

MINIMALITY AND SYMPLECTIC SUMS

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ABSTRACT. Let X_1, X_2 be symplectic 4-manifolds containing symplectic surfaces F_1, F_2 of identical positive genus and opposite squares. Let Z denote the symplectic sum of X_1 and X_2 along the F_i . Using relative Gromov–Witten theory, we determine precisely when the symplectic 4-manifold Z is minimal (*i.e.*, cannot be blown down); in particular, we prove that Z is minimal unless either: one of the X_i contains a (-1) -sphere disjoint from F_i ; or one of the X_i admits a ruling with F_i as a section. As special cases, this proves a conjecture of Stipsicz asserting the minimality of fiber sums of Lefschetz fibrations, and implies that the non-spin examples constructed by Gompf in his study of the geography problem are minimal.

1. INTRODUCTION

Let $(X_1, \omega_1), (X_2, \omega_2)$ be symplectic 4-manifolds, and let $F_1 \subset X_1, F_2 \subset X_2$ be two-dimensional symplectic submanifolds with the same genus whose homology classes satisfy $[F_1]^2 + [F_2]^2 = 0$, with the ω_i normalized to give equal area to the surfaces F_i . For $i = 1, 2$, a neighborhood of F_i is symplectically identified by Weinstein’s symplectic neighborhood theorem [19] with the disc normal bundle ν_i of F_i in X_i . Choose a smooth isomorphism ϕ of the normal bundle to F_1 in X_1 (which is a complex line bundle) with the dual of the normal bundle to F_2 in X_2 . According to [2] (and independently [11]), the *symplectic sum*

$$Z = X_1 \#_{F_1=F_2} X_2 = (X_1 \setminus \nu_1) \cup_{\partial\nu_1 \sim_{\phi} \partial\nu_2} (X_2 \setminus \nu_2)$$

carries a natural deformation class of symplectic structures (note that the diffeomorphism type of Z may depend on the identification ϕ , as seen for instance in Example 3.2 of [2], a feature which is mostly suppressed from the notation hereinafter).

Symplectic sums along S^2 are well-understood; according to pp. 563–566 of [2] such a symplectic sum amounts to either blowing down a sphere of square -1 or -4 in one of the summands, taking the fiber sum of two ruled surfaces, or leaving the diffeomorphism type of one of the summands unchanged, and then possibly blowing up the result. Accordingly, we shall restrict our attention to symplectic sums along surfaces of positive genus.

Recall that a symplectic 4-manifold M is called *minimal* if it contains no symplectically embedded spheres of square -1 , and hence cannot be expressed as a blowup of another symplectic 4-manifold. According to results arising from Seiberg–Witten theory ([18],[8]), this is equivalent to the condition that M not contain any *smoothly* embedded spheres of square -1 .

In this note, we resolve completely the question of under what circumstances the above symplectic sum Z is minimal. Our result may be summarized as follows:

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Theorem 1.1. *Let the symplectic sum $Z = X_1 \#_{F_1=F_2} X_2$ be formed as above, and assume that the F_i have positive genus g . Then:*

- (i) *If either $X_1 \setminus F_1$ or $X_2 \setminus F_2$ contains an embedded symplectic sphere of square -1 , then Z is not minimal.*
- (ii) *If one of the summands X_i (for definiteness, say X_1) admits the structure of an S^2 -bundle over a surface of genus g such that F_1 is a section of this fiber bundle, then Z is minimal if and only if X_2 is minimal.*
- (iii) *In all other cases, Z is minimal.*

Case (i) above should be obvious: if X_1 admits a sphere of square -1 which misses F_1 , then cutting out a small neighborhood of F_1 and replacing the neighborhood with something else will not change that fact. In the situation of Case (ii), note that since F_1 and F_2 have opposite squares, the complement of a neighborhood of F_1 in the ruled surface X_1 is diffeomorphic to a neighborhood of F_2 , and so effectively the symplectic sum operation cuts out a neighborhood of F_2 and then glues it back in via a map which (since $\phi: \partial\nu_1 \rightarrow \partial\nu_2$ is a bundle isomorphism, not just a diffeomorphism) takes a meridian of F_2 to itself; certainly for some choices of the gluing map the result is just diffeomorphic to X_2 and so of course will have the same minimality properties, and indeed the author is unaware of any cases in which the diffeomorphism type of X_2 is changed by summing with a ruled surface along a section with a nonstandard choice of gluing map.

Case (iii) is thus quite broad, and in particular confirms the belief expressed by the authors of [10] that the main theorem of that paper could be generalized. For some prior results asserting the minimality of symplectic sums in various special cases, see [17], Theorem 1.5 of [16], Theorem 2.5 of [14], [10], and [7].

We now turn to some corollaries of Theorem 1.1. Recall that a map $f: X \rightarrow \Sigma$ from a symplectic 4-manifold to a 2-manifold is called a symplectic Lefschetz fibration provided that its smooth fibers are symplectic submanifolds and f has just finitely many critical points, near each of which it is given in orientation preserving complex coordinates by $(z_1, z_2) \mapsto z_1 z_2$. f is called relatively minimal if none of its singular fibers (each of which is a nodal curve) contains a (-1) -sphere as a reducible component. Note that in general a Lefschetz fibration admits compatible symplectic structures provided that its fiber is homologically essential; this condition is automatic if the fiber genus is at least two or if the genus is one and the map has at least one critical point. The fiber sum of two symplectic Lefschetz fibrations whose fibers have the same genus is just the symplectic sum along a smooth fiber. The following was conjectured by A. Stipsicz in [17]:

Corollary 1.2. *Let $f_i: X_i \rightarrow S^2$ ($i = 1, 2$) be relatively minimal symplectic Lefschetz fibrations on 4-manifolds X_1, X_2 whose fibers F_1, F_2 have the same positive genus g , and assume that neither f_i is the projection $\Sigma_g \times S^2 \rightarrow S^2$. Then the fiber sum $X_1 \#_{F_1=F_2} X_2$ is minimal.*

