Spectral invariants, the energy-capacity inequality, and the non-squeezing theorem

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Outline

$$\mathbb{R}^{2m} = \{(x_1, y_1, \ldots, x_m, y_m)\}$$

carries a standard symplectic form

$$\omega_0 = \sum_{i=1}^{m} dx^i \wedge dy^i.$$

If $U, V \subset \mathbb{R}^{2m}$, call $\phi : U \to V$ symplectic if $\phi^* \omega_0 = \omega_0$.
Let

$$B^{2m}(r) = \left\{ \sum_{i=1}^{m} (x_i^2 + y_i^2) < r^2 \right\}.$$

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Gromov's non-squeezing theorem

Theorem (Gromov, 1985)

If there exists a symplectic embedding

$$\phi : B^{2n}(r) \hookrightarrow B^2(R) \times \mathbb{R}^{2n-2},$$

then $r \leq R$.

Here the codomain should be understood as $\{x_1^2 + y_1^2 < R^2\}$; if we used $\{x_1^2 + x_2^2 < R^2\}$ the theorem would be false: if $\varepsilon > 0$ the symplectic map

$$(x_1, y_1, x_2, y_2, \dots) \mapsto (\varepsilon x_1, \varepsilon^{-1} y_1, \varepsilon x_2, \varepsilon^{-1} y_2, \dots)$$

embeds $B^{2n}(1)$ into $\{x_1^2 + x_2^2 < \varepsilon^2\}$.

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Generalizations

Theorem (Lalonde-McDuff, 1994)

If (M,ω) is any $(2n-2)$-dimensional symplectic manifold (i.e., $\omega \in \Omega^2(M)$ satisfies $d\omega = 0$ and $\omega^{\wedge(n-1)}$ is a volume form), and if there exists a symplectic embedding

$$\phi : (B^{2n}(r), \omega_0) \hookrightarrow (B^2(R) \times M, \omega_0 \oplus \omega),$$

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Theorem (U. 2008)

If (P^{2p}, ω_P) and (M^{2m}, ω_M) are symplectic manifolds with P closed and M closed or Stein (e.g., $M = \mathbb{R}^{2m}$) and if there exists a symplectic embedding

$$\phi : (B^{2n-2p}(r) \times P, \omega_0 \oplus \omega_P) \hookrightarrow (B^{2n-2m}(R) \times M, \omega_0 \oplus \omega_M),$$

then $r \leq R$.

This seems most interesting when the volume $\int_P \omega^{\wedge p}$ is very small; contrastingly, it follows from old results of Gromov that for any $r, R > 0$ there is $\varepsilon > 0$ such that $B^{2n-2p}(r) \times B^{2p}(\varepsilon)$ symplectically embeds in $B^{2n-2m}(R) \times M$.

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Hamiltonian flows

If (M, ω) is a symplectic manifold and

$$\begin{aligned} H : \mathbb{R}/\mathbb{Z} \times M &\to \mathbb{R} \\ (t, m) &\mapsto H_t(m) \end{aligned}$$

is smooth and compactly supported, define a time-dependent vector field X_{H_t} by

$$\omega(X_{H_t}, \cdot) = dH_t.$$

One has $\mathscr{L}_{X_{H_t}} \omega = 0$, so where $\phi_H^t \in Diff(M)$ is defined by $\phi_H^0 = I$, $\frac{d}{dt}\phi_H^t(m) = X_{H_t}(\phi_H^t(m))$, we have

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$$Ham^c(M,\omega) = \{\phi_H^1 | H \in C_c^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R})\} \leq Symp(M,\omega) \leq Diff(M).$$

Define:

$$\|H\| = \int_0^1 (\max H_t - \min H_t)\, dt \quad (H \in C_c^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R}));$$

$$\|\phi\| = \inf\{\|H\| : \phi_H^1 = \phi\} \quad (\phi \in Ham^c(M,\omega));$$

$$d(\phi,\psi) = \|\phi\psi^{-1}\| \quad (\phi,\psi \in Ham^c(M,\omega))$$

Theorem (Hofer, Lalonde-McDuff)

*d is a bi-invariant **metric** on $Ham^c(M,\omega)$; in particular if $\phi \neq I$ then $\|\phi\| > 0$.*

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Displacement energy

Definition

If $K \subset M$ is compact, set

$$e(K,M) = \inf\{\|\phi\| : \phi(K) \cap K = \varnothing\}.$$

For general $A \subset M$ set

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It's not hard to see that if $\phi : M \to M$ is a symplectic diffeomorphism with $\phi(A) \subset B$, then $e(A,M) \leq e(B,M)$.
In light of this, since for any $\varepsilon > 0$ there is $\phi \in Symp(\mathbb{R}^{2n}, \omega_0)$ such that

$$\phi(B^{2n}(r)) \subset \left([0,1] \times [0, \pi r^2 + \varepsilon]\right)^n,$$

we have

$$e(B^{2n}(r), \mathbb{R}^{2n}) \leq \pi r^2.$$

In fact:

Theorem (Hofer)

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Let us say that $H \in C_c^\infty(M, \mathbb{R})$ is *admissible* if the Hamiltonian vector field X_H has no nonconstant periodic orbits of period at most 1.

