C^0 stability in Morse theory, Floer theory, and symplectic topology

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Outline

1. Zeros of 1-forms
2. The Novikov complex
3. The Hamiltonian Floer complex
4. Filtration-based invariants

Let $E \overset{\theta}{\underset{}{\rightleftarrows}} M$ be a vector bundle over a closed manifold M, with $\mathrm{rank}(E) = \dim M$, and $\theta \in \Gamma(E)$ a section transverse to 0_E (e.g., $E = T^*M$, $\theta \in \Omega^1(M)$).

Standard facts:

(i) If $\varepsilon\colon M \to E$ is a sufficiently C^1-small section, then $\theta + \varepsilon$ has *exactly* as many zeros as θ.

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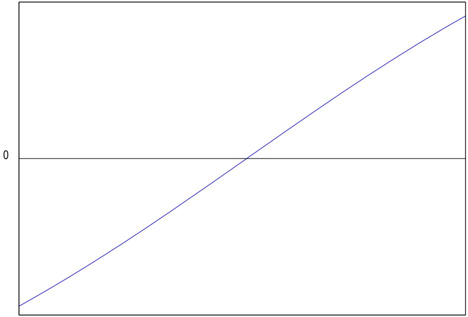

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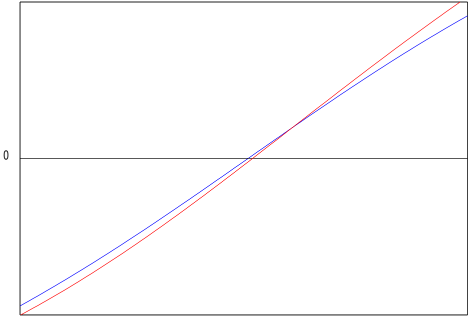
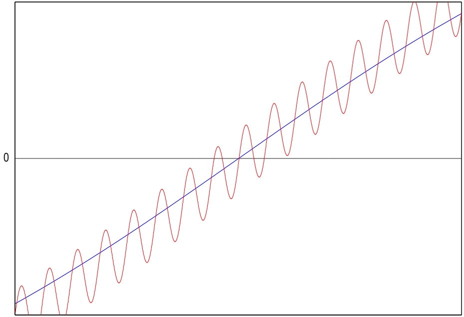


(ii) If $\varepsilon\colon M \to E$ is a sufficiently C^0-small section, then $\theta + \varepsilon$ has *at least* as many zeros as θ.



When $E = T^*M$, a standard choice of C^k-small perturbation of $\theta \in \Omega^1(M)$ is $\theta + df$, where f is a C^{k+1}-small function.

So the preceding shows that, if $\theta \in \Omega^1(M)$ vanishes transversely,

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Curiously, when θ is *closed*, something stronger is true:

Stability Theorem (Cornea-Ranicki, U.)

If $\theta \in \Omega^1(M)$ vanishes transversely and satisfies $d\theta = 0$, there is $\delta > 0$ such that, whenever

$$osc(f) := \max f - \min f < \delta \text{ and } (\theta + df) \pitchfork 0_{T^*M},$$

one has

$$\#(\theta + df)^{-1}(0) \geq \#\theta^{-1}(0).$$

In fact, choosing a metric on M and letting V be the vector field metrically dual to θ, one can take

$$\delta = \inf\left\{ \int_\gamma \theta \,\middle|\, \gamma\colon \mathbb{R} \to M \text{ nonconstant}, \dot{\gamma} = V(\gamma) \right\}$$

