Lefschetz Fibrations and Pseudoholomorphic Curves

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This talk will discuss the equivalence of two pseudoholomorphic invariants. The first is the **Gromov(-Taubes) Invariant**

Let (X, ω) be a symplectic 4-manifold, j a generic tame almost complex structure, and
$\alpha \in H^2(X, \mathbb{Z})$.

$Gr(\alpha)$ counts the (possibly disconnected) j-holomorphic submanifolds Poincaré dual to α that pass through a generic set of
$d(\alpha) = \frac{1}{2}(\alpha^2 - \kappa \cdot \alpha)$ points. $(\kappa = c_1(T^*X))$

Main technical difficulty: *Multiple covers*
Our answer should be independent of j, but as we vary j a family of embedded square-zero tori PD to a class 2α can degenerate into double covers of a family of tori PD to the class α.

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Taubes found a prescription for assigning contributions of the m-fold covers of curves PD to α to $Gr(m\alpha)$ in such a way that Gr does nonetheless give an invariant (which moreover equals the Seiberg-Witten invariant for a corresponding $spin^c$ structure).

Gr counts pseudoholomorphic *sets*, whereas Gromov-Witten (GW) invariants count pseudoholomorphic *maps* $u : \Sigma \to X$.
Using the GW viewpoint, the multiple cover issue is resolved by instead counting solutions to $\bar{\partial}_j u(x) = \nu(x, u(x))$ for a generic inhomogeneous term ν. Still:

Theorem 1. *(Ionel-Parker) Gr may be expressed as a combination of GW invariants.*

Theorem 2. *(Taubes) If $b^+(X) > 1$, then $Gr(\alpha) = \pm Gr(\kappa - \alpha)$.*

Proof. Immediate from the "charge-conjugation symmetry" in Seiberg-Witten theory and from $Gr = SW$. □

Goal: Understand Taubes duality symplectically rather than via gauge theory.

We'll do this by equating Gr with another invariant which has earlier been shown to satisfy the same duality. This invariant is constructed from a *Lefschetz fibration* structure on X.

Symplectic Lefschetz Fibrations

Assume $[\omega] \in H^2(X, \mathbb{Z})$. Where $L \to X$ has $c_1(L) = [\omega]$, Donaldson showed that for $k \gg 0$ $L^{\otimes k}$ has many "approximately holomorphic" sections s_k $(|\bar{\partial} s_k| \ll |\partial s_k|)$

Taking two of these gives a *symplectic Lefschetz pencil*

$$f_k : X \dashrightarrow \mathbb{P}^1$$
$$x \mapsto [s_k^0(x) : s_k^1(x)]$$

defined away from a finite base locus $B_k = \{s_k^0 = s_k^1 = 0\}$.

f_k lifts to the blowup X' along B_k to give a *symplectic Lefschetz fibration* (SLF) $f : X' \to \mathbb{P}^1$.

So after blowing up, X is viewed as a $\mathbb{P}^1$-parametrized family of Riemann surfaces (Σ_t).

$Gr(\alpha) = Gr(\pi_k^* \alpha)$, so from now on we'll replace X with X'.

Let $f : X \to \mathbb{P}^1$ be a SLF.
Assume j is an almost complex structure on X with respect to which f is a pseudoholomorphic map. How does the Lefschetz fibration f see a j-holomorphic curve C?

Let $r = [C] \cdot [fiber]$. C meets each fiber r times (with multiplicities), so C is identified with a family of elements in the symmetric products $S^r\Sigma_t$ $(t \in \mathbb{P}^1)$.

From the SLF $f : X \to \mathbb{P}^1$, Donaldson and Smith constructed the *relative Hilbert scheme*

$$F : X_r(f) \to \mathbb{P}^1$$

whose fibers are $F^{-1}(t) = S^r f^{-1}(t)$.

So

$$\text{curve } C \rightsquigarrow \text{section } s_C \text{ of } X_r(f)$$

From j, we can build an almost complex structure $\mathbb{J}_j$ on $X_r(f)$ by naively "taking the symmetric product" of j, and then

$$C \text{ is } j\text{-holomorphic} \Leftrightarrow \bar{\partial}_{\mathbb{J}_j} s_C = 0.$$

This suggests counting holomorphic submanifolds of X by counting holomorphic sections of $X_r(f)$. $\mathbb{J}_j$ is only Hölder, though, so we can't directly use it to compute GW invariants in $X_r(f)$.

