

SYMPLECTICALLY KNOTTED CODIMENSION-ZERO EMBEDDINGS OF DOMAINS IN \mathbb{R}^4

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ABSTRACT. We show that many toric domains X in \mathbb{R}^4 admit symplectic embeddings ϕ into dilates of themselves which are knotted in the strong sense that there is no symplectomorphism of the target that takes $\phi(X)$ to X . For instance X can be taken equal to a polydisk $P(1, 1)$, or to any convex toric domain that both is contained in $P(1, 1)$ and properly contains a ball $B^4(1)$; by contrast a result of McDuff shows that $B^4(1)$ (or indeed any four-dimensional ellipsoid) cannot have this property. The embeddings are constructed based on recent advances on symplectic embeddings of ellipsoids, though in some cases a more elementary construction is possible. The fact that the embeddings are knotted is proven using filtered positive S^1 -equivariant symplectic homology.

1. INTRODUCTION

Recent years have seen a significant improvement in our understanding of when one region in \mathbb{R}^4 symplectically embeds into another, see *e.g.* [M09], [MS12], [C14]. Complementing this existence question, one can ask whether embeddings are unique up to an appropriate notion of equivalence; in particular, if $A \subset U \subset \mathbb{R}^4$ this entails asking whether every symplectic embedding $A \hookrightarrow U$ is equivalent to the inclusion. Somewhat less is known about this uniqueness question, though there are positive results in [M09],[C14] and negative results in [FHW94], [H13]. We show in this paper that modern techniques of constructing symplectic embeddings $B \hookrightarrow U$ often give rise, when restricted to certain subsets $A \subset B \cap U$, to embeddings $A \hookrightarrow U$ that are distinct from the inclusion in a strong sense.

The subsets of \mathbb{R}^4 (and in some cases more generally in $\mathbb{R}^{2n} \cong \mathbb{C}^n$) that we consider are toric domains; let us set up some notation and recall basic definitions.

Define $\mu: \mathbb{C}^n \rightarrow [0, \infty)^n$ by

$$\mu(z_1, \dots, z_n) = (\pi|z_1|^2, \dots, \pi|z_n|^2).$$

A **toric domain** is by definition a set of the form $X_\Omega = \mu^{-1}(\Omega)$ where Ω is a domain in $[0, \infty)^n$. Throughout the paper the term “domain” will always refer to the closure of a bounded open subset of \mathbb{R}^n or \mathbb{C}^n ; in particular domains are by definition compact.

Given $\Omega \subset [0, \infty)^n$, we define

$$(1.1) \quad \widehat{\Omega} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (|x_1|, \dots, |x_n|) \in \Omega\}.$$

Symplectic embedding problems for toric domains are currently best understood when the domains are concave or convex according to the following definitions, which follow [GH17].

Definition 1.1. A **convex toric domain** is a toric domain X_Ω such that $\widehat{\Omega}$ is a convex domain in \mathbb{R}^n .

Definition 1.2. A **concave toric domain** is a toric domain X_Ω where $\Omega \subset [0, \infty)^n$ is a domain and $[0, \infty)^n \setminus \Omega$ is convex.

Example 1.3. If $n = 2$, a convex or concave toric domain X_Ω arises from a “region under a graph” $\Omega = \{(x, y) | 0 \leq x \leq a, 0 \leq y \leq f(x)\}$ where $f: [0, a] \rightarrow [0, \infty)$ is a monotone decreasing function. The corresponding toric domain X_Ω is convex if f is concave, and is concave if f is convex and $f(a) = 0$.

Example 1.4. If $a_1, \dots, a_n > 0$, the *ellipsoid* $E(a_1, \dots, a_n)$ is defined as X_Ω where $\Omega = \{(x_1, \dots, x_n) \in [0, \infty)^n | \sum_{i=1}^n \frac{x_i}{a_i} \leq 1\}$. As a special case, the ball of capacity a is $B^{2n}(a) = E(a, \dots, a)$. Note that $E(a_1, \dots, a_n)$ is both a concave toric domain and a convex toric domain. We will occasionally find it convenient to extend this to the case that some $a_i = 0$ by taking $E(\dots, 0, \dots) = \emptyset$.

Example 1.5. If $a_1, \dots, a_n > 0$, the *polydisk* $P(a_1, \dots, a_n)$ is defined as X_Ω where $\Omega = \{(x_1, \dots, x_n) \in [0, \infty)^n | (\forall i)(0 \leq x_i \leq a_i)\}$. Equivalently, $P(a_1, \dots, a_n) = B^2(a_1) \times \dots \times B^2(a_n)$. Polydisks are convex toric domains.

We use the following standard notational convention:

Definition 1.6. If $A \subset \mathbb{C}^n$ and $\alpha > 0$, we define $\alpha A = \{\sqrt{\alpha}a | a \in A\}$.

(The square root ensures that any capacity c will obey $c(\alpha A) = \alpha c(A)$, and also that we have $E(\alpha a_1, \dots, \alpha a_n) = \alpha E(a_1, \dots, a_n)$ and similarly for polydisks.)

For any subset $B \subset \mathbb{C}^n$ let B° denote the interior of B . This paper is largely concerned with symplectic embeddings $X \hookrightarrow \alpha X^\circ$ where X is a concave or convex toric domain and $\alpha > 1$. The definitions imply that concave or convex toric domains X always satisfy $X \subset \alpha X^\circ$ for all $\alpha > 1$ (see Proposition 2.20), so one such embedding is given by the inclusion of X into αX° . However we will find that in many cases there are other such embeddings that are inequivalent to the inclusion in the following sense:

Definition 1.7. Let $A \subset U \subset \mathbb{C}^n$, with A closed and U open, and let $\phi: A \rightarrow U$ be a symplectic embedding.¹ We say that ϕ is **unknotted** if there is a symplectomorphism $\Psi: U \rightarrow U$ such that $\Psi(A) = \phi(A)$. We say that ϕ is **knotted** if it is not unknotted.

¹Since A may not be a manifold or even a manifold with boundary we should say what it means for $\phi: A \rightarrow U$ to be a symplectic embedding; our convention will be that it means that there is an open neighborhood of A to which ϕ extends as a symplectic embedding. When A is a manifold with boundary it is not hard to see using a relative Moser argument that this is equivalent to the statement that $\phi: A \rightarrow U$ is a smooth embedding of manifolds with boundary which preserves the symplectic form.

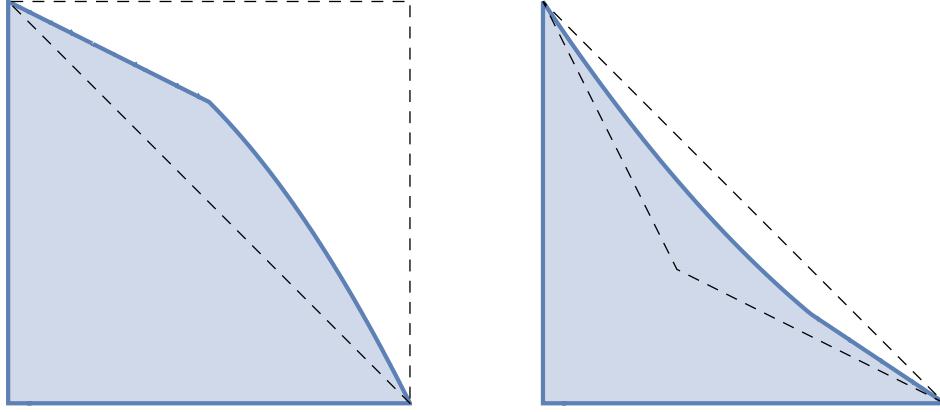


FIGURE 1. The shaded regions are examples of choices of Ω such that Theorem 1.8 gives knotted embeddings $X_\Omega \rightarrow \alpha X_\Omega^\circ$ for suitable $\alpha > 1$. The dashed lines delimit the regions which are assumed to contain $(\partial\Omega) \cap (0, \infty)^2$ in, respectively, Cases (i) and (ii) of the theorem.

Note that we do not require the map Ψ to be compactly supported, or Hamiltonian isotopic to the identity, or even to extend continuously to the closure of U ; accordingly our definition of knottedness is in principle more restrictive than others that one might use.

In Section 1.1 (based on results from Sections 2 and 3) we will prove the existence of knotted embeddings from X to αX° for many toric domains $X \subset \mathbb{C}^2$ and suitable $\alpha > 1$.

Theorem 1.8. Let $X \subset \mathbb{C}^2$ belong to any of the following classes of domains:

- (i) All convex toric domains X such that, for some $c > 0$, $B^4(c) \subsetneq X \subset P(c, c)$.
- (ii) All concave toric domains X_Ω such that, for some $c > 0$, $\{(x, y) \in [0, \infty)^2 \mid \min\{2x+y, x+2y\} \leq c\} \subset \Omega \subsetneq \{(x, y) \in [0, \infty)^2 \mid x+y \leq c\}$.
- (iii) All complex ℓ^p balls $\{(w, z) \in \mathbb{C}^2 \mid |w|^p + |z|^p \leq r^p\}$ for $p > \frac{\log 9}{\log 6} \approx 1.23$, except for $p = 2$.
- (iv) All polydisks $P(a, b)$ for $a \leq b < 2a$.

Then there exist $\alpha > 1$ and a knotted embedding $\phi : X \rightarrow \alpha X^\circ$.

For context, recall that McDuff showed in [M91] that the space of symplectic embeddings from one four-dimensional ball to another is always connected; by the symplectic isotopy extension theorem this implies that symplectic embeddings $B^4(c) \rightarrow \alpha B^4(c)^\circ$ can never be knotted. (In particular the exclusion of $B^4(c)$ from each of the classes (i),(ii),(iii) above is necessary.) McDuff's result was later extended to establish the connectedness of the space of embeddings of one four-dimensional ellipsoid into another [M09] or of a four-dimensional

concave toric domain into a convex toric domain [C14]. So Theorem 1.8 reflects that embeddings from concave toric domains into concave ones, or convex toric domains into convex ones, can behave differently than embeddings from concave toric domains into convex ones.

We do not know whether the bound $b < 2a$ in part (iv) of Theorem 1.8 is sharp. The bound $p > \frac{\log 9}{\log 6}$ in part (iii) is not sharp; we are aware of extensions of our methods that lower this bound slightly, though in the interest of brevity we do not include them. Note that the domains in part (iii) are concave when $p < 2$ and convex when $p > 2$ (in the latter case the result follows directly from part (i)).

While our primary focus in this paper is on domains in \mathbb{R}^4 , we show in Theorem 2.21 that the embeddings from Cases (i) and (iv) of Theorem 1.8 remain knotted after being trivially extended to the product of X_Ω with an ellipsoid of sufficiently large Gromov width. It remains an interesting problem to find knotted embeddings involving broader classes of high-dimensional domains that do not arise from lower-dimensional constructions.