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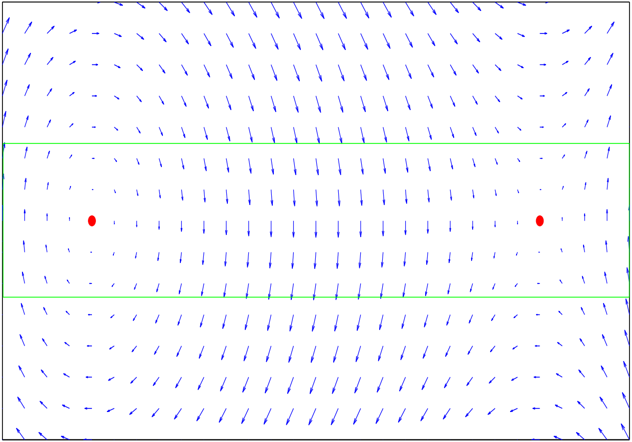
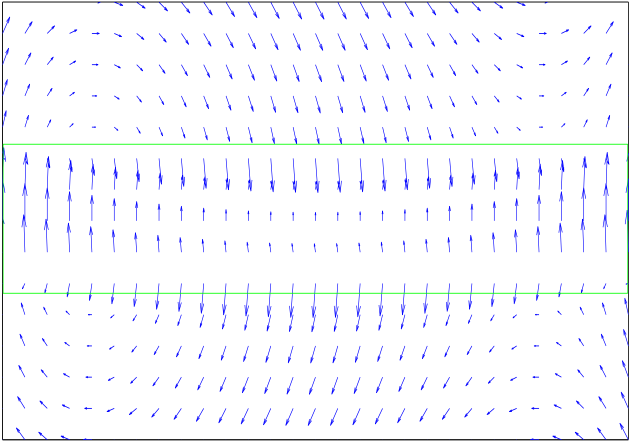
θ does need to be closed for the theorem to hold; for instance the non-closed form

$$ydx + (x^2 - 1)dy$$

can have its two zeros eliminated by adding $d(2y\chi(y))$ where χ is a cutoff function supported near zero





The proof of the stability theorem uses the **filtrations** on the **Novikov complexes** of θ and $\theta + df$.

Given a transversely vanishing closed 1-form θ, choose an abelian cover

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Thus

$$Crit(\mathscr{A}) = (\pi^*\theta)^{-1}(0) \subset \tilde{M}$$

consists of an orbit of Γ for every zero of $\theta \in \Omega^1(M)$.

The **Novikov chain complex** is

$$CN_*(\mathscr{A}) = \left\{ \sum_{i=1}^{\infty} n_i p_i \,\middle|\, n_i \in \mathbb{Z}, p_i \in Crit(\mathscr{A}), \mathscr{A}(p_i) \searrow -\infty \right\}.$$

$CN_*(\mathscr{A})$ is a free module of rank $\#\theta^{-1}(0)$ over the **Novikov ring** $\Lambda_{\Gamma,[\theta]}$ (this is a completion of the group ring of Γ, depending on the de Rham cohomology class $[\theta]$; if $[\theta] = 0$ we can choose $\Gamma = \{0\}$ and then $\Lambda_{\Gamma,[\theta]} = \mathbb{Z}$).

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The boundary operator on $CN_*(\mathscr{A})$ "counts negative gradient flowlines of $\mathscr{A}$":

$$\partial p = \sum_q n(p,q)q$$

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$CN_*(\mathscr{A})$ admits a $\mathbb{R}$-valued **filtration**: take $CN_*^{\lambda}(\mathscr{A}) \leq CN_*(\mathscr{A})$ equal to the set of all formal sums $\sum n_i p_i$ of critical points p_i of $\mathscr{A}$ having $\mathscr{A}(p_i) \leq \lambda$. Note that

$$\partial \colon CN_*^{\lambda}(\mathscr{A}) \to CN_{*-1}^{\lambda-\delta}(\mathscr{A}).$$

The goal is to compare the sizes of the zero sets of θ and $\theta + df$ for f C^0-small; these are the ranks, respectively, of the $\Lambda_{\Gamma,[\theta]}$-modules $CN_*(\mathscr{A})$ and $CN_*(\mathscr{A} + \pi^* f)$.
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One has:

- A chain map

$$\Phi_-^+ \colon CN_*(\mathscr{A}) \to CN_*(\mathscr{A} + \pi^* f)$$

constructed by counting isolated solutions to

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where $\mathscr{A}(s,\cdot) = \begin{cases} \mathscr{A} & s \ll 0 \\ \mathscr{A} + \pi^* f & s \gg 0 \end{cases}$

For suitably-chosen $\mathscr{A}(s,\cdot)$, one has

$$(\mathscr{A} + \pi^* f)(\gamma(\infty)) - \mathscr{A}(\gamma(-\infty)) = \int_{-\infty}^{\infty} \frac{d}{ds}(\mathscr{A}(s,\gamma(s)))ds \leq \max f$$

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But recall that ∂ *strictly lowers* the filtration by δ, so if $osc(f) < \delta$ we have

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But then $\sum_{k=0}^{\infty}(-A)^k$ is a well-defined inverse to $I+A$.