Another consequence of Theorem 1.1 relates to the geography of minimal symplectic 4-manifolds, that is, the question of which pairs of integers (a, b) have the property that there is a minimal symplectic 4-manifold $M_{a,b}$ such that $c_1^2(M_{a,b}) = a$ and $c_2(M_{a,b}) = b$. Any such pair (a, b) necessarily satisfies $a + b \equiv 0 \pmod{12}$ by the Noether formula. By performing symplectic sums on certain manifolds covered by Case (iii) of Theorem 1.1, R. Gompf in [2] realized a great many such pairs as the Chern numbers of symplectic manifolds with prescribed fundamental group; in the

case that the Chern numbers are consistent with Rohlin's theorem (so that (a, b) has form $(8k, 4k + 24l)$) Gompf was able to arrange that the resulting manifold be spin and hence minimal, but in other cases minimality appeared likely but could not be proven. However, Theorem 1.1 in conjunction with Remark 2 after Theorem 6.2 of [2] allow us to deduce:

Corollary 1.3. *Let G be any finitely presentable group. There is a constant $r(G)$ with the property that if $(a, b) \in \mathbb{Z}^2$ satisfies $a + b \equiv 0 \pmod{12}$ and $0 \leq a \leq 2(b - r(G))$ then there is a minimal symplectic 4-manifold $M_{a,b,G}$ such that $\pi_1(M_{a,b,G}) \cong G$, $c_1^2(M_{a,b,G}) = a$, and $c_2(M_{a,b,G}) = b$.*

Note that most of Gompf's examples involve taking symplectic sums in which at least one of the summands is rational, so these examples are not covered by previous results on the minimality of symplectic sums. In the simply connected case, different constructions have been used to obtain symplectic manifolds occupying large parts of the $c_1^2 c_2$ -plane and then to show that they are minimal using gauge theory; see, *e.g.*, Theorem 10.2.14 of [3].

The next two sections are occupied with the proof of Theorem 1.1. This proof splits into two parts: first, we use relative Gromov–Witten theory to give a condition on the pairs $(X_1, F_1), (X_2, F_2)$ in terms of the intersection numbers of the F_i with holomorphic spheres which is sufficient to guarantee the minimality of Z (namely, the F_i should be “rationally K -nef,” defined below). We then see that surfaces of positive genus in symplectic 4-manifolds are always rationally K -nef except in the cases (i) and (ii) in Theorem 1.1; this follows from results of Seiberg–Witten theory concerning the canonical class ([18],[9]) when the ambient manifold is not a blowup of a rational or ruled surface, while the ruled case can be handled fairly directly and the rational case depends in part on the analysis of the chamber structure in the cohomology of rational surfaces that was carried out in [1]. Finally, in the last section we prove Corollaries 1.2 and 1.3.

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2. SYMPLECTIC SUMS ALONG RATIONALLY K -NEF SURFACES ARE MINIMAL

If (X, ω) is a symplectic 4-manifold, $g \geq 0$, $A \in H_2(X; \mathbb{Z})$, and $\alpha \in H^*(X; \mathbb{Z})$, we let $GW_{g,A}^X(\alpha)$ denote the Gromov–Witten invariant ([15]) counting perturbed-pseudoholomorphic maps from a surface of genus g into X , representing the homology class A , and passing through a cycle Poincaré dual to α .

Suppose now that the symplectic sum $Z = X_1 \#_{F_1=F_2} X_2$ is *not* minimal, so that Z contains a symplectic sphere E of square -1 . Using an almost complex structure J generic among those making E pseudoholomorphic, we immediately see that $GW_{0,[E]}^Z(1) = \pm 1$, since the operator $\bar{\partial}_J$ will (for generic J) have nondegenerate linearization at the embedding of E , and positivity of intersections (see, *e.g.*, Theorem E.1.5 of [13]) prevents the existence of any other J -holomorphic spheres homologous to E . We now review how to use this nonvanishing to deduce the existence of nonzero relative Gromov–Witten invariants in certain homology classes in X_1, X_2 by means of the gluing results in [5].

The main theorem of [5] (or, similarly, that of [6]) provides a somewhat complicated formula expressing the Gromov–Witten invariants of Z in terms of the

Gromov–Witten invariants of X_1 and X_2 *relative* to the fibers F_1 and F_2 , the latter invariants having been defined in [4]. Without reproducing the formula, we recall the essential points, referring readers to [5] for details: in forming the symplectic sum, letting x_1 and x_2 be complex coordinates in the normal bundles ν_1, ν_2 , one performs the identification by taking $x_1 x_2 = \lambda$ for some small $\lambda \in \mathbb{C}^*$; this results in a symplectic 6-manifold Z equipped with a projection to the disc whose fiber $(Z_\lambda, \omega_\lambda)$ over $\lambda \in D^2 \setminus \{0\}$ is isotopic to Z but whose fiber over 0 is $Z_0 = X_1 \cup_{F_1=F_2} X_2$. As the parameter λ approaches zero, pseudoholomorphic curves in Z_λ limit to trees of curves in Z_0 consisting of: curves in X_1 meeting F_1 in isolated points; curves contained in the identified surfaces $F = F_1 = F_2$; and curves in X_2 meeting F_2 in isolated points. In fact there is a quantifiable finite-to-one correspondence between these trees (which are counted by a combination of (relative) Gromov–Witten invariants in X_1, F , and X_2) and the curves counted by the Gromov–Witten invariants of Z . This leads to a gluing formula (Theorem 12.4 of [5]) expressing the latter invariants in terms of the former.

