The diameters of Hofer's metrics on Hamiltonian diffeomorphisms and Lagrangian submanifolds

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June 30, 2011

Outline

Let (M,ω) be a symplectic manifold.
A smooth function $H\colon [0,1]\times M\to\mathbb{R}$ (compactly supported in $int(M)$ if M is open) gives rise to a time-dependent Hamiltonian vector field X_H by requiring $\omega(\cdot,X_H)=d_MH$, and then to a Hamiltonian flow $\{\phi_H^t\}_{t\in[0,1]}$.

Write

$$Ham(M,\omega)=\{\phi\in Diff(M)|\phi=\phi_H^1 \text{ for some } H\}.$$

So $Ham(M,\omega)$ is an infinite-dimensional subgroup of the symplectomorphism group of (M,ω).

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So $Ham(M,\omega)$ is an infinite-dimensional subgroup of the symplectomorphism group of (M,ω).

For $H\colon [0,1]\times M\to\mathbb{R}$ define

$$osc(H)=\int_0^1\left(\max_M H(t,\cdot)-\min_M H(t,\cdot)\right)dt.$$

Then if $\phi\in Ham(M,\omega)$ let

$$\|\phi\|=\inf\{osc(H)|\phi=\phi_H^1\}$$

and if $\psi,\phi\in Ham(M,\omega)$ let

$$d(\phi,\psi)=\|\psi^{-1}\phi\|.$$

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Easy Theorem

d satisfies, for any $\alpha, \phi, \psi \in Ham(M,\omega)$:

- $d(\phi,\psi) = d(\psi,\phi)$
- $d(\alpha,\psi) \le d(\alpha,\phi) + d(\phi,\psi)$
- $d(\alpha\phi, \alpha\psi) = d(\phi\alpha, \psi\alpha) = d(\phi,\psi)$

Hard Theorem (Hofer, Lalonde–McDuff)

On any (M,ω), we have

$$d(\phi,\psi) = 0 \Leftrightarrow \phi = \psi.$$

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We still know relatively little about the global geometry of $Ham(M,\omega)$.

For instance, given a path $\gamma\colon [0,1] \to Ham(M,\omega)$, there is work of Bialy–Polterovich, Lalonde–McDuff and others which can tell when γ, or at least a short segment of γ, is length-minimizing *among nearby paths*, but we do not have many tools to tell whether γ is length-minimizing over all paths with given endpoints.
Relatedly, the following question is still open (the answer is expected to be no):

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Nearly all of the closed (M,ω) for which $Ham(M,\omega)$ was known to have infinite diameter are covered by a 2008 theorem of McDuff[1] (though many special cases were proven earlier, e.g. by Lalonde–Mcduff, Schwarz, Entov–Polterovich...); roughly speaking the theorem implies infinite diameter when (M,ω) either:

- has minimal Chern number large in comparison to $\dim M$ (e.g., $M = \mathbb{C}P^n$)
- has few nonvanishing genus-zero Gromov–Witten invariants (e.g., symplectically aspherical or negatively monotone manifolds, at least under a mild topological hypothesis)

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- A construction of Ostrover, valid on *any* closed (M,ω), of a path $\{\phi_t\}_{t\in[0,\infty)}$ so that the *lift* $\{\tilde{\phi}_t\}$ to the universal cover has $\tilde{d}(\tilde{\phi}_0,\tilde{\phi}_t)\to\infty$ with respect to the natural lifted metric;
- A proof that, on the class of manifolds covered by her theorem, certain functions on $\widetilde{Ham}(M,\omega)$ ("asymptotic spectral invariants") which give Ostrover's lower bounds for $\tilde{d}(\tilde{\phi}_0,\tilde{\phi}_t)$ actually descend to functions on $Ham(M,\omega)$ and so give lower bounds (tending to ∞) for $d(\phi_0,\phi_t)$.

The proof involves difficult calculations relating to the Seidel representation of $\pi_1(Ham(M,\omega))$ on quantum homology. McDuff shows that if one even just slightly weakens the hypotheses then the asymptotic spectral invariants can fail to descend, so the argument will not work.

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Here is a new criterion for infinite Hofer diameter:

Theorem (U.)