By the way, embeddings such as those in Theorem 1.8 can only be knotted for a limited range of α , since the extension-after-restriction principle [S, Proposition A.1] implies that for any compact set $X \subset \mathbb{C}^n$ which is star-shaped with respect to the origin and contains the origin in its interior and any symplectic embedding $\phi: X \rightarrow \mathbb{C}^n$, there is $\alpha_0 > 1$ such that $\phi(X) \subset \alpha_0 X^\circ$ and such that ϕ is unknotted when considered as a map to αX° for all $\alpha \geq \alpha_0$. The values for α that we find in the proof of Theorem 1.8 vary from case to case, but in each instance lie between 1 and 2. This suggests the:

Question 1.9. Do there exist a domain $X \subset \mathbb{R}^{2n}$, a number $\alpha > 2$, and a knotted symplectic embedding $\phi: X \rightarrow \alpha X^\circ$?

Theorem 1.8 concerns embeddings of a domain X into the interior of a dilate αX° of X ; of course it is also natural to consider embeddings in which the source and target are not simply related by a dilation. Our methods in principle allow for this, though the proofs that the embeddings are knotted become more subtle. In Section 4 we carry this out for embeddings of four-dimensional polydisks into other polydisks, and in particular we prove the following as Corollary 4.7:

Theorem 1.10. Given any $y \geq 1$, there exist polydisks $P(a, b)$ and $P(c, d)$ and knotted embeddings of $P(a, b)$ into $P(1, y)^\circ$ and of $P(1, y)$ into $P(c, d)^\circ$.

Theorem 1.10 and Case (iv) of Theorem 1.8 should be compared to [FHW94, Section 3.3], in which it is shown that, if $a \leq b < c$ but $a + b > c$, then the embeddings $\phi_1, \phi_2: P(a, b) \rightarrow P(c, c)^\circ$ given by $\phi_1(w, z) = (w, z)$ and $\phi_2(w, z) = (z, w)$ are not isotopic through compactly supported symplectomorphisms of $P(c, c)^\circ$. However our embeddings are different from these; in fact the embeddings from [FHW94] are not even knotted in our (rather strong) sense since there is a symplectomorphism of the open polydisk $P(c, c)^\circ$ mapping $P(a, b)$ to $P(b, a)$. If one instead considers embeddings into $P(c, d)$ with $c < d$ chosen such that $P(c, d)^\circ$ contains both $P(a, b)$ and $P(b, a)$ and $a + b > d$,

then $P(a, b)$ and $P(b, a)$ are inequivalent to each other under the symplectomorphism group of $P(c, d)^\circ$. However in situations where this construction and the construction underlying Theorem 1.8 (iv) and Theorem 1.10 both apply, our knotted embeddings represent different knot types than both $P(a, b)$ and $P(b, a)$, see Remark 4.5.

Let us be a bit more specific about how we prove Theorem 1.8; the proof of Theorem 1.10 is conceptually similar. The knotted embeddings $\phi: X \rightarrow \alpha X^\circ$ described in Theorem 1.8 are obtained as compositions of embeddings $X \rightarrow E \rightarrow \alpha X^\circ$ where E is an ellipsoid. In the cases that X is convex, the first map $X \rightarrow E$ is just an inclusion, while the second map $E \rightarrow \alpha X^\circ$ is obtained by using recent developments from [M09],[C14] that ultimately have their roots in Taubes-Seiberg-Witten theory, see Section 3. (For a limited class of convex toric domains X that are close to a cube $P(c, c)$, we provide a much more elementary and explicit construction in Section 3.2.) In the cases that X is concave the reverse is true: $E \rightarrow \alpha X^\circ$ is an inclusion while $X \rightarrow E$ is obtained from these more recent methods. Meanwhile, we use the properties of transfer maps in filtered S^1 -equivariant symplectic homology to obtain a lower bound on possible values α such that there can exist any unknotted embedding $X \rightarrow \alpha X^\circ$ which factors through an ellipsoid E . In each case in Theorem 1.8, we will find compositions $X \rightarrow E \rightarrow \alpha X^\circ$ arising from the constructions in Section 3 for which α is less than this symplectic-homology-derived lower bound, leading to the conclusion that the composition must be knotted. Figure 2 and its caption explain this more concretely in a representative special case.

To carry this out systematically, let us introduce the following two quantities associated to a star-shaped domain $X \subset \mathbb{C}^n$, where the symbol \hookrightarrow always denotes a symplectic embedding:

$$(1.2) \quad \delta_{\text{ell}}(X) = \inf\{\alpha \geq 1 \mid (\exists a_1, \dots, a_n)(X \hookrightarrow E(a_1, \dots, a_n) \hookrightarrow \alpha X^\circ)\}$$

and

$$(1.3) \quad \delta_{\text{ell}}^u(X) = \inf\left\{\alpha \geq 1 \mid \begin{array}{l} (\exists a_1, \dots, a_n, f: X \hookrightarrow E(a_1, \dots, a_n), \\ g: E(a_1, \dots, a_n) \hookrightarrow \alpha X^\circ)(g \circ f \text{ is unknotted.}) \end{array} \right\}$$

(The u in δ_{ell}^u stands for ‘‘unknotted.’’) To put this into a different context, as was suggested to us by Y. Ostrover and L. Polterovich, one can define a pseudometric on the space of star-shaped domains in \mathbb{C}^n by declaring the distance between two domains X and Y to be the logarithm of the infimal $\alpha \in \mathbb{R}$ such that there is a sequence of symplectic embeddings $\alpha^{-1/2}X \hookrightarrow Y \hookrightarrow \alpha^{1/2}X^\circ$; a more refined version of this pseudometric would additionally ask that neither of the resulting compositions $X \rightarrow \alpha X^\circ$ and $Y \rightarrow \alpha Y^\circ$ be knotted. Then (at least if $n = 2$) the logarithm of $\delta_{\text{ell}}(X)$ or of $\delta_{\text{ell}}^u(X)$ is the distance from X to the set of ellipsoids with respect to such a pseudometric. (In the case of δ_{ell}^u this statement depends partly on the result from [M09] that when E is an ellipsoid in \mathbb{R}^4 a symplectic embedding $E \hookrightarrow \alpha E^\circ$ is never knotted.)

We will prove Theorem 1.8 by proving, for each X as in the statement, a strict inequality $\delta_{\text{ell}}(X) < \delta_{\text{ell}}^u(X)$. This entails finding upper bounds for $\delta_{\text{ell}}(X)$

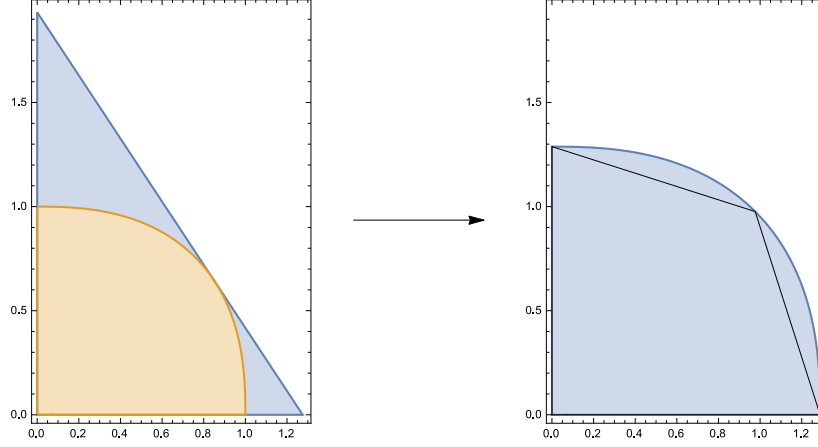


FIGURE 2. The strategy underlying our knotted embedding in the case that X is the ℓ^5 ball of capacity 1, as in Case (i) or (iii) of Theorem 1.8. X is the toric domain associated to the smaller region on the left; the toric domain associated to the triangle on the left is the ellipsoid $E = E((3/2)^{3/5}, 3^{3/5})$, which in particular contains X . The larger region at right is obtained by dilating X by $\alpha = (1 + \varepsilon)(3/2)^{3/5}$ for a small $\varepsilon > 0$, and Proposition 3.5 shows that there is a symplectic embedding $\phi : E \rightarrow \alpha X^\circ$ (in fact, ϕ has image contained in the preimage under μ of the inscribed quadrilateral on the right). Our knotted embedding is $\phi|_X$; Theorem 1.12(a) implies that any unknotted embedding $X \rightarrow \alpha X^\circ$ that extends to a symplectic embedding $E \rightarrow \alpha X^\circ$ would have $\alpha \geq 2^{3/5}$, whereas in this construction α can be taken arbitrarily close to $(3/2)^{3/5}$.

by exhibiting particular compositions of embeddings $X \hookrightarrow E \hookrightarrow \alpha X^\circ$, and finding lower bounds for $\delta_{\text{ell}}^u(X)$ using filtered positive S^1 -equivariant symplectic homology. As it happens, for convex or concave toric domains both our upper bounds and our lower bounds can be conveniently expressed in terms of the following notation:

Notation 1.11. For a domain $\Omega \subset [0, \infty)^n$ we define functions $\|\cdot\|_\Omega^*$ and $[\cdot]_\Omega$ from \mathbb{R}^n to \mathbb{R} as follows:

- For $\vec{a} \in \mathbb{R}^n$, $\|\vec{a}\|_\Omega^* = \sup\{\vec{a} \cdot \vec{v} \mid \vec{v} \in \Omega\}$.
- For $\vec{a} \in \mathbb{R}^n$, $[\vec{a}]_\Omega = \inf\{\vec{a} \cdot \vec{v} \mid \vec{v} \in [0, \infty)^n \setminus \Omega\}$.

The estimates for δ_{ell}^u that are relevant to Theorem 1.8 are given by the following result, proven in Section 2:

Theorem 1.12. (a) If $X_\Omega \subset \mathbb{C}^2$ is a convex toric domain, then

$$\delta_{\text{ell}}^u(X_\Omega) \geq \frac{\|(1, 1)\|_\Omega^*}{\max\{\|(1, 0)\|_\Omega^*, \|(0, 1)\|_\Omega^*\}}.$$

(b) If $X_\Omega \subset \mathbb{C}^2$ is a concave toric domain, then

$$\delta_{\text{ell}}^u(X_\Omega) \geq \frac{\min\{[(2, 1)]_\Omega, [(1, 2)]_\Omega\}}{[(1, 1)]_\Omega}.$$

As for upper bounds on δ_{ell} , in Section 3.1 we prove the following:

Theorem 1.13. (a) Suppose that $\Omega \subset [0, \infty)^2$ is a domain such that $\hat{\Omega}$ is convex and such that Ω contains points $(a, 0), (0, b), (x, y)$ with $0 < x \leq a \leq b \leq x + y$. Then

$$\delta_{\text{ell}}(X_\Omega) \leq \left\| \left(\frac{1}{a}, \frac{1}{x+y} \right) \right\|_\Omega^*.$$

(b) Suppose that $\Omega \subset [0, \infty)^2$ is a domain that contains $(0, 0)$ in its interior and whose complement in $[0, \infty)^2$ is convex, and such that points $(a, 0), (0, b), (x, y)$ with $0 < x + y \leq a \leq b$ all belong to $[0, \infty)^2 \setminus \Omega$. Then

$$\delta_{\text{ell}}(X_\Omega) \leq \frac{1}{\left[\left(\frac{1}{b}, \frac{1}{x+y} \right) \right]_\Omega}.$$

(c) For a polydisk $P(a, b)$ with $a \leq b \leq 2a$ we have

$$\delta_{\text{ell}}(P(a, b)) \leq \left\| \left(\frac{3}{a+b}, \frac{1}{2a+b} \right) \right\|_{[0, a] \times [0, b]}^*.$$

Assuming Theorems 1.12 and 1.13 for the time being, we now show how they lead to Theorem 1.8.

