Applications of filtration-theoretic invariants in Floer homology to symplectic topology

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Outline

Throughout this talk, (M,ω) is a closed symplectic manifold and $S^1 = \mathbb{R}/\mathbb{Z}$.

$H\colon S^1 \times M \to \mathbb{R}$ smooth $\rightsquigarrow$ time-dependent vector field X_H, given by $\iota_{X_H}\omega = d(H(t,\cdot))$

$\rightsquigarrow$ Hamiltonian flow $\{\phi_H^t\}_{t\in\mathbb{R}}$
$\phi_H^0 = Id$, $\frac{d}{dt}(\phi_H^t(p)) = X_H(t,\phi_H^t(p))$.

H is called *non-degenerate* if, for each fixed point $p \in M$ of ϕ_H^1, the linearization $(\phi_H^1)_*\colon T_pM \to T_pM$ has all eigenvalues different from 1.

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Let

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Define a 1-form $\alpha_H \in \Omega^1(\mathscr{L}_0 M)$ by

$$(\alpha_H)_\gamma(\xi) = \int_{S^1} \omega_{\gamma(t)}(\dot{\gamma}(t), \xi(t))dt - \int_{S^1} (dH)_{\gamma(t)}(\xi(t))dt.$$

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The 1-form $\mathfrak{a}_H$ on $\mathscr{L}_0M$ is closed; the pullback of $\mathfrak{a}_H$ to the cover

$$\widetilde{\mathscr{L}_0M} = \frac{\{(\gamma,w)|\gamma\in\mathscr{L}_0M, w\colon D^2\to M, w|_{\partial D^2}=\gamma\}}{(\gamma,w)\sim(\gamma',w') \text{ iff } \gamma=\gamma' \text{ and } \int_{D^2}w^*\omega=\int_{D^2}w'^*\omega}$$

is exact; specifically, the pullback is $d\mathscr{A}_H$ where

$$\mathscr{A}_H([\gamma,w]) = -\int_{D^2}w^*\omega - \int_{S^1}H(t,\gamma(t))dt.$$

Thus the set of critical points of $\mathscr{A}_H$ comprises one orbit of the covering group of $\widetilde{\mathscr{L}_0M}\to\mathscr{L}_0M$ for each fixed point of ϕ_H^1 whose orbit under $\{\phi_H^t\}$ is contractible.

$\mathscr{A}_H$ is a Morse function iff H is nondegenerate, and *Hamiltonian Floer homology* is Morse-Novikov homology for

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As a group, the Floer chain complex is

$$CF_*(H) = \left\{ \sum c_i[\gamma_i, w_i] \,|\, c_i \in \mathbb{Q}, [\gamma_i, w_i] \in Crit(\mathscr{A}_H), \mathscr{A}_H([\gamma_i, w_i]) \searrow -\infty \right\}.$$

The boundary operator (which depends on auxiliary data, in particular on an almost complex structure J) counts negative gradient flowlines of $\mathscr{A}_H$:

$$\partial[\gamma^-, w^-] = \sum n_{[\gamma^-,w^-],[\gamma^+,w^+]}[\gamma^+, w^+]$$

where $n_{[\gamma^-,w^-],[\gamma^+,w^+]}$ is a formal count of index-one solutions $u\colon \mathbb{R}\times S^1 \to M$ to

$$\frac{\partial u}{\partial s} + J(u(s,t))\left(\frac{\partial u}{\partial t} - X_H(t, u(s,t))\right) = 0$$

such that $u(s,\cdot) \to \gamma^{\pm}$ as $s \to \pm\infty$ and $[\gamma^+, w^+] = [\gamma^+, w^- \# u]$.

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Theorem (Floer, Hofer-Salamon, Fukaya-Ono, Liu-Tian)

This can be carried out for a nondegenerate Hamiltonian H on an arbitrary closed symplectic (M,ω) with coefficients in an appropriate (Novikov) ring Λ; one has $\partial^2 = 0$, and the resulting homology $HF_*(H)$ satisfies

$$HF_*(H) \cong H_*(M,\mathbb{Q}) \otimes \Lambda,$$

***independently of** H*

Corollary (variant of Arnold's conjecture)

If $H\colon S^1 \times M \to \mathbb{R}$ is nondegenerate then the number of fixed points of ϕ_H^1 is at least the sum of the Betti numbers of M.

