma(v,w) = \langle v \times \vec{\beta}, w \rangle$$

is closed.

However, passing to the Hamiltonian picture we'll be able to remedy this.

The statement that $d\alpha$ is exact translates to the statement that the magnetic field $\vec{\beta}$ has a global vector potential: $\vec{\beta} = \nabla \times \vec{\gamma}$. One of Maxwell's equations is $\nabla \cdot \vec{\beta} = 0$; this corresponds to the statement that the form $d\alpha$ is closed. Thus our Lagrangian formulation accomodates magnetic fields which have global vector potentials, but not a general magnetic field $\vec{\beta}$, for which we'd just know that the 2-form σ given by

$$\sigma(v,w) = \langle v \times \vec{\beta}, w \rangle$$

is closed.
However, passing to the Hamiltonian picture we'll be able to remedy this.

The Legendre transform of $L(q,v) = \frac{1}{2}|v|^2 - \alpha(v)$ ($\alpha \in \Omega^1(M)$) works out to be $H : T^*M \to \mathbb{R}$ given by

$$H(q,p) = \frac{1}{2}|p + \alpha(q)|^2.$$

So where $K = \frac{1}{2}|p|^2$ is the standard kinetic energy, we have $H = K \circ T_\alpha$ where $T_\alpha : T^*M \to T^*M$ is a fiberwise translation. This implies that T_α sends the Hamiltonian vector field X_H (on the standard cotangent bundle $(T^*M, -d\lambda_{can})$) to the vector field X_K for the Hamiltonian K on the *twisted cotangent bundle* $(T^*M, \omega_{d\alpha})$, where

$$\omega_{d\alpha} = -d\lambda_{can} + \pi^* d\alpha \text{ satisfies } T_\alpha^* \omega_{d\alpha} = -d\lambda_{can}.$$

So we can view the flow as taking place on the symplectic manifold $(T^*M, \omega_{d\alpha})$ with the kinetic energy Hamiltonian $K = \frac{1}{2}|p|^2$; this only involves $d\alpha$ and not α, and so $d\alpha$ can be replaced by an arbitrary closed $\sigma \in \Omega^2(M)$.

The Legendre transform of $L(q,v) = \frac{1}{2}|v|^2 - \alpha(v)$ $(\alpha \in \Omega^1(M))$ works out to be $H : T^*M \to \mathbb{R}$ given by

$$H(q,p) = \frac{1}{2}|p + \alpha(q)|^2.$$

So where $K = \frac{1}{2}|p|^2$ is the standard kinetic energy, we have $H = K \circ T_\alpha$ where $T_\alpha : T^*M \to T^*M$ is a fiberwise translation. This implies that T_α sends the Hamiltonian vector field X_H (on the standard cotangent bundle $(T^*M, -d\lambda_{can})$) to the vector field X_K for the Hamiltonian K on the *twisted cotangent bundle* $(T^*M, \omega_{d\alpha})$, where

$$\omega_{d\alpha} = -d\lambda_{can} + \pi^* d\alpha \text{ satisfies } T_\alpha^* \omega_{d\alpha} = -d\lambda_{can}.$$

So we can view the flow as taking place on the symplectic manifold $(T^*M, \omega_{d\alpha})$ with the kinetic energy Hamiltonian $K = \frac{1}{2}|p|^2$; this only involves $d\alpha$ and not α, and so $d\alpha$ can be replaced by an arbitrary closed $\sigma \in \Omega^2(M)$.

Summary: If $B : TM \to TM$ is a skew-symmetric endomorphism such that $\sigma \in \Omega^2(M)$ given by $\sigma(v,w) = \langle Bv, w\rangle$ is closed, then the twisted geodesic flow

$$\frac{D\dot{q}}{dt} = B\dot{q}$$

arises as the Hamiltonian flow of the function $K(q,p) = \frac{1}{2}|p|^2$ on the symplectic manifold

$$(T^*M, -d\lambda_{can} + \pi^*\sigma).$$

We consider the Hamiltonian flow X_K of $K(q,p) = \frac{1}{2}|p|^2$ on the symplectic manifold $(T^*M, \omega_\sigma = -d\lambda_{can} + \pi^*\sigma)$ where $\sigma \in \Omega^2(M)$ is closed. We'll assume M is compact and without boundary.

Since $dK(X_K) = \omega_\sigma(X_K, X_K) = 0$, this flow preserves the energy levels $\{K = E\}$.

Question

Given $E > 0$, does there exist a closed orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to T^*M$ of X_K on the energy level $\{K = E\}$?