Thus $\Phi_{-}^{+}\colon CN_*(\mathscr{A}) \to CN_*(\mathscr{A} + \pi^* f)$ has a left inverse, so

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Now let (M,ω) be a closed symplectic manifold ($\omega \in \Omega^2(M)$ nondegenerate, $d\omega = 0$). Write $S^1 = \mathbb{R}/\mathbb{Z}$.

Any smooth $H\colon S^1 \times M \to \mathbb{R} \rightsquigarrow$ time-dependent "Hamiltonian vector field" $X_H : \omega(X_H(t,\cdot),\cdot) = d(H(t,\cdot))$

$\rightsquigarrow$ Hamiltonian flow $\{\phi_H^t\}_{t\in\mathbb{R}}$: $\frac{d}{dt}(\phi_H^t(p)) = X_H(t,\phi_H^t(p))$.

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$$\mathscr{L}_0 M = \{\text{contractible loops } \gamma\colon S^1 \to M\}$$

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θ_H is closed, and vanishes transversely if all fixed points of ϕ_H^1 are nondegenerate. Lifting to a suitable cover $\pi\colon \widetilde{\mathscr{L}_0 M} \to \mathscr{L}_0 M$, we have

$$\pi^* \theta_H = d\mathscr{A}_H \text{ where } \mathscr{A}_H([\gamma, u]) = -\int_{D^2} u^*\omega - \int_0^1 H(t, \gamma(t))dt.$$

Formally, the **Floer chain complex** of H is the Novikov chain complex of this "action functional" on $\widetilde{\mathscr{L}_0 M}$.

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Theorem (Floer, Hofer-Salamon, Liu-Tian, Fukaya-Ono)

This construction can be carried out on any closed symplectic manifold, producing a chain complex $CF_*(H)$ whose homology $HF_*(H)$ is, independently of H, canonically isomorphic to $H_*(M,\mathbb{Q})\otimes\Lambda$.

Corollary (Variant of Arnol'd conjecture)

If all fixed points of H are nondegenerate then

$$\#Fix(\phi_H^1) \geq \sum_i b_i(M;\mathbb{Q}).$$

While the just-stated results are about arbitrary (nondegenerate) H, interesting information **specific to** H can be obtained from the $\mathbb{R}$-filtration on the chain complex:

$$CF_*^{\lambda}(H) = \left\{ \sum_{i=1}^{\infty} a_i[\gamma_i, u_i] \;\middle|\; \begin{array}{l} [\gamma_i, u_i] \in Crit(\mathscr{A}_H) \\ \lambda \geq \mathscr{A}_H([\gamma_i, u_i]) \searrow -\infty \end{array} \right\}.$$

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Analogy: In Heegaard Floer homology one constructs a chain complex whose chain homotopy type depends only on the manifold; the choice of a knot induces a filtration which carries significant information about the knot.

The same argument as in case of a closed one-form on a compact manifold shows that (where we set

$$\|H\| = \int_0^1 \left(\max_{p \in M} H(t,p) - \min_{p \in M} H(t,p) \right) dt$$

for $H \colon S^1 \times M \to \mathbb{R}$):

Theorem

If $H \colon S^1 \times M \to \mathbb{R}$ is a nondegenerate Hamiltonian, there is $\delta > 0$ such that for any nondegenerate Hamiltonian K with $\|H - K\| < \delta$, we have

$$\#Fix(\phi_K^1) \geq \#Fix(\phi_H^1).$$

Again, this is somewhat surprising, since ϕ_K^1 is only "C^{-1}-close" to ϕ_H^1.

Theorem (U.)

Given H, the ***filtered chain isomorphism type*** *of $CF_*(H)$ is independent of the other auxiliary data involved in its construction, and (for H suitably normalized) in fact depends only on the homotopy class rel endpoints of the path $\{\phi_H^t\}_{0\leq t\leq 1}$ in the group of Hamiltonian diffeomorphisms.*

Hence invariants of the filtered chain isomorphism type of $CF_*(H)$ are invariants of the representative of $\{\phi_H^t\}_{0\leq t\leq 1}$ in $\widetilde{Ham}$. Because the standard maps $CF_*(H) \to CF_*(K)$ restrict as $CF_*^{\lambda}(H) \to CF_*^{\lambda+\|H-K\|}(K)$ (where $\|H\| = \int_0^1 \left(\max_{p\in M} H(t,p) - \min_{p\in M} H(t,p)\right) dt$), such invariants are often continuous with respect to the C^0 norm on the space of Hamiltonians (and with respect to the "Hofer norm" on $\widetilde{Ham}$).