1.1. Proof of Theorem 1.8. In each of the four cases it suffices to prove a strict inequality $\delta_{\text{ell}}(X) < \delta_{\text{ell}}^u(X)$.

First let $X = X_\Omega$ be a convex toric domain with $B^4(c) \subsetneq X_\Omega \subset P(c, c)$. Thus $\Omega \subset [0, c] \times [0, c]$ (since $X_\Omega \subset P(c, c)$), and Ω is a convex region containing the points $(c, 0), (0, c)$, and (due to the strict inclusion $B^4(c) \subsetneq X_\Omega$) some point (x_0, y_0) having $x_0 + y_0 > c$. The fact that $(c, 0), (0, c) \in \Omega \subset [0, c] \times [0, c]$ implies that $\|(1, 0)\|_\Omega^* = \|(0, 1)\|_\Omega^* = c$. Consequently by Theorem 1.12(a),

$$\delta_{\text{ell}}^u(X_\Omega) \geq \frac{1}{c} \|(1, 1)\|_\Omega^*.$$

Meanwhile Theorem 1.13(a) gives

$$\delta_{\text{ell}}(X_\Omega) \leq \left\| \left(\frac{1}{c}, \frac{1}{x_0 + y_0} \right) \right\|_\Omega^* = \frac{1}{c} \left\| \left(1, \frac{c}{x_0 + y_0} \right) \right\|_\Omega^*.$$

So to prove that $\delta_{\text{ell}}(X_\Omega) < \delta_{\text{ell}}^u(X_\Omega)$ it suffices to show that $\|(1, 1)\|_\Omega^* > \|(1, a)\|_\Omega^*$ where $a := \frac{c}{x_0 + y_0} < 1$. Choose $(v_1, v_2) \in \Omega$ such that $v_1 + av_2 = (1, a) \cdot (v_1, v_2) = \|(1, a)\|_\Omega^*$; it suffices to find $(w_1, w_2) \in \Omega$ with $(1, 1) \cdot (w_1, w_2) = w_1 + w_2 > v_1 + av_2$. Bearing in mind that $(v_1, v_2) \in \Omega \subset [0, \infty)^2$ and $a < 1$, if $v_2 \neq 0$

we can simply take $(w_1, w_2) = (v_1, v_2)$. On the other hand if $v_2 = 0$ then since $\Omega \subset [0, c] \times [0, c]$ we have $v_1 + av_2 \leq c$, so taking $(w_1, w_2) = (x_0, y_0)$ gives $w_1 + w_2 > c \geq v_1 + av_2$. So in any case $\|(1, 1)\|_{\Omega}^* \geq w_1 + w_2 > v_1 + av_2 = \|(1, a)\|_{\Omega}^*$, proving that $\delta_{\text{ell}}^u(X_{\Omega}) > \delta_{\text{ell}}(X_{\Omega})$ and thus completing the proof of Case (i) of Theorem 1.8.

Case (ii) is rather similar. The hypothesis implies that all points (x, y) of $[0, \infty)^2 \setminus \Omega$ have $\min\{2x + y, x + 2y\} \geq c$ and so Theorem 1.12(b) yields $\delta_{\text{ell}}^u(X_{\Omega}) \geq \frac{c}{[(1, 1)]_{\Omega}}$. The hypothesis also implies that $[0, \infty)^2 \setminus \Omega$ contains a point (x_0, y_0) with $x_0 + y_0 < c$, and also contains the points $(c, 0)$ and $(0, c)$, so by Theorem 1.13(b) we have

$$\delta_{\text{ell}}(X_{\Omega}) \leq \frac{1}{\left[\left(\frac{1}{c}, \frac{1}{x_0 + y_0}\right)\right]_{\Omega}} = \frac{c}{\left[\left(1, \frac{c}{x_0 + y_0}\right)\right]_{\Omega}}.$$

So to show that $\delta_{\text{ell}}(X_{\Omega}) < \delta_{\text{ell}}^u(X_{\Omega})$ it suffices to show that $[(1, b)]_{\Omega} > [(1, 1)]_{\Omega}$ where $b := \frac{c}{x_0 + y_0} > 1$. This is established in basically the same way as the similar inequality in Case (i): let $(v_1, v_2) \in \overline{[0, \infty)^2 \setminus \Omega}$ minimize $\vec{v} \mapsto (1, b) \cdot \vec{v}$. Then either $v_2 \neq 0$, in which case $(1, 1) \cdot (v_1, v_2) < (1, b) \cdot (v_1, v_2)$, or else $v_2 = 0$, in which case $v_1 = c$ by our assumptions on Ω , and so $[(1, 1)]_{\Omega} \leq x_0 + y_0 < c = (1, b) \cdot (v_1, v_2)$. So in Case (ii) we indeed have $\delta_{\text{ell}}(X_{\Omega}) < \delta_{\text{ell}}^u(X_{\Omega})$.

We now turn to Case (iii) concerning complex ℓ^p balls $X = \{(w, z) \in \mathbb{C}^2 \mid |w|^p + |z|^p \leq r^p\}$. Using appropriate rescalings it suffices to prove the result in the case that $r = \frac{1}{\sqrt{\pi}}$, so that $X = X_{\Omega}$ where $\Omega = \{(x, y) \in [0, \infty)^2 \mid x^{p/2} + y^{p/2} \leq 1\}$. When $p > 2$, X_{Ω} is a convex toric domain contained in $P(1, 1)$ and strictly containing $B^4(1)$, so the result follows from Case (i). From now on assume that $0 < p < 2$, so that X_{Ω} is a concave toric domain. Since $p/2 < 1$, the reverse Hölder inequality (and the fact that it is sharp) implies that for any $(v, w) \in [0, \infty)^2$ we have $[(v, w)]_{\Omega} = (v^q + w^q)^{1/q}$ where $q = \frac{p}{p-2} < 0$. So from Theorem 1.12(b) we obtain

$$\delta_{\text{ell}}^u(X) \geq \frac{(2^q + 1)^{1/q}}{2^{1/q}} = \left(2^{q-1} + \frac{1}{2}\right)^{-1/|q|}.$$

Meanwhile $\overline{[0, \infty)^2 \setminus \Omega}$ contains the points $(0, 1), (1, 0), (2^{-2/p}, 2^{-2/p})$, so Theorem 1.13(b) yields

$$\delta_{\text{ell}}(X) \leq \frac{1}{\left(1 + \left(\frac{1}{2^{1-2/p}}\right)^q\right)^{1/q}} = \left(1 + \frac{1}{2}\right)^{-1/q} = \left(\frac{2}{3}\right)^{-1/|q|}.$$

So we will have $\delta_{\text{ell}}(X) < \delta_{\text{ell}}^u(X)$ provided that $2^{q-1} + \frac{1}{2} < \frac{2}{3}$, where $q = \frac{p}{p-2}$. This condition is equivalent to $2^q < \frac{1}{3}$, i.e., $\frac{p}{2-p} > \frac{\log 3}{\log 2}$, i.e., $p > \frac{\log 9}{\log 6}$.

Turning finally to Case (iv), let $X = P(a, b) = X_{\Omega}$ where $\Omega = [0, a] \times [0, b]$ and we assume that $a \leq b < 2a$. Clearly for $(v, w) \in [0, \infty)^2$ we have $\|(v, w)\|_{\Omega}^* = av + bw$. Hence Theorem 1.12(a) gives

$$\delta_{\text{ell}}^u(P(a, b)) \geq \frac{a + b}{b}$$

while Theorem 1.13(c) gives

$$\delta_{\text{ell}}(P(a, b)) \leq \frac{3a}{a+b} + \frac{b}{2a+b}.$$

In other words, writing $s = \frac{b}{a}$, we have $\delta_{\text{ell}}^u(P(a, b)) \geq 1 + \frac{1}{s}$ and $\delta_{\text{ell}}(P(a, b)) \leq 1 + \frac{4+s}{2+3s+s^2}$. So $\delta_{\text{ell}}(P(a, b)) < \delta_{\text{ell}}^u(P(a, b))$ provided that $4s + s^2 < 2 + 3s + s^2$, *i.e.* provided that $\frac{b}{a} = s < 2$, as we have assumed. \square

1.2. Organization of the paper. The following Section 2 will recall some facts about S^1 -equivariant symplectic homology and extend these using an inverse limit construction to open subsets of \mathbb{R}^{2n} in order to prove Theorem 1.12, which is the key to showing that our embeddings are indeed knotted. The point of the argument, roughly speaking, is that the filtered positive S^1 -equivariant symplectic homology of an ellipsoid E is “as simple as possible” given the total (unfiltered) homology, while that of the domains in Theorem 1.8 has additional features in the form of elements that persist over certain finite action intervals before disappearing in the total homology. The ratios of the endpoints of these intervals are related to the bounds that we prove on the quantity δ_{ell}^u in Theorem 1.12. We also show that our knotted embeddings remain knotted in certain products in Section 2.1.

The embeddings appearing in our main results are constructed in Section 3 using methods derived from Taubes-Seiberg-Witten theory in work of McDuff [M09] and Cristofaro-Gardiner [C14]. While these sophisticated methods seem to be necessary to obtain results as broad as Theorems 1.8 and 1.10, we show in Section 3.2 that for certain domains that are close to a cube the embeddings can be obtained by much more elementary methods, leading to explicit formulas which we provide. Section 4 extends the work in the rest of the paper to obtain the knotted polydisks from Theorem 1.10.

The appendix contains a proof of a lemma concerning filtered positive S^1 -equivariant symplectic homology, showing that it can be identified as the filtered homology of a certain filtered complex generated by good Reeb orbits. This lemma probably will not surprise experts (in particular it was anticipated in [GH17, Remark 3.2]), and is similar to [GG16, Proposition 3.3], but we have not seen full details of a proof of a result as sharp as this one elsewhere.

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2. OBSTRUCTIONS TO UNKNOTTEDNESS FROM FILTERED POSITIVE S^1 -EQUIVARIANT SYMPLECTIC HOMOLOGY

The goal of this section is to prove Theorem 1.12, which gives lower bounds on the quantity $\delta_{\text{ell}}^{\text{u}}$ defined in (1.3). The main tool for proving this theorem is the positive S^1 -equivariant symplectic homology which was introduced by Viterbo [Vit99] and developed by Bourgeois and Oancea [BO16, BO13a, BO13b, BO10]. We refer to [BO16, BO13a, G15, GH17] for a precise definition, but describe here some of the key features.

Let (X, λ) be a Liouville domain, so that X is a compact smooth manifold with boundary and $\lambda \in \Omega^1(X)$ has the properties that $d\lambda$ is non-degenerate and that $\lambda|_{\partial X}$ is a contact form. We say that (X, λ) is non-degenerate if the linearized return map of the Reeb flow at each closed Reeb orbit on ∂X , acting on the contact hyperplane $\ker \lambda$, does not have 1 as an eigenvalue. We will also assume that the first Chern class of TX vanishes on $\pi_2(X)$.