Main theme of this talk: Although the Floer homology $HF_*(H)$ is independent of H, the underlying chain complex $CF_*(H)$ carries a $\mathbb{R}$-valued **filtration**, and this filtration carries interesting information that is specific to the isotopy $\{\phi_H^t\}_{0\leq t\leq 1}$.

Recall

$$\mathscr{A}_H([\gamma, w]) = -\int_{D^2} w^*\omega - \int_{S^1} H(t, \gamma(t))dt$$

and

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For

$$c = \sum c_i[\gamma_i, w_i] \in CF_*(H),$$

put

$$\mathscr{L}_H(c) = \max_{c_i \neq 0} \mathscr{A}_H([\gamma_i, w_i]).$$

Then, for any $\lambda \in \mathbb{R}$, define

$$CF_*^{\lambda}(H) = \{c \in CF_*(H) | \mathscr{L}_H(c) \leq \lambda\}.$$

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Whenever $u\colon \mathbb{R} \times S^1 \to M$ contributes to the matrix element

$$n_{[\gamma^-, w^-],[\gamma^+, w^+]}$$

for the Floer boundary operator, one has

$$\mathscr{A}_H([\gamma^-, w^-]) - \mathscr{A}_H([\gamma^+, w^+]) = \int \left|\frac{\partial u}{\partial s}\right|^2 > 0.$$

Hence

$$\mathscr{L}_H(\partial c) < \mathscr{L}_H(c),$$

and in particular for any $\lambda \in \mathbb{R}$ the λ-filtered part $CF_*^{\lambda}(H)$ is preserved by the boundary operator.

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Theorem

Given two normalized ($\int_M H(t,\cdot)\omega^n = 0$) Hamiltonians H_0, H_1 such that $\phi^1_{H_0} = \phi^1_{H_1}$ and the paths $\{\phi^t_{H_i}\}_{0\le t\le 1}$ are homotopic rel endpoints in the Hamiltonian diffeomorphism group, and given sets of auxiliary data needed to define the boundary operators on $CF_*(H_i)$, there is an ***isomorphism of chain complexes***

$$\Phi\colon\ CF_*(H_0) \to CF_*(H_1)$$

which, for each $\lambda \in \mathbb{R}$, restricts to an isomorphism

$$CF^\lambda_*(H_0) \to CF^\lambda_*(H_1).$$

Furthermore, the induced isomorphism $\Phi_*\colon\ HF_*(H_0) \to HF_*(H_1)$ commutes with the Piunikhin-Salamon-Schwarz isomorphisms $\Psi_{H_i}\colon\ H_*(M;\mathbb{Q}) \otimes \Lambda \cong HF_*(H_i)$.

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Thus the $\mathbb{R}$**-filtered chain isomorphism type** of the Floer chain complex is an **invariant** of the class of $\{\phi_H^t\}_{0\leq t\leq 1}$ in $\widetilde{Ham}(M,\omega)$. Certain numerical invariants that can be extracted from this filtered chain isomorphism type have proven useful in Hamiltonian dynamics.

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Compare with the situation in Heegaard Floer homology: Given a null-homologous knot K in a 3-manifold Y, one has a chain complex $\widehat{CF}(Y)$ whose chain homotopy type only depends on Y, but with a filtration that carries information about K.

Differences:

- In the Heegaard Floer case the filtration is by $\mathbb{Z}$, rather than $\mathbb{R}$.
- The Hamiltonian Floer differential *strictly* lowers the filtration level.
- In knot Floer homology only the filtered chain homotopy type is a knot invariant, whereas in Hamiltonian Floer theory the filtered chain isomorphism type is an invariant of the Hamiltonian.

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We have the PSS isomorphism

$$\Psi_H \colon H_*(M;\mathbb{Q}) \otimes \Lambda \to HF_*(H).$$

For

$$a \in H_*(M;\mathbb{Q}) \otimes \Lambda,$$

put

$$\rho(H;a) = \inf\{\mathscr{L}_H(c)\,|\,c \text{ represents } \Psi_H(a) \in HF_*(H)\}.$$

This depends only on (H,a) (and not on the auxiliary data used to define the Floer boundary operator), and extends continuously to all C^0 functions $H\colon S^1 \times M \to \mathbb{R}$ (rather than just smooth nondegenerate Hamiltonians).
(This is similar to the τ invariant in Heegaard Floer theory.)

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Example

If $H(t,m) = f(m)$ for a sufficiently C^2-small Morse function $f\colon M \to \mathbb{R}$, then $(CF_*(H), \partial)$ coincides with the Thom-Smale-Morse-Witten complex $(CM_*(-f), \partial_{Morse})$ of the Morse function $-f$.