If $A \subset M$,

$$c_{HZ}(A) = \sup\{\max H | H : M \twoheadrightarrow [0, \max H], suppH \subset A,\ H \text{ is admissible}\}.$$

If $A \subset M$, $B \subset N$, and there is a codimension-zero symplectic embedding $\phi : M \hookrightarrow N$ with $\phi(A) \subset B$, then

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Example

Let

$$A = B^{2n}(r) \subset \mathbb{R}^{2n} = \mathbb{C}^n = \{\vec{x} + i\vec{y}\}.$$

Where $\rho = \sum_{j=1}^{n}(x_j^2 + y_j^2)$, consider H having the special form

$$H(\vec{x} + i\vec{y}) = f(\rho).$$

Then

$$X_H(\vec{v}) = -2f'(\rho)i\vec{v};$$

hence the sphere $\{\rho = \rho_0\}$ is filled by periodic orbits of minimal period $\frac{\pi}{|f'(\rho_0)|}$.
So given $c > 0$, we can construct an admissible such H with $\max H = c$ supported in $B^{2n}(r)$ iff $c < \pi r^2$. This implies that $c_{HZ}(B^{2n}(r)) \geq \pi r^2$.

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The inequality

We've seen the elementary facts

$$e(B^{2n}(r),\mathbb{R}^{2n}) \leq \pi r^2, \quad c_{HZ}(B^{2n}(r)) \geq \pi r^2.$$

Theorem (Hofer–Zehnder 1994 for $M = \mathbb{R}^{2n}$, U. 2008 for M closed or Stein)

For any $A \subset M$ we have

$$c_{HZ}(A) \leq e(A,M).$$

This implies most of the theorems we've discussed earlier. In particular, if $\phi \in Ham^c(M,\omega)$ is unequal to the identity, then there is a symplectomorphic copy B of a ball $B^{2n}(r)$ such that $\phi(B) \cap B = \varnothing$; hence

$$\|\phi\| \geq e(B,M) \geq c_{HZ}(B) \geq \pi r^2 > 0.$$

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For the generalized non-squeezing theorem (saying that if P, M are closed then $P \times B^{2n-2p}(r)$ symplectically embeds in $M \times B^{2n-2m}(R)$ only when $r \leq R$), note that in general, for P closed,

$$c_{HZ}(P \times A) \geq c_{HZ}(A),$$

and for any M, if $B \subset N$,

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So if the embedding exists,

$$\begin{aligned} \pi r^2 &\leq c_{HZ}(P \times B^{2n-2p}(r)) \leq c_{HZ}(M \times B^{2n-2p}(R)) \\ &\leq e(M \times B^{2n-2m}(R), M \times \mathbb{R}^{2n-2m}) \leq e(B^{2n-2m}(R), \mathbb{R}^{2n-2m}) \\ &= \pi R^2. \end{aligned}$$

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To a (nondegenerate) $H : (\mathbb{R}/\mathbb{Z}) \times M \to \mathbb{R}$ one associates a chain complex

$$CF_*(H) = \widehat{\bigoplus}_{p \in Crit(\mathscr{A}_H)} \mathbb{Z}\langle p \rangle$$

with differential counting formal negative gradient flowlines of a certain function $\mathscr{A}_H : \widetilde{\mathscr{L}_0 M} \to \mathbb{R}$.

For distinct H, K, there is a chain homotopy equivalence $\Psi_{HK} : CF_*(H) \to CF_*(K)$; the induced map on homology fits into a commutative diagram of isomorphisms

$$\begin{array}{ccccc} & & H_*(M) \otimes \Lambda & & \\ & \swarrow^{\Phi_H} & & \searrow^{\Phi_K} & \\ HF_*(H) & & \xrightarrow[\Psi_{HK}]{} & & HF_*(K) \end{array}$$

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Although the isomorphism $\Phi_H : H_*(M) \otimes \Lambda \to HF_*(H)$ shows $HF_*(H)$ is independent of H, one can obtain H-dependent information from Floer homology by considering a natural *filtration* on $CF_*(H)$:

If $a \in \mathbb{R}$, let

$$CF_*^a(H) = \widehat{\bigoplus}_{p \in Crit(\mathscr{A}_H), \mathscr{A}_H(p) \leq a} \mathbb{Z}\langle p \rangle$$

For $c \in CF_*(H)$, set

$$\ell(c) = \inf\{a | c \in CF_*^a(H)\}.$$

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Some properties of ρ:

- $|\rho(H)-\rho(K)| \leq \|H-K\|$; hence ρ extends to a continuous function $\rho : C^0((\mathbb{R}/\mathbb{Z})\times M) \to \mathbb{R}$.
- Where $\overline{H}(t,m) = -H(t,\phi_H^t(m))$ generates the path $(\phi_H^t)^{-1}$,
$$0 \leq \rho(H)+\rho(\overline{H}) \leq \|H\|.$$
- Where $H\#K(t,m) = H(t,m)+K(t,(\phi_H^t)^{-1}(m))$ generates $\phi_H^t \circ \phi_K^t$,
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The proof that $c_{HZ}(A) \leq e(A, M)$ depends on:

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If $\phi_K^1(\mathrm{supp}H) \cap \mathrm{supp}H = \varnothing$, then

$$\rho(H) \leq \rho(K) + \rho(\bar{K}) (\leq \|K\|).$$

Proposition

If $H\colon M \to \mathbb{R}$ is admissible (i.e. X_H has no nonconstant periodic orbits of period at most 1), then

$$\rho(-H) = \max H.$$

Thus if H is admissible and supported in A, and if K has the property that $\phi_K^1(A) \cap A = \varnothing$, then $\|K\| \geq \max H$. That this holds for all H and K is precisely the statement that $c_{HZ}(A) \leq e(A, M)$.

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