In this situation, as in [GH17], for each $L \in \mathbb{R}$ we have an L -filtered positive S^1 -equivariant symplectic homology, denoted by $CH^L(X, \lambda)$; these are \mathbb{Q} -vector spaces that come equipped with maps $\iota_{L_1, L_2}: CH^{L_1}(X, \lambda) \rightarrow CH^{L_2}(X, \lambda)$ for $L_1 \leq L_2$ such that $\iota_{L, L}$ is the identity and $\iota_{L_2, L_3} \circ \iota_{L_1, L_2} = \iota_{L_1, L_3}$.² The assumption on $c_1(TX)$ implies that the $CH^L(X, \lambda)$ are \mathbb{Z} -graded. The (unfiltered) positive S^1 -equivariant symplectic homology of (X, λ) is $CH(X, \lambda) = \varinjlim_L CH^L(X, \lambda)$ where the direct limit is constructed using the maps ι_{L_1, L_2} .

The analysis of the spaces $CH^L(X, \lambda)$ is significantly simplified by the following, which is proven in the appendix. A slightly weaker version for a different version of S^1 -equivariant symplectic homology is given in [GG16, Proposition 3.3].

Lemma 2.1. Assume as above that (X, λ) is a non-degenerate Liouville domain with $c_1(TX)|_{\pi_2(X)} = 0$. There is an \mathbb{R} -filtered chain complex $(CC_*(X, \lambda), \partial)$, freely generated over \mathbb{Q} by the good³ Reeb orbits of $\lambda|_{\partial X}$ with the generator corresponding to a Reeb orbit γ having filtration level equal to the action $\int_{\gamma} \lambda$ and grading equal to the Conley-Zehnder index of γ , such that for each $k \in \mathbb{Z}$ and $L \in \mathbb{R}$ the space $CH_k^L(X, \lambda)$ is isomorphic to the k th homology of the subcomplex $CC_*^L(X, \lambda)$ of $CC_*(X, \lambda)$ consisting of elements with filtration level at most L , and such that for $L_1 \leq L_2$ the image of the map $\iota_{L_1, L_2}: CH_k^{L_1}(X, \lambda) \rightarrow CH_k^{L_2}(X, \lambda)$ is isomorphic to the image of the inclusion-induced map $H_k(CC_*^{L_1}(X, \lambda)) \rightarrow H_k(CC_*^{L_2}(X, \lambda))$.

Moreover, the boundary operator ∂ on $CC_*(X, \lambda)$ strictly decreases filtration, in the sense that if $x \in CC_*^L(X, \lambda)$ then there is $\varepsilon > 0$ such that $\partial x \in CC_*^{L-\varepsilon}(X, \lambda)$.

²Warning: In [GH17] the map that we denote by ι_{L_1, L_2} is denoted by ι_{L_2, L_1} .

³Recall that a Reeb orbit γ is bad if it is an even degree multiple cover of another Reeb orbit γ' such that the Conley-Zehnder indices of γ and γ' have opposite parity. Otherwise, γ is good.

Definition 2.2. A **tame domain** in \mathbb{R}^{2n} is a $2n$ -dimensional Liouville domain (X, λ) where:

- X is a compact submanifold with boundary of \mathbb{R}^{2n} ;
- $d\lambda = \omega_0$, where $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ is the (restriction of the) standard symplectic form on \mathbb{R}^{2n} ; and
- the Reeb flow of $\lambda|_{\partial X}$ is non-degenerate.

A **tame star-shaped domain** is a subset $X \subset \mathbb{R}^{2n}$ such that $(X, \lambda_0|_X)$ is a tame domain, where

$$\lambda_0 = \frac{1}{2} \sum_i (x_i dy_i - y_i dx_i).$$

Said differently, a tame star-shaped domain is a smooth star-shaped domain such that the radial vector field on \mathbb{R}^{2n} is transverse to the boundary, and such that the characteristic flow on the boundary is non-degenerate.

Remark 2.3. If $U \subset \mathbb{R}^{2n}$ is open and $\lambda \in \Omega^1(U)$ with $d\lambda = \omega_0$, and if $X \subset U$ has the property that $(X, \lambda|_X)$ is a tame domain, we will typically write $CH^L(X, \lambda)$ instead of $CH^L(X, \lambda|_X)$. It should be noted however that $CH^L(X, \lambda)$ depends only on the restriction of λ to X . In fact, more specifically, given that we always assume that $d\lambda = \omega_0$ the only dependence of $CH^L(X, \lambda)$ on λ (as opposed to $d\lambda$) arises from the germ of $\lambda|_X$ along ∂X ; this feature is part of what allows for the construction of transfer maps associated to generalized Liouville embeddings in [GH17].

Let (X, λ) and (X', λ') be two non-degenerate Liouville domains. If $\phi : X \hookrightarrow (X')^\circ$ is a symplectic embedding with the property that $(\phi^* \lambda' - \lambda)|_{\partial X}$ is exact⁴, then for all $L \in \mathbb{R}$, there exists a map

$$\Phi_\phi^L : CH^L(X', \lambda') \longrightarrow CH^L(X, \lambda)$$

called the transfer map. This map is defined in [GH17, Section 8.1]. If $X \subset (X')^\circ$, we will simply write Φ^L for the transfer map induced by the inclusion of X into X' .

Such a transfer map Φ_ϕ^L also exists in the case that, instead of being a generalized Liouville embedding into the interior of X' , $\phi : X \rightarrow X'$ is simply an isomorphism of Liouville domains (i.e. ϕ is a diffeomorphism with $\phi^* \lambda' = \lambda$). In this case Φ_ϕ^L is an isomorphism.

The transfer map is functorial in the sense that if (X_1, λ_1) , (X_2, λ_2) , and (X_3, λ_3) are tame domains and if $\phi : X_1 \hookrightarrow X_2$ and $\psi : X_2 \hookrightarrow X_3$ are either generalized Liouville embeddings or isomorphisms of Liouville domains, then the following diagram is commutative:

$$(2.1) \quad \begin{array}{ccccc} CH^L(X_3, \lambda_3) & \xrightarrow{\Phi_\psi^L} & CH^L(X_2, \lambda_2) & \xrightarrow{\Phi_\phi^L} & CH^L(X_1, \lambda_1) \\ & & & \searrow \Phi_{\psi \circ \phi}^L & \nearrow \end{array}$$

⁴Such embeddings in general are called “generalized Liouville embeddings” of X into X' .

(This is proven in the unfiltered context for Liouville embeddings in [G15, Theorem 4.12], and the same argument proves the result in our more general situation.)

Recall that a tame star-shaped domain W by definition has the property that (W, λ_0) is a non-degenerate Liouville domain, where λ_0 is the standard Liouville primitive $\frac{1}{2} \sum_i (x_i dy_i - y_i dx_i)$, so in this case we obtain graded vector spaces $CH^L(W, \lambda_0)$. In this case, for any $\zeta > 0$, the scaled domain $\zeta W = \{\sqrt{\zeta} \vec{x} \mid \vec{x} \in W\}$ is likewise a tame domain with respect to λ_0 . By pulling back the ingredients in the construction of $CH^L(W, \lambda_0)$ by appropriate rescalings, we obtain an identification of $CH^L(W, \lambda_0)$ with $CH^{\zeta L}(\zeta W, \lambda_0)$ (on the level of the Reeb orbits that generate the complex $CC_*(W, \lambda_0)$, this sends an orbit $\gamma: S^1 \rightarrow \mathbb{R}^{2n}$ to the orbit $\sqrt{\zeta} \gamma$, which has the effect of multiplying the action by ζ). We call this isomorphism $CH^L(W, \lambda_0) \cong CH^{\zeta L}(\zeta W, \lambda_0)$ the ‘‘rescaling isomorphism.’’ The following gives useful relations between this rescaling isomorphism and the other maps in the theory.

Lemma 2.4. Let W be a tame star-shaped domain, $\zeta > 1$, and $0 < s < t$. Then the diagrams

$$(2.2) \quad \begin{array}{ccc} CH^{\zeta^{-1}t}(W, \lambda_0) & \xrightarrow{I_{\zeta^{-1}t,t}} & CH^t(W, \lambda_0) \\ \cong \downarrow \text{rescaling} & \nearrow \Phi^t & \\ CH^t(\zeta W, \lambda_0) & & \end{array}$$

and

$$(2.3) \quad \begin{array}{ccc} CH^s(W, \lambda_0) & \xrightarrow{I_{s,t}} & CH^t(W, \lambda_0) \\ \cong \uparrow \text{rescaling} & & \text{rescaling} \downarrow \cong \\ CH^{\zeta s}(\zeta W, \lambda_0) & \xrightarrow{I_{\zeta s, \zeta t}} & CH^{\zeta t}(\zeta W, \lambda_0) \end{array}$$

are both commutative.

Proof. The commutativity of (2.3) follows by conjugating the various ingredients involved in the construction of CH by rescalings, see [G15, Lemma 4.15]. The commutativity of (2.2) follows from the description of the transfer morphism $\Phi^t: CH^t(\zeta W, \lambda_0) \rightarrow CH^t(W, \lambda_0)$ in [G15, Lemma 4.16]; indeed it is shown there that the chain map which induces Φ^t on filtered homology can be chosen to be the one that sends an orbit near the boundary of ζW to its image under the rescaling $\zeta W \rightarrow W$, and this correspondence multiplies actions by ζ^{-1} . \square

Lemma 2.5. Let X be a tame star-shaped domain. Let $b \geq a > 0$. Then the following diagram is commutative:

$$\begin{array}{ccc} CH^L(bX, \lambda_0) & \xrightarrow{\Phi^L} & CH^L(aX, \lambda_0) \\ \simeq \downarrow \text{rescaling} & & \text{rescaling} \downarrow \simeq \\ CH^{b^{-1}L}(X, \lambda_0) & \xrightarrow{i_{b^{-1}L, a^{-1}L}} & CH^{a^{-1}L}(X, \lambda_0) \end{array} .$$

Proof. Consider the diagram

$$\begin{array}{ccccc} CH^L(bX, \lambda_0) & \xrightarrow{\Phi^L} & CH^L(aX, \lambda_0) & & \\ & \searrow \cong & & \nearrow i_{\frac{a}{b}L, L} & \\ & & CH^{\frac{a}{b}L}(aX, \lambda_0) & & \\ & \nearrow \cong & & \searrow \cong & \\ CH^{b^{-1}L}(X, \lambda_0) & \xrightarrow{i_{b^{-1}L, a^{-1}L}} & CH^{a^{-1}L}(X, \lambda_0) & & \end{array}$$

where all of the indicated isomorphisms are given by rescaling. The left triangle commutes trivially, the upper triangle commutes as a special case of (2.2), and the lower right quadrilateral commutes as a special case of (2.3). Hence the entire diagram commutes, which implies the result since the left map is an isomorphism. \square

Lemma 2.6. Let $X, X' \subset \mathbb{R}^{2n}$ and $\lambda \in \Omega^1(X')$ be such that $X \subset (X')^\circ$ and both $(X, \lambda|_X)$ and $(X', \lambda|_{X'})$ are tame domains, and let Ψ be a symplectomorphism between open subsets of \mathbb{R}^{2n} whose domain contains X' . Then the following diagram is commutative:

$$(2.4) \quad \begin{array}{ccc} CH^L(X', \lambda) & \xrightarrow{\Phi^L} & CH^L(X, \lambda) \\ \simeq \downarrow \Phi_{\Psi^{-1}}^L & & \Phi_{\Psi^{-1}}^L \downarrow \simeq \\ CH^L(\Psi(X'), \Psi^{-1*}\lambda) & \xrightarrow{\Phi^L} & CH^L(\Psi(X), \Psi^{-1*}\lambda) \end{array}$$

Proof. This is a direct consequence of the functoriality (2.1): writing $i: X \rightarrow X'$ and $j: \Psi(X) \rightarrow \Psi(X')$ for the inclusions, we have a commutative diagram

$$\begin{array}{ccc} \Psi(X) & \xrightarrow{j} & \Psi(X') \\ \downarrow \Psi^{-1} & & \downarrow \Psi^{-1} \\ X & \xrightarrow{i} & X' \end{array}$$

and (2.4) is obtained by taking transfer maps. \square

In proving Theorem 1.12 it will be helpful to know that the image of the map ι_{L_1, L_2} is not too small in certain situations. The following two lemmas give our first results in this direction.