Let us return to our case, where Z admits a sphere E of square -1 , and consider what the gluing result of [5] allows us to deduce about the Gromov–Witten invariants of X_1 and X_2 . First, notice that since E is a sphere the limiting tree discussed in the previous paragraph will consist only of genus-zero curves, and so will not have any components mapped into F , since $\pi_2(F) = 0$. So the gluing formula will express the nonzero Gromov–Witten invariant $GW_{0,[E]}^Z(1)$ in terms of invariants which count pairs (C_1, C_2) of possibly-disconnected curves (each of whose components are spheres) representing elements of the intersection-homology spaces $\mathcal{H}_{X_1}^{F_1}, \mathcal{H}_{X_2}^{F_2}$ defined in Section 5 of [4], and which have matching intersections with the fiber $F = F_1 = F_2$. So we obtain nonzero relative invariants in both X_1 and X_2 , which in particular count (generally disconnected) holomorphic curves in certain classes $A_i \in H_2(X_i; \mathbb{Z})$ ($i = 1, 2$) (subject to some additional constraints on their intersections with F_i). We note also that while in the general situation considered in [5] the Gromov–Witten invariants are counts of maps satisfying a perturbed Cauchy-Riemann equation $\bar{\partial}_J u = \nu$, in our case we can take $\nu = 0$ by virtue of the fact that $[E]$ is a primitive class in $H_2(Z; \mathbb{Z})$ (so that the relevant moduli space has no strata consisting of multiple covers). Thus the curves C_1 and C_2 will be genuinely pseudoholomorphic curves for almost complex structures J_1, J_2 on X_1, X_2 , which may be chosen generically among those pairs of almost complex structures on the X_i which preserve TF_i and agree on the identified neighborhoods ν_i of F_i . Incidentally, in our proof of Theorem 1.1 we will in fact only need the fact that a nonvanishing invariant $GW_{0,[E]}^Z(1)$ gives rise via a Gromov-type compactness theorem to (generally reducible, non-reduced) J_i -holomorphic curves C_i for some almost complex structures J_i which make the F_i pseudoholomorphic. The full strength of the Ionel-Parker theorem, which shows that the C_i are in fact enumerated by nonvanishing relative invariants, is not needed for this conclusion.

Let $d = A_1 \cap [F_1] = A_2 \cap [F_2]$. Then according to Lemma 2.2 of [5], we have, where for a symplectic manifold M we denote the canonical class of M by κ_M ,

$$\langle \kappa_Z, [E] \rangle = \langle \kappa_{X_1}, A_1 \rangle + \langle \kappa_{X_2}, A_2 \rangle + 2d.$$

But E is an embedded (-1) sphere and so satisfies $\langle \kappa_Z, [E] \rangle = -1$ by the adjunction formula; thus we may suggestively rewrite the above equation as

$$(1) \quad \langle \kappa_{X_1} + PD[F_1], A_1 \rangle + \langle \kappa_{X_2} + PD[F_2], A_2 \rangle = -1.$$

Accordingly, we make the following definition (recall from [13] that a pseudoholomorphic curve $u: \Sigma \rightarrow X$ is called simple if no two disjoint open sets in its domain Σ have the same image; in this case the map u is generically injective):

Definition 2.1. *Let (X, ω) be a symplectic four-manifold. An embedded symplectic surface $F \subset X$ is called rationally K -nef if, whenever J is an almost complex structure preserving TF and $A \in H_2(X; \mathbb{Z})$ is represented by a simple J -holomorphic sphere, we have*

$$(2) \quad \langle \kappa_X + PD[F], A \rangle \geq 0,$$

where κ_X is the canonical class of X .

The classes A_1, A_2 in (1) are both represented by J_i -holomorphic curves for appropriate almost complex structures J_i which make the respective F_i holomorphic (indeed, the A_i are the homology classes underlying elements of $\mathcal{H}_{X_i}^{F_i}$ having non-vanishing (possibly disconnected) relative Gromov–Witten invariants), and each component of these pseudoholomorphic curves has genus zero and is either simple or a multiple cover of a simple curve. So if F_1 and F_2 are both rationally K -nef then (1) cannot hold. Thus:

Proposition 2.2. *If $F_1 \subset X_1, F_2 \subset X_2$ are rationally K -nef surfaces, then the symplectic sum $X_1 \#_{F_1=F_2} X_2$ is minimal.*

3. MANY SURFACES ARE RATIONALLY K -NEF

To prove Case (iii) of our main theorem, we need to demonstrate that, if $F \subset X$ is an embedded symplectic surface of positive genus which intersects all symplectically embedded (-1) -spheres in X , and which is not a section of a ruled surface, then F is rationally K -nef. Let us begin with some general observations. We wish to show that if J preserves TF and $A \in H_2(Z; \mathbb{Z})$ is represented by a simple J -holomorphic sphere, then $\langle \kappa_X + PD(F), A \rangle \geq 0$. Now by positivity of intersections $[F] \cap A \geq 0$, so in the case that $\langle \kappa_X, A \rangle \geq 0$ the desired inequality obviously holds. Further, since F meets all embedded symplectic spheres of square -1 , if A is represented by such a sphere then positivity of intersections implies that $[F] \cap A \geq 1$, and so since in this case $\langle \kappa_X, A \rangle = -1$ the inequality holds when A is represented by an embedded (-1) -sphere as well.

Now in general the adjunction formula (*e.g.*, Corollary E.1.7 of [13]) shows that $A^2 + \langle \kappa_X, A \rangle \geq -2$, with equality if and only if A is represented by an embedded sphere. From this we see that $A^2 \geq 0$ unless either $\langle \kappa_X, A \rangle \geq 0$ or $A^2 = \langle \kappa_X, A \rangle = -1$ (in which case A is represented by an embedded (-1) -sphere), and the latter two cases have already been dispensed with. Hence:

Lemma 3.1. *An embedded symplectic surface $F \subset X$ which intersects all symplectically embedded (-1) -spheres in X is rationally K -nef provided that, whenever J is an almost complex structure preserving TF and $A \in H_2(X; \mathbb{Z})$ is represented by a simple J -holomorphic sphere such that*

$$A^2 \geq 0 \quad \text{and} \quad \langle \kappa_X, A \rangle < 0,$$

we have

$$\langle \kappa_X + PD[F], A \rangle \geq 0.$$

The task of showing that the surfaces F in question are rationally K -nef now naturally splits up into several cases.