Suppose that a closed symplectic manifold (M,ω) admits a nonconstant autonomous Hamiltonian $H\colon M\to\mathbb{R}$ such that all contractible periodic orbits of X_H are constant. Then the diameter of $Ham(M,\omega)$ is infinite. In fact, there is a homomorphism

$$\Phi\colon \mathbb{R}^\infty \to Ham(M,\omega)$$

such that, for all $v,w\in\mathbb{R}^\infty$,

$$\|v-w\|_{\ell_\infty} \le d(\Phi(v),\Phi(w)) \le osc(v-w).$$

(To the second part compare a 2007 result of Py, which showed that in the special case that M contains a π_1-injective Lagrangian submanifold which admits a metric of nonpositive curvature, $Ham(M,\omega)$ contains quasi-isometrically embedded copies of $\mathbb{R}^N$ for all N.)

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There are plenty of examples of manifolds satisfying the hypothesis, for instance:

- Any positive genus surface Σ, as well as many examples built out of Σ such as symplectic fiber bundles over Σ and (at least for many Kähler forms constructed by Perutz) symmetric products of Σ. In these cases the nontrivial orbits of X_H go around noncontractible loops.
- (U. 2011) Many symplectic four-manifolds with $b^+ > 1$ (e.g. the K3 surface, elliptic surfaces $E(n)_{p,q}$ ($n \geq 2$), Gompf's manifolds X_G with $\pi_1 = G$ for any finitely-presented G....) with irrational symplectic forms. In these cases the nontrivial orbits go along irrational lines on 3-tori.
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Let

$$\mathscr{L}(L_0) = \{\phi(L_0) | \phi \in Ham(M,\omega)\}$$

be the orbit of L_0 (as an unparametrized submanifold) under $Ham(M,\omega)$.

Chekanov showed that if for $L, L' \in \mathscr{L}(L_0)$ we put

$$\delta(L,L') = \inf\{\|\phi\| | \phi \in Ham(M,\omega), \phi(L) = L'\},$$

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There are not many results about the diameter of this metric. For L_0 equal to the zero section of the cotangent bundle work of Oh and Milinković implies that the diameter is infinite. Khanevsky proved infinite diameter for $L_0 = S^1 \times \{0\} \subset S^1 \times (-1,1)$, and noted that methods of Lalonde–McDuff imply infinite diameter when L_0 is a noncontractible curve on a closed surface. Leclercq gave another approach to infinite diameter for the meridian on the torus, and his methods probably could be extended to linear Lagrangian $T^n \subset T^{2n}$ and a few other cases.

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If (M,ω) is a closed symplectic manifold, let $(P,\Omega) = (M \times M, (-\omega) \times \omega)$. Then where $\Delta_M \subset P$ is the diagonal, we have an embedding

$$Ham(M,\omega) \hookrightarrow \mathscr{L}(\Delta_M)$$
$$\phi \mapsto \Gamma_\phi = graph(\phi).$$

Theorem (U.)

Let (M,ω), as in the previous theorem, be a closed symplectic manifold admitting an autonomous Hamiltonian all of whose contractible periodic orbits are constant. Then the same homomorphism $\Phi\colon \mathbb{R}^\infty \to Ham(M,\omega)$ from earlier again obeys

$$\|v-w\|_{l_\infty} \le \delta(\Gamma_{\Phi(v)}, \Gamma_{\Phi(w)}) \le osc(v-w).$$

Contrast: Ostrover showed that his path $\{\phi_t\}_{t\in[0,\infty)}$ which goes to ∞ in $\widetilde{Ham}$ (and also in Ham for the manifolds in McDuff's theorem) have the property that $\delta(\Delta_M, \Gamma_{\phi_t})$ is constant.

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Now where $S^1=\mathbb{R}/\mathbb{Z}$, let $M_0=S^1\times\mathbb{R}$ or $S^1\times S^1$, and for $f\colon S^1\to\mathbb{R}$ let $L_f=\{(t,f'(t))|t\in S^1\}$ (or its projection to $S^1\times S^1$). Given $m\in\mathbb{Z}_+$ let

$$C^\infty_m(S^1,\mathbb{R})=\{f\in C^\infty(S^1,\mathbb{R})|(\forall t)(f(t+1/m)=f(t))\}.$$

Theorem

Let $L\subset M$ be a Lagrangian submanifold of another symplectic manifold such that the Floer homology $HF(L,L)$ is well-defined and nonzero. Then[a] in the space $\mathscr{L}(L_0\times L)$, for all $f,g\in C^\infty_m$ where $m\geq 2$ we have

$$\delta(L_f\times L,L_g\times L)\geq osc(f-g)-C$$

for some $C\geq 0$, with $C=0$ if $HF(L,L)\cong H_*(L)$.