Lemma 2.7. Let X_Ω be a convex toric domain in \mathbb{C}^2 . Then for any $\delta, \varepsilon > 0$ there is a tame star-shaped domain $X_\Omega^{\delta, \varepsilon}$ such that $(1 - \varepsilon)X_\Omega \subset X_\Omega^{\delta, \varepsilon} \subset X_\Omega^\circ$ and such that, for any L_1, L_2 with

$$\max\{\|(1, 0)\|_\Omega^*, \|(0, 1)\|_\Omega^*\} + \delta \leq L_1 < L_2 \leq \|(1, 1)\|_\Omega^* - \delta,$$

the map

$$\iota_{L_1, L_2} : CH_3^{L_1}(X_\Omega^{\delta, \varepsilon}, \lambda_0) \longrightarrow CH_3^{L_2}(X_\Omega^{\delta, \varepsilon}, \lambda_0)$$

is an isomorphism of two-dimensional vector spaces.

Proof. The constructions in steps 1, 2, and 3 of [GH17, Proof of Lemma 2.5] use a Morse-Bott perturbation of a suitable smoothing of X_Ω to obtain a tame star-shaped domain $X_\Omega^{\delta, \varepsilon}$ that can be arranged to have the properties that $(1 - \varepsilon)X_\Omega \subset X_\Omega^{\delta, \varepsilon} \subset X_\Omega^\circ$ and such that the Reeb orbits of $\lambda_0|_{\partial X_\Omega^{\delta, \varepsilon}}$ having action at most $\|(1, 1)\|_\Omega^*$ and Conley-Zehnder index at most 4 consist of:

- no orbits of index 2;
- two orbits of index 3, with actions in the intervals $(\|(1, 0)\|_\Omega^* - \delta, \|(1, 0)\|_\Omega^* + \delta)$ and $(\|(0, 1)\|_\Omega^* - \delta, \|(0, 1)\|_\Omega^* + \delta)$, respectively; and
- at most one orbit of index 4, with action greater than $\|(1, 1)\|_\Omega^* - \delta$.

So letting $CC_*^L(X_\Omega^{\delta, \varepsilon}, \lambda_0)$ be as in Lemma 2.1 (so that in particular $CH_k^L(X_\Omega^{\delta, \varepsilon}, \lambda_0) \cong H_k(CC_*^L(X_\Omega^{\delta, \varepsilon}, \lambda_0), \partial)$), for any L in $[\max\{\|(1, 0)\|_\Omega^*, \|(0, 1)\|_\Omega^*\} + \delta, \|(1, 1)\|_\Omega^* - \delta]$ we have $CC_2^L(X_\Omega^{\delta, \varepsilon}, \lambda_0) = CC_4^L(X_\Omega^{\delta, \varepsilon}, \lambda_0) = \{0\}$ and $CC_3^L(X_\Omega^{\delta, \varepsilon}, \lambda_0) \cong \mathbb{Q}^2$, and moreover if L_1, L_2 both lie in this interval with $L_1 < L_2$ then the inclusion of complexes $CC_3^{L_1}(X_\Omega^{\delta, \varepsilon}, \lambda_0) \rightarrow CC_3^{L_2}(X_\Omega^{\delta, \varepsilon}, \lambda_0)$ is an isomorphism. So passing to homology shows that, for $\max\{\|(1, 0)\|_\Omega^*, \|(0, 1)\|_\Omega^*\} + \delta \leq L_1 < L_2 \leq \|(1, 1)\|_\Omega^* - \delta$, the inclusion-induced map $\iota_{L_1, L_2} : CH_3^{L_1}(X_\Omega^{\delta, \varepsilon}, \lambda_0) \rightarrow CH_3^{L_2}(X_\Omega^{\delta, \varepsilon}, \lambda_0)$ is an isomorphism of two-dimensional vector spaces. \square

Lemma 2.8. Let X_Ω be a concave toric domain in \mathbb{C}^2 . Then for any $\delta, \varepsilon > 0$ there is a tame star-shaped domain $X_\Omega^{\delta, \varepsilon}$ such that $(1 - \varepsilon)X_\Omega \subset X_\Omega^{\delta, \varepsilon} \subset X_\Omega^\circ$ and such that, if

$$[(1, 1)]_\Omega + \delta \leq L_1 < L_2 \leq \min\{[(1, 2)]_\Omega, [(2, 1)]_\Omega\} - \delta,$$

the map

$$\iota_{L_1, L_2} : CH_4^{L_1}(X_\Omega^{\delta, \varepsilon}, \lambda_0) \longrightarrow CH_4^{L_2}(X_\Omega^{\delta, \varepsilon}, \lambda_0)$$

is an isomorphism of one-dimensional vector spaces.

Proof. We argue analogously to the proof of Lemma 2.7. By [GH17, Proof of Lemma 2.7], there is a tame star-shaped domain $X_\Omega^{\delta, \varepsilon}$ such that $(1 - \varepsilon)X_\Omega \subset X_\Omega^{\delta, \varepsilon} \subset X_\Omega^\circ$ and such that the part of $CC_*(X_\Omega^{\delta, \varepsilon}, \lambda_0)$ of filtration level at most $\max\{[(1, 2)]_\Omega, [(2, 1)]_\Omega\}$ and degree at most five is generated by:

- one generator, denoted $a_{1,1}$, in degree 3, with filtration level in the interval $([(1,1)]_\Omega - \delta, [(1,1)]_\Omega + \delta)$;
- one generator, denoted $b_{1,1}$, in degree 4, with filtration level in the interval $([(1,1)]_\Omega - \delta, [(1,1)]_\Omega + \delta)$; and
- at most two generators $c_{1,2}$ and $c_{2,1}$ in degree 5, with respective filtration levels in the intervals $([(1,2)]_\Omega - \delta, [(1,2)]_\Omega + \delta)$ and $([(2,1)]_\Omega - \delta, [(2,1)]_\Omega + \delta)$.

Moreover it is a standard fact (see *e.g.* [GH17, Proposition 3.1]) that the full degree-3 homology $CH_3(X_\Omega^{\delta,\varepsilon}, \lambda_0)$ of this complex is isomorphic to \mathbb{Q} ; indeed this statement holds for arbitrary tame star-shaped domains in \mathbb{R}^4 . Also, [GH17, Theorem 1.14] shows that a generator for $CH_3(X_\Omega^{\delta,\varepsilon}, \lambda_0)$ is represented by a chain having filtration level at most $[(1,1)]_\Omega$. So since the generator $a_{1,1}$ spans the part of $CC_3(X_\Omega^{\delta,\varepsilon}, \lambda_0)$ with filtration level at most $\max\{[(1,2)]_\Omega, [(2,1)]_\Omega\}$ (which is greater than $[(1,1)]_\Omega$), it follows that $a_{1,1}$ must not be in the image of the boundary operator ∂ . Since the boundary operator preserves the filtration, we must then have $\partial b_{1,1} = 0$.

Thus for $[(1,1)]_\Omega + \delta \leq L \leq \min\{[(1,2)]_\Omega, [(2,1)]_\Omega\} - \delta$, the element $b_{1,1}$ is a degree-four cycle in the subcomplex $CC_*^L(X_\Omega^{\delta,\varepsilon}, \lambda_0)$, which is not a boundary for the trivial reason that, for this range of L , $CC_5^L(X_\Omega^{\delta,\varepsilon}, \lambda_0) = \{0\}$. Thus $b_{1,1}$ descends to homology to generate the one-dimensional vector space $CH_4^L(X_\Omega^{\delta,\varepsilon}, \lambda_0)$ for any such L , and the map $\iota_{L_1, L_2}: CH_4^{L_1}(X_\Omega^{\delta,\varepsilon}, \lambda_0) \rightarrow CH_4^{L_2}(X_\Omega^{\delta,\varepsilon}, \lambda_0)$ is an isomorphism whenever $[(1,1)]_\Omega + \delta \leq L_1 < L_2 \leq \min\{[(1,2)]_\Omega, [(2,1)]_\Omega\} - \delta$. \square

We are now going to extend the definition of CH^L to open subsets of \mathbb{R}^{2n} . This is part of what makes it possible to prove knottedness in the strong sense of Definition 1.7, which considers arbitrary symplectomorphisms of the open set that serves as the codomain for the embedding. Working with open sets also allows us to consider domains with poorly-behaved boundaries, to which the standard definition of CH^L does not apply.

We continue to denote by ω_0 the standard symplectic form $\sum_{i=1}^n dx_i \wedge dy_i$ on open subsets of \mathbb{R}^{2n} .

Definition 2.9. Let U be an open subset of \mathbb{R}^{2n} and let $\lambda \in \Omega^1(U)$ be such that $d\lambda = \omega_0$. We define the positive S^1 -equivariant symplectic homology of (U, λ) as

$$(2.5) \quad CH^L(U, \lambda) := \varprojlim_{\substack{(X, \lambda|_X) \text{ tame domain} \\ X \subset U}} CH^L(X, \lambda).$$

Here the inverse limit is taken over transfer maps Φ^L associated to inclusions.

Given open sets $U \subset V \subset \mathbb{R}^{2n}$ and $\lambda \in \Omega^1(V)$ with $d\lambda = \omega_0$, we define a transfer map $\Phi^L: CH^L(V, \lambda) \rightarrow CH^L(U, \lambda)$ as the inverse limit of transfer maps $\Phi^L: CH^L(Y, \lambda) \rightarrow CH^L(X, \lambda)$ as X, Y vary through sets such that $(X, \lambda), (Y, \lambda)$ are both tame with $X \subset U \cap Y^\circ$ and $Y \subset V$. This construction will be extended

to certain other symplectic embeddings of open subsets (not just inclusions) in Lemma 2.18.