Consequently, in this case, setting $a = [M] \in H_*(M;\mathbb{Q}) \otimes \Lambda$,

$$\rho(H;[M]) = \max_{S^1 \times M}(-H).$$

Theorem (Oh, U.)

If $H\colon S^1 \times M \to \mathbb{R}$ is autonomous (i.e., independent of the S^1-variable), and if X_H has no nonconstant contractible periodic orbits of period ≤ 1, then

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This allows one to use the spectral invariant to estimate the (π_1-sensitive) **Hofer-Zehnder capacity**: by definition, if $U \subset M$ is open,

$$c^{\circ}_{HZ}(U) = \sup\left\{ \max H \,\middle|\, \begin{array}{l} H \text{ is autonomous, } suppH \Subset S^1 \times U, \\ \text{and H has no nonconstant contractible} \\ \text{periodic orbits of period } \leq 1 \end{array} \right\}$$

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Meanwhile:

Theorem (Frauenfelder-Ginzburg-Schlenk, U.)

If $L \subset M$ is compact, $\mathrm{supp} H \subset S^1 \times L$, and $\phi_K^1(L) \cap L = \varnothing$, then

$$\rho(H;[M]) \leq \int_0^1 \left(\max_M K(t,\cdot) - \min_M K(t,\cdot) \right) dt =: \|K\|.$$

Where the *displacement energy* is defined by

$$e(L,M) = \inf\{\|K\| \,|\, \phi_K^1(L) \cap L = \varnothing\}$$

for L compact and

$$e(S,M) = \sup_{L \Subset S} e(L,M)$$

in general, it follows that:

Theorem (U.)

For any subset $S \subset M$ (where M is any closed symplectic manifold), we have

$$c^{\circ}_{HZ}(S) \leq e(S,M).$$

Hofer–Zehnder proved this for $M = \mathbb{R}^{2n}$ in the early '90s, but for general M it had only been proven up to a constant.

The result is sharp: for S equal to a small Darboux ball $B^{2n}(r)$ one has $c^{\circ}_{HZ}(S) = e(S,M) = \pi r^2$.

Sample non-squeezing consequence: If Σ is any (possibly very low-area) closed surface, N any closed or Stein 4-manifold, and $r < R$, there is no symplectic embedding $B^4(R) \times \Sigma \hookrightarrow B^2(r) \times N$.

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Definition

The **boundary depth** of a nondegenerate Hamiltonian H on M is

$$\beta(H) = \inf\{\beta \geq 0 | (\forall \lambda > 0)(CF_*^{\lambda}(H) \cap \partial(CF_*(H)) \subset \partial\left(CF_*^{\lambda+\beta}(H)\right)\}.$$

Non-obviously, $\beta(H)$ is finite (U. '07); in fact one has (Oh, '07)

$$\beta(H) \leq \|H\|.$$

Theorem (U.)

(i) $\|\beta(H) - \beta(K)\| \leq \|H - K\|$; *hence* β *extends continuously to all continuous* $H\colon S^1 \times M \to \mathbb{R}$ *(and not just nondegenerate Hamiltonians)*

(ii) *If* $H \leq 0$, $suppH \subset S^1 \times L$, *and* $\phi_K^1(L) \cap L = \varnothing$, *then*

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Some applications can be obtained by combining bounds on β such as (1) with properties of the spectral invariants in order to deduce the existence of **low-energy** solutions $u\colon \mathbb{R}\times S^1 \to M$ to $\frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_H) = 0$ having certain asymptotics.

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Applications of the boundary depth:

Theorem (U., generalizing Schwarz, *et al.*)

If $e_{stable}(S,M)=0$ (e.g., if S is a non-Lagrangian submanifold with $\dim S \leq \frac{1}{2}\dim M$, or if S is a symplectic submanifold) and if $\langle[\omega],\pi_2(S)\rangle = 0$, then there is a neighborhood W of S such that if $H\colon S^1\times M\to\mathbb{R}$ is a Hamiltonian with $suppH \Subset S^1\times W$ and satisfying a technical condition, then ϕ_H^1 has infinitely many geometrically distinct nontrivial periodic points.

Theorem (U., generalizing Ginzburg, Kerman)

If $N\subset M$ is a stable coisotropic submanifold, and if $\langle[\omega],\pi_2(N)\rangle$ is discrete, then $e(N,M)>0$. In particular, any coisotropic submanifold of contact type (in the sense of Bolle) has positive displacement energy.

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