3.1. Case 1: $b^+(X) > 1$. In this case, according to results of [18], the Gromov–Taubes invariant of the canonical class is nonzero, and so for generic almost complex structures among those preserving TF the class $PD(\kappa_X)$ is represented by a (possibly disconnected) pseudoholomorphic curve, whose only spherical components are embedded (-1) -spheres. For an arbitrary complex structure J preserving TF , then, Gromov compactness shows that $PD(\kappa_X)$ is represented at least by a union of pseudoholomorphic bubble trees. So if A is represented by a simple J -holomorphic sphere with nonnegative square, then positivity of intersections between the representatives of $PD(\kappa_X)$ and A shows that $\langle \kappa_X, A \rangle \geq 0$ (of course, if A had negative square the representative of $PD(\kappa_X)$ might contain the representative of A as a component, allowing the intersection to be negative; indeed when A is the class of an embedded (-1) -sphere this is precisely what happens). So the sufficient condition provided by Lemma 3.1 is vacuously satisfied and:

Proposition 3.2. *If $b^+(X) > 1$ and F is an embedded symplectic surface in X which intersects all symplectically embedded (-1) -spheres in X , then F is rationally K -nef.*

3.2. Case 2: X is a (possibly trivial) blowup of an irrational ruled surface. First suppose $X = (\Sigma \times S^2) \# n\overline{\mathbb{C}P^2}$, where Σ has genus $h > 0$. Let σ denote the homology class of the strict transform of $\Sigma \times \{pt\} \subset \Sigma \times S^2$, f the homology class of the strict transform of $\{pt\} \times S^2$, and e_1, \dots, e_n the classes of the n exceptional divisors. We then have

$$PD(\kappa_X) = -2\sigma + (2h - 2)f + \sum_{i=1}^n e_i.$$

Now assuming that $n > 0$ the (-1) -spheres e_i all have nonvanishing Gromov–Witten invariant, so if F meets all embedded (-1) -spheres we have $[F] \cap e_i \geq 1$ for each i ; further we get (-1) -spheres in each of the classes $f - e_i$ by choosing a symplectic sphere homologous to $\{pt\} \times S^2$ in $\Sigma \times S^2$ which meets the i th blow-up point and none of the others, and taking its strict transform, so that $[F] \cap (f - e_i) \geq 1$ as well. Also, regardless of whether n is positive, the class f has a nonvanishing Gromov–Witten invariant with a single point constraint, and so by considering an almost complex structure preserving TF we see that by positivity of intersections $[F] \cap f \geq 0$ (since F has positive genus, no rational curve can share a component with it). Hence

$$[F] = c\sigma + df - \sum_{i=1}^n a_i e_i$$

where $c \geq 0$ and, for each i , $1 \leq a_i < c$. Since $h > 0$, the only rational curves which appear will be contained in the fiber of the ruling and so, if their Chern number is positive, will have homology class in the cone spanned by the classes e_i and $f - e_i$, if $n > 0$, or will be a multiple of f if $n = 0$. If $n > 0$, then since e_i and $f - e_i$ have square -1 and so pair positively with F and as -1 with κ_X , $\kappa_X + PD[F]$ is nonnegative on each of them and hence also on any element of the cone spanned by them. So if $n > 0$ F is rationally K -nef. In case $n = 0$, to verify that F is rationally K -nef we just need to ensure that $\kappa_X + PD[F]$ pairs positively with f , which amounts to the statement that $c \geq 2$. Now we have $[F] = c\sigma + df \in H_2(\Sigma \times S^2; \mathbb{Z})$ with $c \geq 0$; observe that if $c = 0$, F would be homologous to a multiple of the fiber of the projection $\Sigma \times S^2 \rightarrow \Sigma$, and this multiple would be positive since both F

and the fiber are symplectic. So if $c = 0$ we would have $[F] = df$ with $d > 0$, and so $2g(F) - 2 = -2d < 0$, contradicting the fact that F has positive genus. If $c = 1$, we see that

$$2g(F) - 2 = [F]^2 + \langle \kappa_X, [F] \rangle = 2d - 2d + 2h - 2,$$

i.e., $h = g(F)$ and X is the total space of a (trivial) fibration over a surface of genus g . Thus F is rationally K -nef unless X is a trivial S^2 -fibration over a surface of genus $g = g(F)$ whose fiber has homological intersection number 1 with F .

Now suppose that X is a nontrivial sphere bundle $\Sigma \tilde{\times} S^2$ over a surface of genus $h > 0$. In this case, the homology is generated by sections $s^+, s^- \in H_2(X; \mathbb{Z})$ such that $(s^\pm)^2 = \pm 1$ and $s^+ \cap s^- = 0$. The fiber of the fibration represents the homology class $f = s^+ - s^-$, and the canonical class is given by $PD(\kappa_X) = (2h - 2)f - (s^+ + s^-) = (2h - 3)s^+ - (2h - 1)s^-$. Again, since $h > 0$, the only classes represented by J -holomorphic rational curves are multiples of the fiber class $s^+ - s^-$; as such, positivity of intersections implies that our surface F represents a class of form $[F] = cs^+ + ds^-$ where $c + d \geq 0$. The desired rational K -nef condition for $[F]$ is that $c + d \geq 2$, so we just need to rule out $c + d \in \{0, 1\}$. We find

$$2g(F) - 2 = [F]^2 + \langle \kappa_X, [F] \rangle = (c + d)(c - d) + (2h - 2)(c + d) + (d - c).$$

If $c + d = 0$, then for some $a \in \mathbb{Z}$, $[F] = a(s^+ - s^-) = af$, with again $a > 0$ since F is symplectic; then the adjunction formula would give $2g(F) - 2 = -2a < 0$, a contradiction. If $c + d = 1$, the adjunction formula reads $2g(F) - 2 = 2h - 2$, so as before $g(F) = h$ and F has intersection number 1 with the fibers.