[a] assuming a Künneth formula for Lagrangian Floer complexes, which is well-known in the monotone case and is the subject of work in progress by Amorim in much greater generality in the Fukaya–Oh–Ohta–Ono setup

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So paths of form $t \mapsto L_{tf} \times L$ are globally length-minimizing for all time if $HF(L,L) \cong H_*(L)$.

Hamiltonian Floer theory results from formally doing Morse theory on a cover of the loopspace $\mathscr{L}M$ of a symplectic manifold (M,ω), for the action functional

$$\mathscr{A}_H([\gamma,v]) = -\int_{[0,1]\times S^1} v^*\omega + \int_0^1 H(t,\gamma(t))dt.$$

Critical points are those $[\gamma,v]$ where the loop $\gamma\colon S^1 \to M$ is a flowline of the Hamiltonian vector field, and so $\phi_H^1(\gamma(0)) = \gamma(0)$.

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Lagrangian Floer theory for two Hamiltonian-isotopic Lagrangians L and $(\phi_H^1)^{-1}(L)$ results from formally doing Morse theory on a cover of the space $\mathscr{P}(L,L)$ of paths from L to itself, for the action functional

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Below we will always assume that $HF(L,L)$ is well-defined, and that we have already fixed a specific relative spin structure and bounding cochain, if necessary.

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In either case, one forms a Floer complex $CF(H)$ which, as a vector space, consists of certain formal linear combinations

$$\sum_{[\gamma,v]\in Crit(\mathcal{A}_H)} a_{[\gamma,v]}[\gamma,v]$$

and defines the boundary operator by counting appropriate negative gradient flowlines of $\mathcal{A}_H$.

If $c=\sum a_{[\gamma,v]}[\gamma,v]\in CF(H)$, define

$$\ell(c)=\max\{\mathcal{A}_H([\gamma,v])|a_{[v,w]}\neq 0\}.$$

Because $\mathcal{A}_H$ decreases along its gradient flowlines, for any $\lambda\in\mathbb{R}$ the subgroups

$$CF^{\lambda}(H)=\{c\in CF(H)|\ell(c)\leq\lambda\}$$

are preserved by the boundary operator—hence they provide a $\mathbb{R}$-**valued filtration** on $CF(H)$.

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The **boundary depth** of $CF(H)$ is

$$\beta(CF(H)) = \inf\left\{\beta \geq 0 \,\middle|\, (\forall\lambda \in \mathbb{R})\left(CF^{\lambda}(H) \cap (Im\partial) \subset \partial(CF^{\lambda+\beta}(H))\right)\right\}.$$

Said differently,

$$\beta(CF(H)) = \begin{cases} 0 & \text{if } \partial = 0 \\ \sup_{0\neq x\in Im\partial} \inf\{\ell(y) - \ell(x) | \partial y = x\} & \text{otherwise} \end{cases}$$

Key Lemma

- *In the Hamiltonian case, $\beta(CF(H))$ depends only on the time-one map $\phi = \phi_H^1$, not on the specific Hamiltonian function generating it.*
- *In the Lagrangian case, $\beta(CF(H))$ depends only on the unparametrized Lagrangian submanifold $L' = (\phi_H^1)^{-1}(L)$, not on the particular Hamiltonian whose flow maps L to L'.*

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For $\phi \in Ham(M,\omega)$ write $\beta(\phi)$ for $\beta(CF(H))$ whenever $\phi = \phi_H^1$, and similarly for $L' \in \mathscr{L}(L)$ write $\beta_L(L')$ for $\beta(CF(H))$ whenever $(\phi_H^1)^{-1}(L) = L'$

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Note that one can read off Chekanov's theorem on displacement energy of Lagrangian submanifolds (at least when $HF(L,L)$ is well-defined) from the last part, since if $L \cap L' = \varnothing$ then $\beta_L(L') = 0$. (If you unpack the proofs of the various lemmas, though, you realize that this isn't actually a new proof in view of similar arguments by e.g. Cornea–Lalonde.)

Other useful properties

- *If* $\phi \in Ham(M,\omega)$ *then*

$$\beta_{\Delta_M}(\Gamma_\phi) = \beta(\phi).$$

- *Suppose that* $HF(L_1, L_1) \neq 0$*. Then for* $K_0 \in \mathscr{L}(L_0)$ *and* $K_1 \in \mathscr{L}(L_1)$*, assuming a Künneth formula we have*

$$\beta_{L_0 \times L_1}(K_0 \times K_1) \geq \beta_{L_0}(K_0).$$

- *For* $L_f = \{(t, f'(t)) | t \in S^1\} \subset S^1 \times \mathbb{R}$ *or* $S^1 \times S^1$*, we have*