We consider the Hamiltonian flow X_K of $K(q,p) = \frac{1}{2}|p|^2$ on the symplectic manifold $(T^*M, \omega_\sigma = -d\lambda_{can} + \pi^*\sigma)$ where $\sigma \in \Omega^2(M)$ is closed. We'll assume M is compact and without boundary. Since $dK(X_K) = \omega_\sigma(X_K, X_K) = 0$, this flow preserves the energy levels $\{K = E\}$.

Question

Given $E > 0$, does there exist a closed orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to T^*M$ of X_K on the energy level $\{K = E\}$?

We consider the Hamiltonian flow X_K of $K(q,p) = \frac{1}{2}|p|^2$ on the symplectic manifold $(T^*M, \omega_\sigma = -d\lambda_{can} + \pi^*\sigma)$ where $\sigma \in \Omega^2(M)$ is closed. We'll assume M is compact and without boundary. Since $dK(X_K) = \omega_\sigma(X_K, X_K) = 0$, this flow preserves the energy levels $\{K = E\}$.

Question

Given $E > 0$, does there exist a closed orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to T^*M$ of X_K on the energy level $\{K = E\}$?

The untwisted case

If $\sigma = 0$, we're just asking for closed geodesics.

Theorem (Fet-Lyusternik, 1952)

On any closed Riemannian manifold (M, g) there is at least one (nonconstant) closed geodesic.

In the untwisted case, since replacing $\gamma(t)$ by $t \mapsto \gamma(at)$ preserves the geodesic condition and changes the energy from E to a^2E, it's superfluous to prescribe the energy.
Note that there need not exist any *contractible* geodesics, and there never exist any geodesics which are shorter than the injectivity radius.

The untwisted case

If $\sigma = 0$, we're just asking for closed geodesics.

Theorem (Fet-Lyusternik, 1952)

On any closed Riemannian manifold (M, g) there is at least one (nonconstant) closed geodesic.

In the untwisted case, since replacing $\gamma(t)$ by $t \mapsto \gamma(at)$ preserves the geodesic condition and changes the energy from E to a^2E, it's superfluous to prescribe the energy.
Note that there need not exist any *contractible* geodesics, and there never exist any geodesics which are shorter than the injectivity radius.

The untwisted case

If $\sigma = 0$, we're just asking for closed geodesics.

Theorem (Fet-Lyusternik, 1952)

On any closed Riemannian manifold (M, g) there is at least one (nonconstant) closed geodesic.

In the untwisted case, since replacing $\gamma(t)$ by $t \mapsto \gamma(at)$ preserves the geodesic condition and changes the energy from E to $a^2 E$, it's superfluous to prescribe the energy.

Note that there need not exist any *contractible* geodesics, and there never exist any geodesics which are shorter than the injectivity radius.

The untwisted case

If $\sigma = 0$, we're just asking for closed geodesics.

Theorem (Fet-Lyusternik, 1952)

On any closed Riemannian manifold (M,g) there is at least one (nonconstant) closed geodesic.

In the untwisted case, since replacing $\gamma(t)$ by $t \mapsto \gamma(at)$ preserves the geodesic condition and changes the energy from E to a^2E, it's superfluous to prescribe the energy.
Note that there need not exist any *contractible* geodesics, and there never exist any geodesics which are shorter than the injectivity radius.

The exact case

When $\sigma = d\alpha$ is exact, the problem has a Lagrangian formulation, which has been used to prove, *e.g.*,

Theorem (Contreras-Macarini-Paternain, 2002)

If $\sigma = d\alpha$ *and* $\dim M = 2$ *then periodic orbits exist on all energy levels.*

Theorem (Contreras, 2003)

If $\sigma = d\alpha$*, periodic orbits exist on almost all energy levels.*

The exact case

When $\sigma = d\alpha$ is exact, the problem has a Lagrangian formulation, which has been used to prove, *e.g.*,

Theorem (Contreras-Macarini-Paternain, 2002)

If $\sigma = d\alpha$ *and* $\dim M = 2$ *then periodic orbits exist on all energy levels.*

Theorem (Contreras, 2003)

If $\sigma = d\alpha$*, periodic orbits exist on almost all energy levels.*

When σ is inexact the story is more complicated; for instance if (M,g) is a hyperbolic surface and σ is the area form, then the flow restricts to a certain energy level $\{E = E_0\}$ as the horocycle flow, which is aperiodic (indeed, uniquely ergodic).

Also, if $E > E_0$, the flow has no *contractible* periodic orbits on energy level E.

However, for *low* energy levels, symplectic topology has provided methods to prove the existence of contractible periodic orbits.