Lemma 2.10. If X is a tame star-shaped domain and if L is not the action of any periodic Reeb orbit on ∂X then the natural map $CH^L(X, \lambda_0) \rightarrow CH^L(X^\circ, \lambda_0)$ is an isomorphism.

Proof. The system of tame star-shaped domains $\{(1 - \varepsilon)X \mid \varepsilon > 0\}$ is cofinal in the system of all tame star-shaped domains Y with $Y \subset X^\circ$, so there is a natural isomorphism

$$CH^L(X^\circ, \lambda_0) \cong \varprojlim_{\varepsilon > 0} CH^L((1 - \varepsilon)X, \lambda_0).$$

Lemma 2.5 then induces a natural isomorphism

$$\varprojlim_{\varepsilon > 0} CH^L((1 - \varepsilon)X, \lambda_0) \cong \varprojlim_{\varepsilon > 0} CH^{(1 - \varepsilon)^{-1}L}(X, \lambda_0)$$

where the inverse limit on the right is constructed from the maps $\iota_{s,t}$ that are identified by Lemma 2.1 with the maps induced by inclusions of subcomplexes $CC_*^s \hookrightarrow CC_*^t$. Since L is not the action of any periodic Reeb orbit on ∂X , it follows from Lemma 2.1 that the map $\iota_{L, (1 - \varepsilon)^{-1}L} : CH^L(X, \lambda_0) \rightarrow CH^{(1 - \varepsilon)^{-1}L}(X, \lambda_0)$ is an isomorphism for all sufficiently small ε , from which the lemma immediately follows. \square

Let U be an open subset of \mathbb{C}^n and $\lambda \in \Omega^1(U)$ with $d\lambda = \omega_0$, and let $L_1 < L_2 \in \mathbb{R}$. We define the map $\iota_{L_1, L_2} : CH^{L_1}(U, \lambda) \rightarrow CH^{L_2}(U, \lambda)$ as the inverse limit of the maps $\iota_{L_1, L_2} : CH^{L_1}(X_U, \lambda|_{X_U}) \rightarrow CH^{L_2}(X_U, \lambda|_{X_U})$ where $(X_U, \lambda|_{X_U})$ is a tame domain, $X_U \subset U$.

Since the inverse limit is a functor from the category of diagrams of abelian groups to the category of abelian groups (see [Wei94, Application 2.6.7]), we have a similar statement to Lemma 2.4:

Lemma 2.11. Let U be an open set in \mathbb{R}^{2n} , let $\zeta > 1$, and let $\lambda \in \Omega^1(U)$ with $d\lambda = \omega_0$. Then the following diagram is commutative:

$$(2.6) \quad \begin{array}{ccc} CH^{\zeta^{-1}L}(U, \lambda) & \xrightarrow{\iota_{\zeta^{-1}L, L}} & CH^L(U, \lambda) \\ \text{rescaling} \downarrow & \nearrow \Phi^L & \\ CH^L(\zeta U, \lambda) & & \end{array}$$

The following calculation related to the maps ι_{L_1, L_2} will be very helpful.

Lemma 2.12. Let X_Ω be a convex toric domain in \mathbb{C}^2 .

- (i) If $\max\{\|(1, 0)\|_\Omega^*, \|(0, 1)\|_\Omega^*\} < L < \|(1, 1)\|_\Omega^*$, then $CH_3^L(X_\Omega^\circ, \lambda_0)$ is a two-dimensional vector space.
- (ii) If $\max\{\|(1, 0)\|_\Omega^*, \|(0, 1)\|_\Omega^*\} < L_1 < L_2 < \|(1, 1)\|_\Omega^*$, then $\iota_{L_1, L_2} : CH_3^{L_1}(X_\Omega^\circ, \lambda_0) \rightarrow CH_3^{L_2}(X_\Omega^\circ, \lambda_0)$ is an isomorphism.

Proof. Choose $\delta > 0$ such that

$$\max\{\|(1, 0)\|_{\Omega}^*, \|(0, 1)\|_{\Omega}^*\} + 2\delta < L, L_1, L_2 < \|(1, 1)\|_{\Omega}^* - 2\delta.$$

For this fixed value of δ and varying $\varepsilon > 0$, the non-degenerate domains $X_{\Omega}^{\delta, \varepsilon}$ from Lemma 2.7 form a cofinal system in the inverse system defining $CH^L(X_{\omega}^{\circ}, \lambda_0)$.

Choose a sequence $\varepsilon_m \searrow 0$ such that $X_{\Omega}^{\delta, \varepsilon_m} \subset (X_{\Omega}^{\delta, \varepsilon_{m+1}})^{\circ}$ for each m , so we have transfer maps $(\Phi^L)_m: CH_3^L(X_{\Omega}^{\delta, \varepsilon_{m+1}}, \lambda_0) \rightarrow CH_3^L(X_{\Omega}^{\delta, \varepsilon_m}, \lambda_0)$; this gives a cofinal subsystem within our inverse system. We claim that these transfer maps are isomorphisms of two-dimensional vector spaces once m is sufficiently large (and hence ε_m is sufficiently small).

To prove this, we first note that the domain and codomain both have dimension two by Lemma 2.7, so it is enough to show that $(\Phi^L)_m$ is injective for all large m . But we have inclusions

$$(1 - \varepsilon_m)X_{\Omega}^{\delta, \varepsilon_{m+1}} \subset (1 - \varepsilon_m)X_{\Omega}^{\circ} \subset (X_{\Omega}^{\delta, \varepsilon_m})^{\circ} \subset X_{\Omega}^{\delta, \varepsilon_m} \subset (X_{\Omega}^{\delta, \varepsilon_{m+1}})^{\circ}$$

and so the transfer map $(\Phi^L)_m$ fits into a sequence of transfer maps (2.7)

$$CH_3^L(X_{\Omega}^{\delta, \varepsilon_{m+1}}, \lambda_0) \xrightarrow{(\Phi^L)_m} CH_3^L(X_{\Omega}^{\delta, \varepsilon_m}, \lambda_0) \longrightarrow CH_3^L((1 - \varepsilon_m)X_{\Omega}^{\delta, \varepsilon_{m+1}}, \lambda_0)$$

whose composition is identified up to isomorphism by Lemma 2.4 with the inclusion-induced map

$$\iota_{L, (1 - \varepsilon_m)^{-1}L}: CH_3^L(X_{\Omega}^{\delta, \varepsilon_{m+1}}, \lambda_0) \rightarrow CH_3^{(1 - \varepsilon_m)^{-1}L}(X_{\Omega}^{\delta, \varepsilon_{m+1}}, \lambda_0).$$

Provided that m is chosen so large that $(1 - \varepsilon_m)^{-1}L < \|(1, 1)\|_{\Omega}^* - \delta$, Lemma 2.7 shows that the above map $\iota_{L, (1 - \varepsilon)^{-1}L}$ is an isomorphism. Thus for m sufficiently large the first map $(\Phi^L)_m$ in the sequence (2.7) must be injective, and hence is also an isomorphism by counting dimensions.

Since the $(\Phi^L)_m$ are all isomorphisms for m sufficiently large, and since they form the structure maps in a cofinal system within the inverse system defining $CH^L(X_{\Omega}^{\circ}, \lambda_0)$, it follows that the canonical map $CH^L(X_{\Omega}^{\circ}, \lambda_0) \rightarrow CH_3^L(X_{\Omega}^{\delta, \varepsilon_m}, \lambda_0)$ is an isomorphism for m sufficiently large. So by Lemma 2.7 $CH_3^L(X_{\Omega}^{\circ}, \lambda_0)$ is two-dimensional, proving statement (i) of the lemma. Moreover this argument works uniformly for all L in the interval from $\max\{\|(1, 0)\|_{\Omega}^*, \|(0, 1)\|_{\Omega}^*\} + 2\delta$ to $\|(1, 1)\|_{\Omega}^* - 2\delta$, and in particular for $L = L_1$ or $L = L_2$ where L_1, L_2 are as in statement (ii) of the lemma. So for sufficiently large m we have a commutative diagram

$$\begin{array}{ccc} CH_3^{L_1}(X_{\Omega}^{\circ}, \lambda_0) & \xrightarrow{\iota_{L_1, L_2}} & CH_3^{L_2}(X_{\Omega}^{\circ}, \lambda_0) \\ \downarrow & & \downarrow \\ CH_3^{L_1}(X_{\Omega}^{\delta, \varepsilon_m}, \lambda_0) & \xrightarrow{\iota_{L_1, L_2}} & CH_3^{L_2}(X_{\Omega}^{\delta, \varepsilon_m}, \lambda_0) \end{array}$$

where the vertical arrows are isomorphisms by what we have just shown, and the bottom horizontal arrow is an isomorphism by Lemma 2.7. Hence the top horizontal arrow is an isomorphism, proving statement (ii) of the lemma. \square

Lemma 2.13. Let X_Ω be a concave toric domain in \mathbb{C}^2 such that $[(1, 1)]_\Omega < \max\{[(1, 2)]_\Omega, [(2, 1)]_\Omega\}$. Then for $[(1, 1)]_\Omega < L_1 < L_2 < \min\{[(1, 2)]_\Omega, [(2, 1)]_\Omega\}$,

$$\iota_{L_1, L_2}: CH_4^{L_1}(X_\Omega^\circ, \lambda_0) \longrightarrow CH_4^{L_2}(X_\Omega^\circ, \lambda_0)$$

is an isomorphism of one-dimensional vector spaces.

Proof. This follows by the exact same argument as Lemma 2.12, using Lemma 2.8 instead of Lemma 2.7. \square

Remark 2.14. In the case that X_Ω is an ellipsoid $E(a, b)$ (and hence in particular is both a concave toric domain and a convex toric domain), Lemmas 2.12 and 2.13 have no content when applied to X_Ω . Indeed in this case, assuming without loss of generality that $a \leq b$,

$$\|(1, 1)\|_\Omega^* = \max\{\|(1, 0)\|_\Omega^*, \|(0, 1)\|_\Omega^*\} = b, \quad [(1, 1)]_\Omega = \min\{[(1, 2)]_\Omega, [(2, 1)]_\Omega\} = a$$

and so there are no choices of L_1, L_2 that satisfy the hypotheses. For each of the domains appearing in our main theorem, on the other hand, Lemma 2.12 or Lemma 2.13 gives important information.

Proposition 2.15. Let $U \subset \mathbb{R}^{2n}$ be a star-shaped open set, and let $\phi: U \rightarrow V$ be a symplectomorphism where V is an open subset of \mathbb{R}^{2n} . Then ϕ determines an isomorphism $\Phi_\phi^L: CH^L(V, \phi^{-1*}\lambda_0) \rightarrow CH^L(U, \lambda_0)$ such that the diagram

$$(2.8) \quad \begin{array}{ccc} CH^L(V, \phi^{-1*}\lambda_0) & \xrightarrow{\Phi^L} & CH^L(\phi(W), \phi^{-1*}\lambda_0) \\ \downarrow \Phi_\phi^L & & \downarrow \Phi_{\phi|_W}^L \\ CH^L(U, \lambda_0) & \xrightarrow{\Phi^L} & CH^L(W, \lambda_0) \end{array}$$

commutes when $W \subset U$ is an open subset.