We now make the following observation:

Proposition 3.3. *Suppose that (X, ω) is the total space of a ruled surface over a surface of genus $g > 0$, and that $F \subset X$ is a symplectic submanifold having genus g and homological intersection number one with the fibers of the ruling. Then X admits a (possibly different) ruling $\pi: X \rightarrow \Sigma$ over a surface of genus g such that F is a section of π .*

Proof. This essentially follows from the techniques of [12]: let J be an ω -compatible almost complex structure on X which preserves TF . The mere existence of an embedded symplectic sphere of square zero in X ensures that, letting Σ denote the moduli space of (unparametrized) J -holomorphic spheres homologous to the fiber of the (original) ruling, Σ is two-real dimensional, and the map $\pi: X \rightarrow \Sigma$ which takes a point $x \in X$ to the point of Σ representing the unique J -holomorphic sphere on which x lies is an S^2 -bundle with symplectic (indeed J -holomorphic) fibers. Since F is also J -holomorphic, the assumption on the intersection number ensures that each of these fibers meets F transversely and just once, so letting $s: \Sigma \rightarrow F$ denote the map which takes a J -curve $C \in \Sigma$ to the unique point of $C \cap F$, we see that s is a section of π with image F . Of course, consideration of $b_1(X)$ reveals that Σ , like F , has genus g . \square

Hence since, for $n > 0$, $(\Sigma \tilde{\times} S^2) \# n \overline{\mathbb{C}P^2}$ is symplectomorphic to $(\Sigma \times S^2) \# n \overline{\mathbb{C}P^2}$, we deduce:

Proposition 3.4. *If the minimal model of X is an irrational ruled surface, and if $F \subset X$ is an embedded surface of positive genus which intersects each embedded (-1) -sphere, then F is rationally K -nef unless X is already minimal and admits a ruling with F as a section.*

3.3. Case 3: $X = S^2 \times S^2$. Let $[F] = c\sigma + df$ where $\sigma = [S^2 \times \{pt\}]$ and $f = [\{pt\} \times S^2]$. Then $PD(\kappa_X) = -2\sigma - 2f$. Now there are nonzero Gromov–Witten invariants counting holomorphic spheres in both the classes σ and f , so since F is symplectic by considering an almost complex structure preserving TF we see that $c, d \geq 0$ by positivity of intersections between F and the holomorphic spheres representing f and σ ; of course F is homologically essential, so c and d cannot both be zero. By the adjunction formula,

$$0 \leq 2g(F) - 2 = [F]^2 + \langle \kappa_X, [F] \rangle = 2cd - 2(c+d) = 2((c-1)(d-1) - 1),$$

and this then forces $c, d \geq 2$. This implies the desired property for $[F] = c\sigma + df$, since if $a\sigma + bf$ were some other class represented by a J -holomorphic curve for some J preserving TF , positivity of intersections with the J -holomorphic representatives of f and σ would imply that $a, b \geq 0$, and so $\langle \kappa_X + PD[F], a\sigma + bf \rangle = a(c-2) + b(d-2) \geq 0$. We have shown:

Proposition 3.5. *If $X = S^2 \times S^2$ and $F \subset X$ is an embedded symplectic surface of positive genus then F is rationally K -nef.*

Before proceeding to the remaining cases, we recall a basic fact about the intersection forms of symplectic 4-manifolds (X, ω) with $b^+(X) = 1$. Where $n = b^-(X)$, the intersection form on $H^2(X; \mathbb{Q})$ has type $(1, n)$. As such, the “positive cone” $\{\beta \in H^2(X; \mathbb{Q}) \mid \beta^2 > 0\}$ has two connected components. A consequence of the Cauchy-Schwarz inequality often called the “light cone lemma” then asserts that the product of any two elements lying in the closure of the same component of the positive cone is nonnegative, and in fact is positive unless the elements are proportional and both have square zero. In particular, one of the components (called the “forward positive cone”) is characterized by the property that all of its elements pair positively with $[\omega]$, and so we obtain the following fact which we shall make use of on two occasions:

Lemma 3.6. *If $b^+(X) = 1$ and $\alpha, \beta \in H^2(X; \mathbb{Q})$ satisfy $\alpha^2 \geq 0$, $\beta^2 \geq 0$, $\alpha \cdot [\omega] \geq 0$, and $\beta \cdot [\omega] \geq 0$, then $\alpha \cdot \beta \geq 0$.*

3.4. Case 4: $X = \mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$. Let $H \in H_2(X; \mathbb{Z})$ be the class of the preimage of a generic line in $\mathbb{C}P^2$ under the blowdown map $X \rightarrow \mathbb{C}P^2$, and let $E_1, \dots, E_n \in H_2(X; \mathbb{Z})$ be the homology classes of the exceptional divisors of the blowups.