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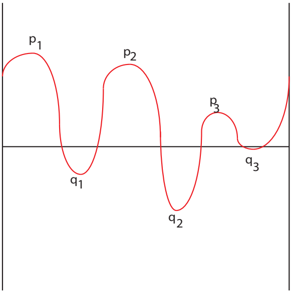
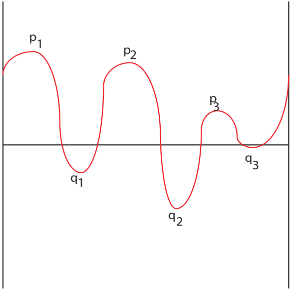
The final two points on the last slide reduce the proof of infinite diameter for $\mathscr{L}(S^1 \times L)$ when $HF(L,L) \neq 0$ to a calculation of the boundary depth for Morse functions on S^1.
A Morse function on S^1 has maxima $p_1, \ldots, p_r$, minima $q_1, \ldots, q_r$, and the Morse boundary operator is given by

$$\partial p_i = q_i - q_{i-1}$$

(where subscripts are evaluated mod r, so $\partial p_1 = q_1 - q_r$)



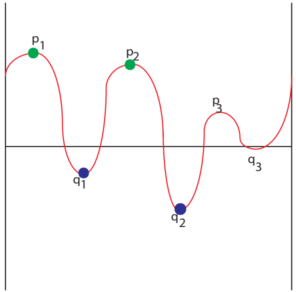
$$\partial p_i = q_i - q_{i-1}$$
$$\ker \partial = \langle \sum p_i \rangle$$
$$Im\partial = \langle \sum n_i q_i | \sum n_i = 0 \rangle$$
$$\beta(f) = \sup_{0 \neq x \in Im\partial} \inf\{\ell(y) - \ell(x) | \partial y = x\}$$



In this case $x := q_2 - q_1$ has primitives $y_0 = p_2$ and $y_1 = -p_1 - p_3$. So

$$\inf\{l(y) - l(x) | \partial y = x\} = f(p_2) - f(q_1)$$

(and taking the sup over x one finds that this is indeed the value of β).



Proposition

For any Morse function $f\colon S^1 \to \mathbb{R}$, $\beta_{Morse}(f)$ is equal to

$$\sup\left\{\min\{f(w),f(y)\} - \max\{f(x),f(z)\} \;\middle|\; \begin{array}{c} w,x,y,z \text{ are in} \\ \text{cyclic order on } S^1 \end{array}\right\}$$

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So $\beta_{Morse}(f)$ detects the presence of *linked* copies C_0, C_1 of S^0 in S^1 such that $f|_{C_0} > f|_{C_1}$. There exists a higher-dimensional generalization of this.

The theorem about the diameter of $\mathscr{L}(S^1 \times L)$ quickly follows: if f is periodic of period $1/m$ with $m \geq 2$ then we can take w and y equal to successive global maxima and x and z equal to successive global minima to see that in this case

$$\beta_{Morse}(f) = \max f - \min f.$$

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Let $H\colon M \to \mathbb{R}$ be a nonconstant autonomous Hamiltonian so that all contractible periodic orbits of X_H are constant.
The boundary depth may do a very poor job of estimating $\|\phi_H^1\|$, but for suitably chosen $f\colon \mathbb{R} \to \mathbb{R}$ it does quite well with $\|\phi_{f\circ H}^1\|$.
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For $f\colon \mathbb{R} \to \mathbb{R}$ with compact support in $]0,1[$, define

$\mathrm{minmax} f = \inf\{f(t) | t$ is a (maybe not strict) local maximum of $f\}$.

(So $\mathrm{minmax} f \le 0$.)

Theorem

For f and H as above,

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This cycle must be a boundary because $H^{-1}(t_0)$ bounds in M, but it can't be the boundary of anything with filtration level smaller than approximately $\text{minmax} f$.

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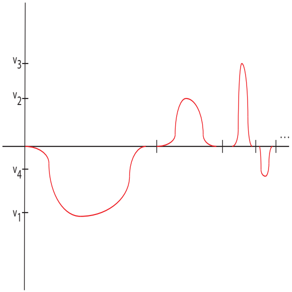
To get the quasi-isometrically embedded copy of $\mathbb{R}^\infty$, associate to $(v_1, v_2, \ldots)$ the Hamiltonian $f_{\vec{v}} \circ H$ where the graph of $f_{\vec{v}}$ looks like:



(One also uses a duality theorem saying that $\beta(\phi) = \beta(\phi^{-1})$.)