Proof. For $X \subset U$, it is straightforward to see that (X, λ_0) is a non-degenerate Liouville domain if and only if $(\phi(X), \phi^{-1*}\lambda_0)$ is a non-degenerate Liouville domain. So in view of Lemma 2.6, we obtain an isomorphism of the inverse systems defining $CH^L(V, \phi^{-1*}\lambda_0)$ and $CH^L(U, \lambda_0)$. This induces the desired isomorphism Φ_ϕ^L between the inverse limits $CH^L(V, \phi^{-1*}\lambda_0)$ and $CH^L(U, \lambda_0)$, and the fact that (2.8) commutes follows by taking inverse limits of the diagrams (2.4) from Lemma 2.6. \square

Definition 2.16. Let $U \subset \mathbb{R}^{2n}$ be an open subset and let $\lambda \in \Omega^1(U)$ obey $d\lambda = \omega_0$. We say that the pair (U, λ) is **tamely exhausted** if for every compact subset $K \subset U$ there is a set X with $K \subset X \subset U$ such that (X, λ) is a tame Liouville domain and such that the natural map $H^1(X; \mathbb{R}) \rightarrow H^1(\partial X; \mathbb{R})$ is zero.

Example 2.17. In any dimension m , let us say that a nonempty compact subset $X \subset \mathbb{R}^m$ is **strictly star-shaped** if for all $x \in X$ and all $t \in [0, 1)$ it holds that $tx \in X^\circ$. We claim that if $X \subset \mathbb{R}^{2n}$ is strictly star-shaped then (X°, λ_0) is tamely exhausted.

To see this, first note that for any $a \in S^{2n-1}$ the set $I_a = \{t \geq 0 \mid ta \in X\}$ is a closed interval of the form $[0, f(a)]$ where $0 < f(a) < \infty$. Indeed I_a contains all sufficiently small positive numbers because the definition implies that $0 \in X^\circ$, and I_a is closed and bounded because X is compact. So we can take $f(a) = \sup I_a = \max I_a$; the fact that I_a contains all numbers between 0 and $f(a)$ is an obvious consequence of the assumption that X is star-shaped. Moreover we then have $tf(a)a \in X^\circ$ for all $t \in [0, 1)$.

So we have defined a function $f : S^{2n-1} \rightarrow (0, \infty)$ with the properties that

$$X = \{sa \mid a \in S^{2n-1}, 0 \leq s \leq f(a)\}$$

and

$$X^\circ = \{sa \mid a \in S^{2n-1}, 0 \leq s < f(a)\}.$$

We will now show that f is continuous. Let $a \in S^{2n-1}$ and let $\varepsilon > 0$ be small enough that $f(a) > \varepsilon$. Then $(f(a) - \varepsilon)a \in X^\circ$, so by considering a small ball around $(f(a) - \varepsilon)a$ that is contained in X° we see that, for $b \in S^{2n-1}$ sufficiently close to a , it will hold that $(f(a) - \varepsilon)b \in X^\circ$ and hence that $f(b) > f(a) - \varepsilon$. Thus f is lower semi-continuous. To see that f is upper semi-continuous note that if it were not then we could find $a_k, a \in S^{2n-1}$ with $a_k \rightarrow a$ and each $f(a_k) \geq f(a) + \varepsilon$ for some $\varepsilon > 0$ independent of k . Since X is compact, after passing to a subsequence the $f(a_k)a_k$ would converge to a point of the form sa where both $s > f(a)$ and $sa \in X$, contradicting the defining property of f . So f is indeed continuous.

With this in hand it is not hard to see that our strictly star-shaped domain X is tamely exhausted. Indeed, if K is a compact subset of X° then there is $\varepsilon > 0$ such that, for all $a \in S^{2n-1}$ and $t \geq 0$ with $ta \in K$, we have $t < f(a) - \varepsilon$. Choose a C^∞ function $g : S^{2n-1} \rightarrow (0, \infty)$ such that, for all $a \in S^{2n-1}$, $f(a) - \varepsilon < g(a) < f(a)$. Then defining $Y = \{ta \mid a \in S^{2n-1}, 0 \leq t \leq g(a)\}$, Y will be a smooth manifold with boundary such that $\lambda_0|_{\partial Y}$ is a contact form and such that $K \subset Y \subset X^\circ$. Possibly after a further perturbation of g , the Reeb flow of $\lambda_0|_{\partial Y}$ will be non-degenerate so that (Y, λ_0) is tame. Because Y is star-shaped, it obviously has $H^1(Y; \mathbb{R}) = 0$. Since K is an arbitrary compact subset of X° this proves our claim that (X°, λ_0) is tamely exhausted.

Lemma 2.18. To each symplectic embedding $\phi : U \hookrightarrow V$ between open subsets $U, V \subset \mathbb{R}^{2n}$ equipped with one-forms λ, λ' such that (U, λ) and (V, λ') are tamely exhausted, we may associate a map $\Phi_\phi^L : CH^L(V, \lambda') \rightarrow CH^L(U, \lambda)$ such that:

- (i) In the case that ϕ is the inclusion of U into V and $\lambda = \lambda'|_U$, Φ_ϕ^L coincides with the transfer map $\Phi^L : CH^L(V, \lambda') \rightarrow CH^L(U, \lambda)$ described just before Lemma 2.10.

- (ii) If $(U, \lambda), (V, \lambda'), (W, \lambda'')$ are tamely exhausted and if $\phi: U \hookrightarrow V$ and $\psi: V \hookrightarrow W$ are symplectic embeddings then we have a commutative diagram

$$(2.9) \quad \begin{array}{ccc} CH^L(W, \lambda'') & \xrightarrow{\Phi_{\psi \circ \phi}^L} & CH^L(U, \lambda) \\ & \searrow \Phi_{\psi}^L & \nearrow \Phi_{\phi}^L \\ & CH^L(V, \lambda') & \end{array}$$

Proof. Since (U, λ) is tamely exhausted, the subsets $X \subset U$ with (X, λ) tame and $H^1(X; \mathbb{R}) \rightarrow H^1(\partial X; \mathbb{R})$ zero form a cofinal system in the inverse system defining $CH^L(U, \lambda)$. So in order to construct Φ_{ϕ}^L it suffices to define maps $\Phi_{VX}^L: CH^L(V, \lambda') \rightarrow CH^L(X, \lambda)$ for all such X in such a way that the diagrams

$$(2.10) \quad \begin{array}{ccc} CH^L(V, \lambda') & & \\ \Phi_{VX}^L \downarrow & \searrow \Phi_{VX'}^L & \\ CH^L(X, \lambda) & \xrightarrow{\Phi^L} & CH^L(X', \lambda) \end{array}$$

commute for subsets $X, X' \subset U$ as above with $X' \subset X^\circ$.

To define Φ_{VX}^L , note that the fact that (V, λ') is tamely exhausted implies that there is Y with $\phi(X) \subset Y^\circ \subset Y \subset V$ such that (Y, λ') is tame, and define Φ_{VX} as a composition $CH^L(V, \lambda') \rightarrow CH^L(Y, \lambda') \rightarrow CH^L(X, \lambda)$ where the first map is the structure map of the inverse limit and the second map is the transfer map associated to $\phi|_X: X \hookrightarrow Y^\circ$. (The fact that $\phi|_X$ is a generalized Liouville embedding follows from the facts that ϕ preserves ω_0 and that $H^1(X; \mathbb{R}) \rightarrow H^1(\partial X; \mathbb{R})$ vanishes.)

We claim that this map Φ_{VX}^L is independent of the choice of Y involved in its construction. Indeed if $Y' \subset V$ is another set satisfying the same properties, then the fact that (V, λ') is tamely exhausted shows that there is Z such that $Y \cup Y' \subset Z^\circ \subset Z \subset V$ and such that (Z, λ') is tame. We can then form a commutative diagram

$$\begin{array}{ccccc} CH^L(V, \lambda') & & & & \\ & \searrow & & & \\ & & CH^L(Z, \lambda') & \longrightarrow & CH^L(Y, \lambda') \\ & & \downarrow & & \downarrow \\ & & CH^L(Y', \lambda') & \longrightarrow & CH^L(X, \lambda) \end{array}$$

Every piece of the above diagram (the square and the two triangles) is commutative by definition of the inverse limit and by functoriality of the transfer map. Therefore the two compositions $CH^L(V, \lambda') \rightarrow CH^L(X, \lambda)$ passing respectively through $CH^L(Y, \lambda')$ and $CH^L(Y', \lambda')$ are equal to each other.

To see that (2.10) commutes, notice that, by what we have just shown, we may use the same subdomain $Y \subset V$ in the constructions of Φ_{VX}^L and of $\Phi_{VX'}^L$, yielding a commutative diagram

$$\begin{array}{ccc}
 & CH^L(V, \lambda') & \\
 & \downarrow & \\
 \Phi_{VX}^L \nearrow & CH^L(Y, \lambda') & \searrow \Phi_{VX'}^L \\
 & \downarrow & \\
 CH^L(X, \lambda) & \xrightarrow{\Phi^L} & CH^L(X', \lambda)
 \end{array}$$

where the bottom triangle is an instance of (2.1). So passing to the inverse limit over X indeed yields our desired map $\Phi_\phi^L : CH^L(V, \lambda') \rightarrow CH^L(U, \lambda)$.

It remains to show that the various maps Φ_ϕ^L constructed in this way satisfy properties (i) and (ii) in the statement of the lemma. However, given the validity of the above construction of Φ_ϕ^L and the functoriality (2.1) for transfer maps associated to generalized Liouville embeddings, both of these are straightforward exercises with inverse limits and so we leave them to the reader. \square

Corollary 2.19. Let $X, E, Y \subset \mathbb{R}^{2n}$ be strictly star-shaped domains with $X \subset Y^\circ$, and let $f : X \rightarrow E$, $g : E \rightarrow Y^\circ$ be symplectic embeddings. If the composition $g \circ f$ is unknotted, then for all $k \in \mathbb{Z}$, $L \in \mathbb{R}$ it holds that

$$\text{Rank}(\Phi^L : CH_k^L(Y^\circ, \lambda_0) \rightarrow CH_k^L(X^\circ, \lambda_0)) \leq \dim CH_k^L(E^\circ, \lambda_0).$$

Proof. The assumption that $g \circ f$ is unknotted implies that there is a symplectomorphism $\phi : Y^\circ \rightarrow Y^\circ$ such that $\phi(g(f(X))) = X$. Example 2.17 shows that each of (X°, λ_0) , (E°, λ_0) , and (Y°, λ_0) is tamely exhausted. It is clear from the definition that if $\psi : Z \hookrightarrow \mathbb{R}^{2n}$ is a symplectic embedding with image Z' , then $(Z, \psi^* \lambda)$ is tamely exhausted if and only if (Z', λ) is tamely exhausted. So since our symplectomorphism ϕ maps Y° to Y° and $g(f(X^\circ))$ to X° it follows that $(Y^\circ, \phi^* \lambda_0)$ and $(g(f(X^\circ)), \phi^* \lambda_0)$ are also tamely exhausted. Consider the diagram

$$\begin{array}{ccc}
 & CH_k^L(E^\circ, \lambda_0) & \\
 \Phi_g^L \nearrow & & \searrow \Phi_{(g|_{f(X^\circ)})^{-1}}^L \\
 CH_k^L(Y^\circ, \phi^* \lambda_0) & \xrightarrow{\Phi^L} & CH_k^L(g(f(X^\circ)), \phi^* \lambda_0) \\
 \downarrow \Phi_{\phi^{-1}}^L & & \downarrow \Phi_{\phi^{-1}}^L \\
 CH_k^L(Y^\circ, \lambda_0) & \xrightarrow{\Phi^L} & CH_k^L(X^\circ, \lambda_0)
 \end{array}$$