Lemma 3.7. *With the notation as above, let $\lambda \in H^2(X; \mathbb{Z})$ satisfy $\lambda^2 \geq \kappa_X \cdot \lambda$, $\langle \lambda, H \rangle \geq 0$ and $\langle \lambda, [S] \rangle \geq 0$ for every embedded symplectic (-1) -sphere S . Then*

$$\lambda^2 \geq 0.$$

Proof. If $n = 0$, the intersection form is positive definite and so the result is trivial. If $n = 1$, write $\lambda = aPD(H) - bPD(E_1)$. That $\langle \lambda, H \rangle, \langle \lambda, E_1 \rangle \geq 0$ shows that $a, b \geq 0$, and that $\lambda^2 \geq \kappa_X \cdot \lambda$ shows that $a^2 + 3a \geq b^2 + b$. We wish to show that $a \geq b$; write $b = a + k$, so we need to show that $k \leq 0$. Expanding out and rearranging $a^2 + 3a \geq b^2 + b$ gives

$$2a \geq 2ak + k + k^2;$$

noting that $k \in \mathbb{Z}$ if k were positive we would have $k \geq 1$ and so $2ak + k + k^2 \geq 2a + 2$, a contradiction. So indeed $k \leq 0$ and so $\lambda^2 \geq 0$.

Assume for the rest of the proof that $n \geq 2$. We hereinafter view λ as a real cohomology class rather than an integral one. Of course, if $\kappa_X \cdot \lambda \geq 0$ then the hypothesis of the lemma immediately shows $\lambda^2 \geq 0$, so we may assume that

$$-\kappa_X \cdot \lambda > 0.$$

Give X an integrable complex structure J which makes it a good generic surface in the sense of [1], *i.e.* such that $-\kappa_X$ is Poincaré dual to a smooth elliptic curve and all rational curves of negative square on X are smooth and have square -1 . (Such a J exists by results from section I.2 of [1].) Let \mathcal{E} denote the set of all classes in $H_2(X; \mathbb{Z})$ which are represented by smooth rational curves of square -1 ; of course these spheres are symplectic, so we have

$$\lambda \in \mathcal{S} := \{\alpha \in H^2(X; \mathbb{R}) \mid -\kappa_X \cdot \alpha \geq 0, \langle \alpha, H \rangle \geq 0, \text{ and } (\forall E \in \mathcal{E})(\langle \alpha, E \rangle \geq 0)\},$$

and so the lemma will be proven if we can show that the above set \mathcal{S} consists only of elements of nonnegative square, which we now set about doing.

Following section II.3 of [1], let $\mathbf{H}(X) \subset H_2(X; \mathbb{R})$ denote the set of elements of square $+1$, and let $\mathcal{K}(X, J) \subset \mathbf{H}(X)$ denote the closure (in $\mathbf{H}(X)$) of the set of cohomology classes of square 1 which are represented by Kähler forms. In [1], $\mathcal{K}(X, J)$ is given two descriptions which are relevant to us. On the one hand, according to Proposition II.3.4 of that paper translated into our notation, we have

$$\mathcal{K}(X, J) = \{\alpha \in \mathcal{S} \mid \alpha^2 = 1\}.$$

On the other hand, Proposition II.3.6 shows that there is a *chamber* $C_0(X)$, *i.e.* the closure of a connected component of

$$\mathbf{H}(X) \setminus \bigcup_{\{\alpha \in H^2(X; \mathbb{Z}) : \alpha^2 = -1\}} \alpha^\perp,$$

with the property that

$$\mathcal{K}(X, J) \subset C_0(X)$$

(in fact, $\mathcal{K}(X, J)$ has a more explicit description as a “ P -cell” of $C_0(X)$, but this will not be important to us). We hence have $\{\alpha \in \mathcal{S} \mid \alpha^2 > 0\} \subset \mathbb{R}_+ \cdot C_0(X)$, whence (since \mathcal{S} is convex, and so if $\alpha \in \mathcal{S}$ has $\alpha^2 = 0$ a line segment from α to an element of \mathcal{S} with positive square gives a sequence of elements of $\mathbb{R}_+ \cdot C_0(X)$ approximating α)

$$\{\alpha \in \mathcal{S} \mid \alpha^2 \geq 0\} \subset \overline{\mathbb{R}_+ \cdot C_0(X)}.$$

We may now appeal to Corollary II.1.20 of [1], which asserts that, *because* $n \geq 2$, the set $\{\alpha \in \overline{\mathbb{R}_+ \cdot C_0(X)} \mid \alpha^2 = 0\}$ (and hence also $\{\alpha \in \mathcal{S} \mid \alpha^2 = 0\}$) has no interior in $\{\alpha \in H^2(X; \mathbb{R}) \mid \alpha^2 = 0\}$. So arguing just as in the proof of Corollary II.1.21 of [1], if \mathcal{S} contained some element β of negative square, then since the intersection of \mathcal{S} with $\{\alpha^2 > 0\}$ (namely $\mathbb{R}_+ \cdot \mathcal{K}(X, J)$) has nonempty interior, the set of points of square zero lying on line segments from β to elements of $\mathbb{R}_+ \cdot \mathcal{K}(X, J)$ would sweep out a set with nonempty interior in $\{\alpha^2 = 0\}$, which gives a contradiction since \mathcal{S} is convex. Thus indeed all elements of \mathcal{S} have nonnegative square. \square

We now assume that F is an embedded symplectic surface of positive genus which intersects all embedded symplectic (-1) -spheres. Write $[F] = aH - \sum_{i=1}^n b_i E_i$, so positivity of intersections with the holomorphic representatives of E_i for an almost

complex structure preserving TF shows $b_i \geq 1$, and the fact that F has positive area shows that $a > 0$. By the adjunction formula, we have

$$0 \leq [F]^2 + \langle \kappa_X, [F] \rangle = a(a-3) + \sum (b_i - b_i^2) \leq a(a-3),$$

and so since $a > 0$ we in fact have $a \geq 3$.

Set

$$\lambda = \kappa_X + PD[F] = (a-3)PD(H) + \sum_{i=1}^n (1-b_i)E_i \in H^2(X; \mathbb{Z})$$

So $\langle \lambda, H \rangle = a-3 \geq 0$, and if S is an embedded symplectic (-1) -sphere, then $\langle \lambda, [S] \rangle = [F] \cap [S] - 1 \geq 0$. Further by the adjunction formula $\lambda^2 - \kappa_X \cdot \lambda = [F]^2 + \langle \kappa_X, [F] \rangle \geq 0$. Hence by Lemma 3.7 $\lambda^2 \geq 0$.