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One example of such an invariant is the **Oh–Schwarz spectral invariant**: Recall there is a canonical isomorphism $HF_*(H) \cong H_*(M) \otimes \Lambda$. So for $a \in H_*(M) \otimes \Lambda$ we can put

$$\rho(H;a) = \inf\{\text{filtration level of } c \,|\, c \in CF_*(H) \text{ represents } a\}.$$

Remark

- The above makes sense when ϕ_H^1 is nondegenerate, but the definition can be extended to arbitrary H by continuity.
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ρ has diverse applications (e.g., in Entov-Polterovich's theory of quasimorphisms on $\widetilde{Ham}$); I'll focus here on its applications to **displacement energy**.

For compact $K \subset M$, put

$$e(K,M) = \inf\{\|H\| \,|\, \phi_H^1(K) \cap K = \varnothing\}.$$

Now

$$\|H\| = \int_0^1 \left(\max_{p \in M} H(t,p) - \min_{p \in M} H(t,p) \right) dt,$$

whereas ϕ_H^1 is constructed from dH, so it's a bit surprising that $e(K,M)$ would ever be finite but positive.

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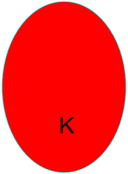

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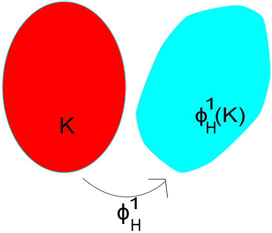

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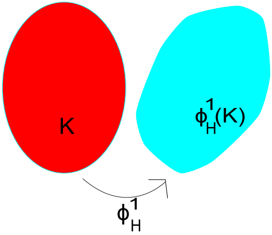

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Write $p_2\colon S^1 \times M \to M$ for the projection.

Theorem (U., generalizing Frauenfelder-Ginzburg-Schlenk)

For any Hamiltonian $G\colon S^1 \times M \to \mathbb{R}$ we have

$$\rho(G;[M]) \leq e(p_2(supp(G)), M).$$

Theorem (U., generalizing Oh)

If G is independent of the S^1 factor, and if all nonconstant contractible periodic orbits of X_G have period > 1, then

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Corollary

Where the Hofer–Zehnder capacity $c_{HZ}(K)$ is defined as

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This inequality $c_{HZ} \leq e$ is sharp, as can be seen by explicit examples with K equal to a closed ball.
Gromov's nonsqueezing theorem follows as a quick consequence, as do some (new) generalizations, e.g., if Σ^{2n-2k} is closed and N^{2n-2} is closed or Stein then $\Sigma \times B^{2k}(r)$ symplectically embeds in $N \times B^2(R)$ only if $r \leq R$.

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Another filtration-based invariant: the **boundary depth**. Set

$$\beta(H) = \inf\left\{\beta \geq 0 | (\forall\lambda > 0)(CF_*^{\lambda}(H) \cap \partial(CF_*(H)) \subset \partial\left(CF_*^{\lambda+\beta}(H)\right)\right\}$$

Non-obviously, $\beta(H)$ is finite (U.); in fact one has (Oh)

$$\beta(H) \leq \|H\|.$$

As with the spectral invariant, the definition above assumes H nondegenerate, but then extends continuously to degenerate H.

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If $H \geq 0$ everywhere or $H \leq 0$ everywhere, then

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An application of the boundary depth:

Theorem (U., generalizing Ginzburg, Kerman)

If $N \subset M$ is a coisotropic submanifold ($(TN)^{\perp_\omega} \subset TN$) satisfying certain intrinsic conditions[a], then $e(N,M) > 0$.

[a] namely, N is stable (in the sense of Ginzburg) and $\langle \omega, \pi_2(N) \rangle$ is discrete. Alternately, $\langle [\omega], \pi_2(N) \rangle$ is discrete and N admits a metric making its characteristic foliation totally geodesic and having no contractible leafwise geodesics.

Idea of proof: For $U \supset N$ consider a Hamiltonian H_U supported in U whose associated X_{H_U} generates a reparametrization of the leafwise geodesic flow of N. If $e(N,M) = 0$, then the $\beta(H_U)$ become arbitrarily small, and this forces the existence of a geodesic that violates the assumptions on N.

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