Theorem 3. *(Smith) 1) Given $\alpha \in H^2(X, \mathbb{Z})$, there is at most one homotopy class c_α of sections of $X_r(f)$ which descend to sets Poincaré dual to α.*

2) $d(\alpha)$:=virdim(holomorphic submanifolds of X PD to α) is also the virtual dimension of the space of J-holomorphic sections of $X_r(f)$ in class c_α (for a generic smooth *J on $X_r(f)$ which will generally have nothing to do with any almost complex structure on X).*

Definition 1. *The standard surface count* $\mathcal{DS}_{(X,f)}(\alpha)$ *is the Gromov-Witten invariant counting sections in class* c_α *which descend to curves passing through a generic set of* $d(\alpha)$ *points in* X.

Theorem 4. *(Smith) If* $f_k : X \dashrightarrow \mathbb{P}^1$ *is a symplectic Lefschetz pencil lifting to a SLF* $f : X' \to \mathbb{P}^1$ *via the blowup* $\pi_k : X' \to X$, *then*

$$\mathcal{DS}_{(X',f)}(\pi_k^*\beta) = \pm\mathcal{DS}_{(X',f)}(\pi_k^*(\kappa_X - \beta))$$

for sufficiently large k, *assuming that* $b^+(X) > b_1(X) + 1$.

Theorem 5. *(U) If* $f : (X, \omega) \to \mathbb{P}^1$ *is a SLF and* $\omega \cdot \alpha < \omega \cdot (\text{fiber})$ *then*

$$Gr(\alpha) = \mathcal{DS}_{(X,f)}(\alpha).$$

Remarks: 1) The assumption on the fibers allows us to rule out the possibility of bubbling for the sections of $X_r(f)$. In Donaldson's construction, the size of the fibers tends to infinity with k, so the hypothesis can always be satisfied in that context.
2) Since $Gr(\pi_k^*\beta) = Gr(\beta)$, this combines with Smith's result to give a new proof of Taubes duality under the stronger assumption $b^+ > b_1 + 1$.
3) It follows that $\mathcal{DS}_{(X,f)}$ is independent of the fibration f, which was previously not known.

Outline of Proof

Because of the correspondence $\mathcal{M}_j^X(\alpha) = \mathcal{M}_{\mathbb{J}_j}^{X_r(f)}(c_\alpha)$, we'd like to come as close to using the (only Hölder!) almost complex structure $\mathbb{J}_j$ to evaluate $\mathcal{DS}$ as possible.

Why $\mathbb{J}_j$ is only Hölder: Suppose j is given locally by $T_j^{0,1} = \langle \partial_{\bar z} + b(z,w)\partial_w, \partial_{\bar w}\rangle$. Recall that the natural coordinates on $S^r\mathbb{C}$ are the elementary symmetric polynomials

$$\sigma_k = \sum_{i_1<\cdots<i_k} w_{i_1}\cdots w_{i_k}.$$

$\mathbb{J}_j$ is given locally by

$$T_{\mathbb{J}_j}^{0,1} = \langle \partial_{\bar z} + \sum_{d=1}^{r} b_d(z,\sigma_1,\ldots,\sigma_r)\partial_{\sigma_d}, \partial_{\bar\sigma_1},\ldots,\partial_{\bar\sigma_r}\rangle$$

where the functions b_d on $\mathbb{C}\times S^r\mathbb{C}$ descend from

$$\hat b_d(z,w_1,\ldots,w_r) = \sum_{k=1}^{r} \sigma_{d-1}(w_1,\ldots,\widehat{w_k},\ldots,w_r)b(z,w_k)$$

on $\mathbb{C}\times\mathbb{C}^r$. Usually, smooth symmetric functions on $\mathbb{C}^r$ only descend to Hölder-continuous functions on $S^r\mathbb{C}$, so for general j, $\mathbb{J}_j$ will only be Hölder.

Note, though, that if b is *holomorphic* in w, then so is $\hat{b}_d$, so since holomorphic symmetric functions on $\mathbb{C}^r$ descend to holomorphic functions on $S^r\mathbb{C}$, $\mathbb{J}_j$ will be smooth in this case. b being holomorphic corresponds to j being *integrable*, so if j is integrable, then so is $\mathbb{J}_j$.

Pushing this a little farther, for a **non-multiply covered** j-holomorphic curve C, if j happens to be integrable near all of the branch points of the restriction $f|_C$, then $\mathbb{J}_j$ will be smooth on some neighborhood of s_C. It turns out to be possible to compute Gr using an almost complex structure j that has this property for all of the curves we wish to count, so that $\mathbb{J}_j$ will be smooth near all of the sections in the relevant moduli space, allowing us to use $\mathbb{J}_j$ directly to compute $\mathcal{DS}$.