Assume that $A \in H_2(X; \mathbb{Z})$ is represented by a simple J -holomorphic sphere for some almost complex structure J preserving TF ; in accordance with Lemma 3.1, we assume that $A^2 \geq 0$. The class H has a nonvanishing Gromov–Witten invariant with two point constraints, so positivity of intersections guarantees that $A \cap H \geq 0$. So since $(\kappa_X + PD[F])^2 \geq 0$, $A^2 \geq 0$, and $\langle \kappa_X + PD[F], H \rangle$ and $A \cap H$ are both nonnegative, it follows from Lemma 3.6 that also $\langle \kappa_X + PD[F], A \rangle \geq 0$. Thus:

Proposition 3.8. *If $F \subset \mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ is an embedded symplectic surface of positive genus g which intersects all embedded (-1) -spheres, then F is rationally K -nef.*

Of course, since $(S^2 \times S^2) \# n\overline{\mathbb{C}P^2} = \mathbb{C}P^2 \# (n+1)\overline{\mathbb{C}P^2}$ for $n \geq 1$, we've now exhausted all rational cases.

3.5. Case 5: $b^+(X) = 1$ and the minimal model of X is neither rational nor ruled. We claim that no $A \in H_2(X; \mathbb{Z})$ satisfying $A^2 \geq 0$ and $\langle \kappa_X, A \rangle < 0$ is represented by a simple J -holomorphic sphere (or indeed any J -holomorphic curve) for any almost complex structure J . To see this, let X_0 be a minimal model for X , and let $\pi: X \rightarrow X_0$ be the blowdown map. Let E_1, \dots, E_k be homology classes of the exceptional divisors of the blowups, so that we can write $A = \pi^! A_0 + \sum a_i E_i$ and $PD(\kappa_X) = PD(\pi^* \kappa_{X_0}) + \sum E_i$. Now if our claim were false positivity of intersections of A with the E_i would show that $a_i \leq 0$, so that $\langle \kappa_X, A \rangle = \langle \kappa_{X_0}, A_0 \rangle - \sum a_i \geq \langle \kappa_{X_0}, A_0 \rangle$; also $A^2 = A_0^2 - \sum a_i^2 \leq A_0^2$. So the assumption that $A^2 \geq 0$ implies that also $A_0^2 \geq 0$, and so A_0 lies in the closure of the forward light cone determined by the symplectic form on X_0 . But Theorems A and B of [9] show that, since X_0 is minimal and neither rational nor ruled, κ_{X_0} is also in the closure of the forward light cone, and so $\langle \kappa_{X_0}, A_0 \rangle \geq 0$, whence $\langle \kappa_X, A \rangle \geq 0$, contradicting our assumption on $\langle \kappa_X, A \rangle$.

Thus in this case, just as in Case 1, the sufficient condition in Lemma 3.1 is vacuously satisfied. Combining this with the results in Cases 1,2,3, and 4 now shows:

Proposition 3.9. *Let (X, ω) be a symplectic 4-manifold, and F an embedded symplectic surface in F with genus $g > 0$. Assume that there are no symplectic spheres of square -1 contained in $X \setminus F$, and that X is not the total space of an S^2 -bundle over a surface of genus g having F as a section. Then F is rationally K -nef.*

Combining Proposition 3.9 with Proposition 2.2 thus immediately proves Case (iii) of Theorem 1.1.

Since Case (i) of Theorem 1.1 is trivial, all that remains now is to prove the statement of Case (ii). So assume that X_1 is a ruled surface with section F_1 . If X_2 is not minimal, let E be a symplectic (-1) -sphere in X_2 . If necessary, perturb E so that its intersections with F_2 are transverse, say at p_1, \dots, p_k . So $E \setminus \nu_2 \subset X_2 \setminus \nu_2$ is a sphere with k boundary components, which lie over p_1, \dots, p_k in the circle bundle over F_2 which is the boundary of $X_2 \setminus \nu_2$. Now F_1 contains symplectic spheres f_1, \dots, f_k of square zero meeting $F_1 \cong F_2$ at just the points p_1, \dots, p_k , respectively, and we may (perhaps after a small isotopy) use the discs $f_i \setminus \nu_1$ to cap off the boundary components of $E \setminus \nu_2$ in the symplectic sum Z , thus obtaining a sphere which has square -1 since $[E]^2 = -1$ and each $[f_i]^2 = 0$; one can see that the sphere can be taken symplectic by means of the pairwise sum construction from Theorem 1.4 of [2].

Conversely, if the symplectic sum Z is not minimal we can simply note that if we take the symplectic sum of Z with the ruled surface X_1 by identifying the image of F_1 in $Z = X_1 \#_{F_1=F_2} X_2$ with $F_1 \subset X_1$ using the inverse of the gluing map that was used to form Z , then the resulting manifold is deformation equivalent to X_2 ; effectively, in forming Z we have removed a neighborhood of $F_2 \subset X_2$ and then glued it back in a possibly new way, and in taking this second symplectic sum with X_1 we are just regluing this neighborhood of F_2 in its original configuration. So since by the previous paragraph non-minimality is preserved under symplectic sums with X_1 along F_1 , we deduce that if Z is not minimal then neither is X_2 . Another way of seeing this fact is to note that in the relative Gromov–Witten theory argument in Section 2 one or the other of the disconnected curves representing the classes A_1 and A_2 in equation (1) must necessarily include a sphere of square -1 as a component (for a virtual dimension count precludes spheres of lower square than -1 from contributing), and so since X_1 is minimal X_2 would have to contain a sphere of square -1 . This concludes the proof of Theorem 1.1.