When σ is inexact the story is more complicated; for instance if (M,g) is a hyperbolic surface and σ is the area form, then the flow restricts to a certain energy level $\{E = E_0\}$ as the horocycle flow, which is aperiodic (indeed, uniquely ergodic).
Also, if $E > E_0$, the flow has no *contractible* periodic orbits on energy level E.
However, for *low* energy levels, symplectic topology has provided methods to prove the existence of contractible periodic orbits.

When σ is inexact the story is more complicated; for instance if (M,g) is a hyperbolic surface and σ is the area form, then the flow restricts to a certain energy level $\{E = E_0\}$ as the horocycle flow, which is aperiodic (indeed, uniquely ergodic).
Also, if $E > E_0$, the flow has no *contractible* periodic orbits on energy level E.
However, for *low* energy levels, symplectic topology has provided methods to prove the existence of contractible periodic orbits.

After work of Gürel, Kerman, Lu, Macarini, and most notably Ginzburg, the current state of the art is:

Theorem (Schlenk, 2005)

If $\sigma \neq 0$ then contractible periodic orbits exist on almost all sufficiently small energy levels (i.e., on each of the energy levels $\{K = E\}$ for E in a full-measure subset of some $[0, E_0]$).

Theorem (U., 2008)

If σ is symplectic (i.e, if $B : TM \to TM$ is nonsingular) then contractible periodic orbits exist on all sufficiently small energy levels. Moreover, these periodic orbits can be taken to have bounded period, and hence length tending to zero with the energy.

After work of Gürel, Kerman, Lu, Macarini, and most notably Ginzburg, the current state of the art is:

Theorem (Schlenk, 2005)

If $\sigma \neq 0$ then contractible periodic orbits exist on almost all sufficiently small energy levels (i.e., on each of the energy levels $\{K = E\}$ for E in a full-measure subset of some $[0, E_0]$).

Theorem (U., 2008)

If σ is symplectic (i.e, if $B : TM \to TM$ is nonsingular) then contractible periodic orbits exist on all sufficiently small energy levels. Moreover, these periodic orbits can be taken to have bounded period, and hence length tending to zero with the energy.

Both of these theorems exploit facts about certain classes of submanifolds of symplectic manifolds.

In Schlenk's theorem, the key is that $M \subset T^*M$ is half-dimensional and non-Lagrangian (i.e., ω_σ doesn't vanish identically on it). A result of Polterovich then says that $M \times S^1$ is a displaceable subset of $T^*(M \times S^1)$, and then some facts about the Hofer–Zehnder capacity prove the result.

In the case of my theorem, the key is that $M \subset T^*M$ is a symplectic submanifold. In fact, one has the following more general result:

Theorem (U.)

If (P, Ω) is a symplectic manifold with closed symplectic submanifold M such that $c_1(TP)|_M = 0$, and if $K\colon P \to \mathbb{R}$ attains a Morse-Bott nondegenerate minimum (say 0) along M, then in any given tubular neighborhood of M X_K has periodic orbits with bounded period on every sufficiently low energy level.

This was proven by Ginzburg-Gürel when $\{\int_A \Omega | A \in \pi_2(M)\}$ is discrete; they also gave examples showing that the theorem fails without some kind of nondegeneracy condition on K.

In the case of my theorem, the key is that $M \subset T^*M$ is a symplectic submanifold. In fact, one has the following more general result:

Theorem (U.)

If (P,Ω) is a symplectic manifold with closed symplectic submanifold M such that $c_1(TP)|_M = 0$, and if $K\colon P \to \mathbb{R}$ attains a Morse-Bott nondegenerate minimum (say 0) along M, then in any given tubular neighborhood of M X_K has periodic orbits with bounded period on every sufficiently low energy level.

This was proven by Ginzburg-Gürel when $\{\int_A \Omega | A \in \pi_2(M)\}$ is discrete; they also gave examples showing that the theorem fails without some kind of nondegeneracy condition on K.

For suitable Hamiltonians $H\colon P \to \mathbb{R}$ (having a standard form outside a neighborhood of M) and sufficiently small intervals $[a,b]$, a version of Morse-Novikov homology on (a cover of) the space of contractible loops in P gives rise to a chain complex $CF_*^{[a,b]}(H)$, generated by (lifts of) contractible 1-periodic orbits of X_H.

If $H_0 \leq H_1$ one obtains a chain map $CF_*^{[a,b]}(H_0) \to CF_*^{[a,b]}(H_1)$, and if $H_0 \leq H_1 \leq H_2$ the induced diagram on homology

$$\begin{array}{ccc} HF_*^{[a,b]}(H_0) & \longrightarrow & HF_*^{[a,b]}(H_2) \\ & \searrow \quad \nearrow & \\ & HF_*^{[a,b]}(H_1) & \end{array}$$

commutes.