C then contributes to Gr and s_C to $\mathcal{DS}$ each with some weight ± 1, and we need to determine whether these weights are the same.

The weights are determined by *spectral flows*: Where D_C^j is the (Fredholm, index-zero) linearization of $\bar{\partial}_j$ at the embedding of C, acting on sections which preserve the incidence conditions, we take a path of operators D^t connecting D_C^j to a surjective, complex-linear operator $\tilde{D}$ and count the number of t for which D^t has a kernel; the contribution of C is -1 raised to that number.

Observation: If $\tilde{j}$ is integrable near C and makes both C and the fibration holomorphic, then
1) $D_C^{\tilde{j}}$ is complex-linear, and
2) $\mathbb{J}_{\tilde{j}}$ is also integrable, so $D_{s_C}^{\mathbb{J}_{\tilde{j}}}$ is also complex-linear.
Strategy: Find a path j_t of almost complex structures making both C and the fibration holomorphic, starting at j and ending at an *integrable* almost structure $\tilde{j}$; for the spectral flows, use the paths $D_C^{j_t}$ and $D_{s_C}^{\mathbb{J}_{j_t}}$.
The special choice of j we made earlier makes it straightforward to find such a path, and:

Proposition 1. *In this situation, for all t*

$$\ker D_C^{j_t} = 0 \Leftrightarrow \ker D_{s_C}^{\mathbb{J}_{j_t}} = 0.$$

So the eigenvalue crossings for the two flows occur at exactly the same times, and we conclude that *if C is not multiply covered, the contribution of C to Gr is the same as that of s_C to $\mathcal{DS}$.*

For **multiply covered** curves, this direct comparison via spectral flows is not possible. If C is an embedded j-holomorphic curve and $m > 1$, we obtain a $\mathbb{J}_j$- holomorphic section s_{mC} which is *completely contained* in the diagonal stratum Δ where $\mathbb{J}_j$ fails to be C^1, and this issue can't be evaded by the methods I discussed earlier.

Taubes found weights $r(C, m)$ (generally not equal to ± 1) with which m-fold covers of C should contribute to Gr in order for Gr to be an invariant.

Even though $\mathbb{J}_j$ doesn't fit into the usual GW invariant machinery because it's not C^1, we can still make sense of "the contribution of s_{mC} to $\mathcal{DS}$" (denoted $r'(C, m)$) thanks to *Gromov compactness*: If we perturb $\mathbb{J}_j$ to a smooth, regular, almost complex structure J which is Hölder-close to $\mathbb{J}_j$, the J-holomorphic sections will all be close to the $\mathbb{J}_j$-holomorphic sections. So s_{mC} will have a well-defined "entourage" of J-holomorphic sections nearby, and counting the members of this entourage with signs in the usual way gives the contribution $r'(C, m)$.

We'd like to say, then, that $r(C, m) = r'(C, m)$. Both of these numbers are determined by the restriction of j to a small neighborhood of C, so we consider the effect of varying the almost complex structure on such a small neighborhood among those which make both C and the fibration holomorphic.

The two basic ingredients:
1) When $\tilde{j}$ is *integrable* and suitably nondegenerate, $r_{\tilde{j}}(C,m) = r'_{\tilde{j}}(C,m) = 1$.
2) When we vary j along a generic path, r and r' both remain constant except at certain points, where they both transform according to certain *identical* wall-crossing formulas.
For (1), that $r_{\tilde{j}}(C,m) = 1$ is just part of the definition of Gr (the only reason that this is the natural choice, as far as I know, is that it causes Gr to equal SW.) That $r'_{\tilde{j}}(C,m) = 1$ is easy to see: since $\tilde{j}$ is integrable, $\mathbb{J}_{\tilde{j}}$ is as well (in particular it's smooth, so we can use it directly to evaluate $\mathcal{DS}$), and at least assuming nondegeneracy s_{mC} will contribute $+1$ to $\mathcal{DS}$.

For (2), when Taubes introduced Gr he discovered the ways in which the moduli spaces in X could change under a generic variation in the complex structure, and the specific wall crossing formulas for the $r(C, m)$ were *forced* on him in order that Gr be independent of the choice of almost complex structure. Since $\mathcal{DS}$ is likewise independent of the almost complex structure used to define it, the $r'(C, m)$ necessarily satisfy the same formulas.

By taking a path of almost complex structures joining our original structure j to a structure $\tilde{j}$ which is integrable near C, we conclude that because r and r' are the same at the end of path and change in the same way along it, they must agree at the start as well. That $Gr = \mathcal{DS}$ then follows.