4. PROOF OF THE COROLLARIES

Proof of Corollary 1.2. Let $f: X \rightarrow S^2$ be a symplectic Lefschetz fibration of positive genus g on a symplectic 4-manifold which is relatively minimal and is not the projection $\Sigma_g \times S^2 \rightarrow S^2$, and let F be a smooth fiber of X . Choose J generically from the set of almost complex structures on X which make f a pseudoholomorphic map. If $E \in H_2(X; \mathbb{Z})$ is represented by an embedded symplectic (-1) -sphere, then it has a J -holomorphic representative S , and then $f|_S$ is a holomorphic map from S^2 to itself of degree equal to $E \cap [F]$. Hence $[F] \cap E > 0$ unless S is contained in a fiber of f , but this latter possibility is forbidden by relative minimality.

If X admits a ruling with F as a section, then since F has even square $X = \Sigma \times S^2$. Again, let J be an almost complex structure making f pseudoholomorphic, and as in the proof of Proposition 3.3 let Σ denote the space of unparametrized pseudoholomorphic curves in the class $[\{pt\} \times S^2]$. We then obtain an S^2 -fibration $g: X \rightarrow \Sigma$ having F as a section by sending $x \in X$ to the point in Σ representing the unique J -holomorphic sphere on which x lies. Then $x \mapsto (g(x), f(x))$ gives an identification $X \cong \Sigma \times S^2$ with respect to which f appears as the projection $\Sigma \times S^2 \rightarrow S^2$, a contradiction.

So if f_1, f_2 are as in the statement of the Corollary, then their smooth fibers F_i ($i = 1, 2$) meet each embedded symplectic (-1) -sphere in X_i and are not sections of rulings on X_i . Hence by Theorem 1.1 the fiber sum of f_1 and f_2 is minimal. \square

Proof of Corollary 1.3. Gompf constructs 4-manifolds $M_{a,b,G}$ with $\pi_1 = G$, $c_1^2 = a$, $c_2 = b$ in Section 6 of [2]; we just need to see that these are minimal. The $M_{a,b,G}$ are obtained by fiber sums involving the following building blocks:

- (i) A manifold M_G which is spin (and hence minimal) and satisfies $\pi_1(M_G) = G$, $c_1^2(M_G) = 0$, $c_2(M) > 0$. The summation occurs along a certain torus T of square zero in M_G .
- (ii) $P_1 = \mathbb{C}P^2 \# 13\overline{\mathbb{C}P^2}$, which contains a square zero, genus 2 curve F which is obtained from an irreducible quartic in $\mathbb{C}P^2$ having a single ordinary double point by blowing up at the double point and at 12 other points on the curve. Note that where S is the strict transform of a line in $\mathbb{C}P^2$ which passes through the first blown-up point and none of the others, we have $PD[F] = c_1(P_1) + PD[S]$. If E is an embedded symplectic sphere of square -1 in P_1 , we have $[S] \cap [E] \geq 0$ (using positivity of intersections) and $\langle c_1(P_1), [E] \rangle = 1$, so F meets E .
- (iii) $P_2 = \mathbb{C}P^2 \# 12\overline{\mathbb{C}P^2}$, which contains a square zero, genus 2 curve F which is obtained from a sextic in $\mathbb{C}P^2$ having eight ordinary double points by blowing up at the double points and at four other points on the curve. Again, since $PD[F] - c_1(P_2)$ is represented by an effective divisor (namely the strict transform of a cubic passing through the first eight blown-up points), we deduce as in the preceding case that F meets every embedded symplectic sphere of square -1 in P_2 .
- (iv) $Q_1 = (T^2 \times T^2) \# 2\overline{\mathbb{C}P^2}$, which contains a square zero, genus 2 curve F obtained by symplectically resolving the double point in $\mathbb{T}^2 \times \{p\} \cup \{p\} \times T^2 \subset T^2 \times T^2$ and then blowing up two points on the resulting surface. Since $T^2 \times T^2$ is aspherical the only embedded symplectic (-1) -spheres in Q_1 are the two exceptional divisors of the blowup, each of which meets F .
- (v) Q_2 , which is a torus bundle over a surface of genus 2 (hence is aspherical) and contains a square zero surface F of genus 2.
- (vi) $S_{1,1}$, which is constructed as follows: let $T_1 = T^2 \times \{p\}$, $T_2 = \{p\} \times T^2 \subset T^2 \times T^2$, and blow up at (p, p) and at 8 points on $T_1 \setminus T_2$, and take the symplectic sum with $\mathbb{C}P^2$ by identifying (the strict transform of) T_1 with a cubic; the result is minimal by Theorem 1.1. Now blow up at 8 more points on the strict transform of T_2 and form $S_{1,1}$ as the symplectic sum with $\mathbb{C}P^2$ by identifying the new strict transform of T_2 with a cubic; again this is minimal by Theorem 1.1.
- (vii) $\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$, which contains a square 1, genus 1 curve which is Poincaré dual to c_1 and hence meets each sphere of square -1 .

None of these building blocks are ruled, and each is either minimal or has the property that each of its embedded symplectic (-1) -spheres pass through the genus-1 or 2 surface along which the fiber sum is taken. Further any symplectic sum of two or more of them along the relevant surfaces has positive Euler characteristic and so is not a ruled surface over a curve of positive genus. So if $M_{a,b,G} = X_1 \#_{F_1=F_2} X_2 \# \cdots \# X_{k-1} \#_{F_{k-1}=F_k} X_k$ where the X_i are as above, applying Theorem 1.1 inductively shows that for each $l \geq 2$ $X_1 \#_{F_1=F_2} X_2 \# \cdots \# X_{l-1} \#_{F_{l-1}=F_l} X_l$ is minimal, and in particular $M_{a,b,G}$ is minimal. \square

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