For suitable Hamiltonians $H\colon P \to \mathbb{R}$ (having a standard form outside a neighborhood of M) and sufficiently small intervals $[a,b]$, a version of Morse-Novikov homology on (a cover of) the space of contractible loops in P gives rise to a chain complex $CF_*^{[a,b]}(H)$, generated by (lifts of) contractible 1-periodic orbits of X_H.

If $H_0 \le H_1$ one obtains a chain map $CF_*^{[a,b]}(H_0) \to CF_*^{[a,b]}(H_1)$, and if $H_0 \le H_1 \le H_2$ the induced diagram on homology

$$\begin{array}{ccc} HF_*^{[a,b]}(H_0) & \longrightarrow & HF_*^{[a,b]}(H_2) \\ & \searrow \quad \nearrow & \\ & HF_*^{[a,b]}(H_1) & \end{array}$$

commutes.

Step 1

Given a sufficiently small E and any $\delta > 0$, find a periodic orbit of X_K with energy in $(E-\delta, E+\delta)$ as follows. Where $f : \mathbb{R} \to \mathbb{R}$ is a certain function which is flat outside the interval $(E-\delta, E+\delta)$ and where H_0 and H_2 are certain standard Hamiltonians with $H_0 \leq f \circ K \leq H_2$, establish properties of

$$\begin{array}{ccc} HF_*^{[a,b]}(H_0) & \longrightarrow & HF_*^{[a,b]}(H_2) \\ & \searrow \quad \nearrow & \\ & HF_*^{[a,b]}(f \circ K) & \end{array}$$

which yield the existence of a nonconstant 1-periodic orbit of $X_{f\circ K}$, representing a particular grading k independent of E and δ.

Reparametrizing this orbit gives a periodic orbit of X_K with energy in $(E-\delta, E+\delta)$, with undetermined period.

Step 1

Given a sufficiently small E and any $\delta > 0$, find a periodic orbit of X_K with energy in $(E-\delta, E+\delta)$ as follows. Where $f : \mathbb{R} \to \mathbb{R}$ is a certain function which is flat outside the interval $(E-\delta, E+\delta)$ and where H_0 and H_2 are certain standard Hamiltonians with $H_0 \leq f \circ K \leq H_2$, establish properties of

$$\begin{array}{ccc} HF_*^{[a,b]}(H_0) & \longrightarrow & HF_*^{[a,b]}(H_2) \\ & \searrow \quad \nearrow & \\ & HF_*^{[a,b]}(f \circ K) & \end{array}$$

which yield the existence of a nonconstant 1-periodic orbit of $X_{f\circ K}$, representing a particular grading k independent of E and δ.

Reparametrizing this orbit gives a periodic orbit of X_K with energy in $(E-\delta, E+\delta)$, with undetermined period.

Step 1

Given a sufficiently small E and any $\delta > 0$, find a periodic orbit of X_K with energy in $(E-\delta, E+\delta)$ as follows. Where $f : \mathbb{R} \to \mathbb{R}$ is a certain function which is flat outside the interval $(E-\delta, E+\delta)$ and where H_0 and H_2 are certain standard Hamiltonians with $H_0 \leq f \circ K \leq H_2$, establish properties of

$$\begin{array}{ccc} HF_*^{[a,b]}(H_0) & \longrightarrow & HF_*^{[a,b]}(H_2) \\ & \searrow \qquad \nearrow & \\ & HF_*^{[a,b]}(f \circ K) & \end{array}$$

which yield the existence of a nonconstant 1-periodic orbit of $X_{f\circ K}$, representing a particular grading k independent of E and δ.

Reparametrizing this orbit gives a periodic orbit of X_K with energy in $(E-\delta, E+\delta)$, with undetermined period.

Step 2 (Ginzburg-Gürel)

When $c_1(TP)|_M = 0$, the periods of the periodic orbits of X_K can be bounded in terms of their Floer homological grading.

So since we have, for all $\delta > 0$, a periodic orbit with energy in $(E - \delta, E + \delta)$ in a given grading (and hence with bounded period), the Arzelà-Ascoli theorem (using a sequence $\delta_k \to 0$) gives an orbit with the desired energy.

Step 2 (Ginzburg-Gürel)

When $c_1(TP)|_M = 0$, the periods of the periodic orbits of X_K can be bounded in terms of their Floer homological grading.
So since we have, for all $\delta > 0$, a periodic orbit with energy in $(E-\delta, E+\delta)$ in a given grading (and hence with bounded period), the Arzelà-Ascoli theorem (using a sequence $\delta_k \to 0$) gives an orbit with the desired energy.