We see that the top triangle commutes since it is an instance of (2.9) (as $g \circ (g|_{f(X^\circ)})^{-1}$ is just the inclusion of $g(f(X^\circ))$ into Y°); the square commutes by

Corollary 2.15; and the vertical arrows are isomorphisms. Hence for each $k \in \mathbb{Z}$,

$$\begin{aligned} \text{Rank}(\Phi^L : CH_k^L(Y^\circ, \lambda_0) \rightarrow CH_k^L(X^\circ, \lambda_0)) &= \\ \text{Rank}(\Phi^L : CH_k^L(Y^\circ, \phi^* \lambda_0) \rightarrow CH_k^L(g(f(X^\circ)), \phi^* \lambda_0)). \end{aligned}$$

But since $\Phi^L : CH_k^L(Y^\circ, \phi^* \lambda_0) \rightarrow CH_k^L(g(f(X^\circ)), \phi^* \lambda_0)$ factors through $CH_k^L(E^\circ, \lambda_0)$, its rank is at most the dimension of $CH_k^L(E^\circ, \lambda_0)$. \square

Throughout the rest of the paper Corollary 2.19 will be our main tool for showing that embeddings are knotted. First we need the following to show that it applies to the domains appearing in our main theorems.

Proposition 2.20. Let X be either a convex toric domain or a concave toric domain in \mathbb{R}^{2n} . Then X is strictly star-shaped.

Proof. First suppose that $X = X_\Omega$ is a convex toric domain; thus $\Omega \subset [0, \infty)^n$ has the property that $\hat{\Omega}$ (as defined in (1.1)) is a convex domain in \mathbb{R}^n . It is easy to see that X_Ω is strictly star-shaped if and only if $\hat{\Omega}$ is strictly star-shaped.

Let us re-emphasize that “domains” are by definition closures of bounded open sets. Consequently if $x = (x_1, \dots, x_n) \in \hat{\Omega}$ and $0 < \varepsilon < 1$, we can find $(y_1, \dots, y_n) \in \hat{\Omega}^\circ$ such that $\frac{y_i}{x_i} > 1 - \varepsilon$ for all i such that $x_i \neq 0$. Now $\hat{\Omega}^\circ$ is convex and is invariant under reversal of the sign of any subset of the coordinates of \mathbb{R}^n , so it follows that $\{(z_1, \dots, z_n) \mid |z_i| \leq y_i\} \subset \hat{\Omega}^\circ$. In particular this implies that $(1 - \varepsilon)x \in \hat{\Omega}^\circ$. Since ε can be taken arbitrarily small this proves that $\hat{\Omega}$ is strictly star-shaped and hence that X_Ω is strictly star-shaped.

Now let us turn to the case that $X = X_\Omega$ is a concave toric domain, so that $\Omega \subset [0, \infty)^n$ has the property that $[0, \infty)^n \setminus \Omega$ is convex. It is easy to see that X_Ω is strictly star-shaped if and only if Ω has the property that $t\Omega \subset \Omega^\circ$ for all $t \in [0, 1)$, where the interior Ω° is taken relative to $[0, \infty)^n$. Suppose for contradiction that $x = (x_1, \dots, x_n) \in \Omega$ and $tx \notin \Omega^\circ$ where $0 \leq t < 1$. Then

$$tx \in [0, \infty)^n \setminus \Omega^\circ = \overline{[0, \infty)^n \setminus \Omega}.$$

Now $\overline{[0, \infty)^n \setminus \Omega}$ is a convex set which (since Ω is compact) contains all points sufficiently far from the origin in addition to containing tx , in view of which

$$\{(y_1, \dots, y_n) \mid y_i \geq tx_i \text{ for all } i\} \subset \overline{[0, \infty)^n \setminus \Omega}.$$

The set on the left hand side above contains our point x in its interior, so we would have

$$x \in \Omega \cap \left(\overline{[0, \infty)^n \setminus \Omega} \right)^\circ.$$

But

$$\left(\overline{[0, \infty)^n \setminus \Omega} \right)^\circ = ([0, \infty)^n \setminus \Omega^\circ)^\circ = [0, \infty)^n \setminus \overline{\Omega^\circ},$$

so we would have $x \in \Omega \cap ([0, \infty)^n \setminus \overline{\Omega^\circ})$, which is impossible since Ω is the closure of an open subset. \square

We now fulfill the main goal of this section by proving Theorem 1.12.

Proof of Theorem 1.12 (a). We will show that $\alpha > \delta_{\text{ell}}^u(X)$ implies that $\alpha \geq \frac{\|(1,1)\|_{\Omega}^*}{\max\{\|(1,0)\|_{\Omega}^*, \|(0,1)\|_{\Omega}^*\}}$. Let $\alpha > \delta_{\text{ell}}^u(X)$. Then there is an ellipsoid E and embeddings $f: X \hookrightarrow E$ and $g: E \hookrightarrow \alpha X^\circ$ such that $g \circ f$ is unknotted. By slightly perturbing E we may assume that E is irrational (i.e. $E = E(a, b)$ where $\frac{b}{a} \notin \mathbb{Q}$); this ensures that E is a tame star-shaped domain. We will apply Corollary 2.19 with $k = 3$. Note that, for each $L \in \mathbb{R}$, $\dim CH_3^L(E^\circ, \lambda_0) \leq 1$ by Lemma 2.10 and [BCE07, Section 3]. So by Corollary 2.19, we must have $\text{Rank}(\Phi^L: CH_3^L(\alpha X^\circ, \lambda_0) \rightarrow CH_3^L(X^\circ, \lambda_0)) \leq 1$ for all $L \in \mathbb{R}$. By Lemma 2.11, then, $\iota_{\alpha^{-1}L, L}: CH^{\alpha^{-1}L}(X^\circ, \lambda_0) \rightarrow CH^L(X^\circ, \lambda_0)$ has rank at most one.

If we had $\alpha < \frac{\|(1,1)\|_{\Omega}^*}{\max\{\|(1,0)\|_{\Omega}^*, \|(0,1)\|_{\Omega}^*\}}$, then Lemma 2.12 would allow us to find a real number L such that $\iota_{\alpha^{-1}L, L}: CH^{\alpha^{-1}L}(X^\circ, \lambda_0) \rightarrow CH^L(X^\circ, \lambda_0)$ is an isomorphism of two-dimensional vector spaces, a contradiction which proves that $\alpha \geq \frac{\|(1,1)\|_{\Omega}^*}{\max\{\|(1,0)\|_{\Omega}^*, \|(0,1)\|_{\Omega}^*\}}$, as desired. \square

Proof of Theorem 1.12 (b). This follows by essentially the same argument, using $k = 4$ in the application of Corollary 2.19 in place of $k = 3$, and appealing to Lemma 2.13 instead of Lemma 2.12. This yields the result since any irrational ellipsoid E has no periodic Reeb orbits on its boundary with Conley-Zehnder index equal to 4 and hence obeys $CH_4^L(E^\circ, \lambda_0) = \{0\}$ for all $L \in \mathbb{R}$. \square

2.1. Products. The goal of this section is to show that Theorem 1.8 extends to products of convex toric domains with large ellipsoids of arbitrary even dimension.

Theorem 2.21. Let $X \subset \mathbb{C}^2$ belong to any of the following classes of domains:

- (i) All convex toric domains X such that, for some $c > 0$, $B^4(c) \subsetneq X \subset P(c, c)$.
- (ii) All polydisks $P(a, b)$ for $a \leq b < 2a$.

Then there exist numbers $\alpha > 1$ and $R > 0$ and a knotted symplectic embedding $\phi: X \times E(b_1, \dots, b_{n-2}) \rightarrow \alpha(X \times E(b_1, \dots, b_{n-2}))^\circ$ for any b_1, \dots, b_{n-2} with each $b_i \geq R$.

(Specific values for R in the various cases will appear in the proof.)

In order to prove this we will first establish some basic facts concerning the relationship of the filtered positive S^1 -equivariant symplectic homology of a product of two convex toric domains to that of the factors. Observe that the product of two convex toric domains is a convex toric domain: we have $X_{\Omega_1} \times X_{\Omega_2} = X_{\Omega_1 \times \Omega_2}$. Also notice that, if $\Omega_1 \subset \mathbb{R}^m$ and $\Omega_2 \subset \mathbb{R}^n$ and if we express general elements of \mathbb{R}^{n+m} as (α, β) where $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^n$, then $\|(\alpha, \beta)\|_{\Omega_1 \times \Omega_2}^* = \|\alpha\|_{\Omega_1}^* + \|\beta\|_{\Omega_2}^*$.

Proposition 2.22. Let $X_{\Omega_1} \subset \mathbb{C}^2$ and $X_{\Omega_2} \subset \mathbb{C}^{n-2}$ be two convex toric domains, and assume that $\min\{\|e_i\|_{\Omega_2}^*\} > \|(1, 1)\|_{\Omega_1}^*$ where $\{e_1, \dots, e_{n-2}\}$ is the standard basis for \mathbb{R}^{n-2} . Then for any $\delta, \varepsilon > 0$ there is a tame star-shaped domain $Z^{\delta, \varepsilon}$

such that $(1 - \varepsilon)X_{\Omega_1 \times \Omega_2} \subset Z^{\delta, \varepsilon} \subset X_{\Omega_1 \times \Omega_2}^\circ$ and such that, for

$$\max \left\{ \|(1, 0)\|_{\Omega_1}^*, \|(0, 1)\|_{\Omega_1}^* \right\} + \delta \leq L_1 < L_2 \leq \|(1, 1)\|_{\Omega_1}^* - \delta,$$

the map

$$\iota_{L_1, L_2} : CH_{n+1}^{L_1}(Z^{\delta, \varepsilon}, \lambda_0) \longrightarrow CH_{n+1}^{L_2}(Z^{\delta, \varepsilon}, \lambda_0)$$

is an isomorphism of two-dimensional vector spaces.

Proof. As in the proof of Lemma 2.7, Steps 1, 2, and 3 of the proof of [GH17, Lemma 2.5] provide a tame star-shaped domain $Z^{\delta, \varepsilon}$ such that $(1 - \varepsilon)X_{\Omega_1 \times \Omega_2} \subset Z^{\delta, \varepsilon} \subset X_{\Omega_1 \times \Omega_2}^\circ$ and such that the Reeb orbits of $\lambda_0|_{\partial Z^{\delta, \varepsilon}}$ having action at most $\|(1, 1)\|_{\Omega_1}^*$ and Conley-Zehnder index at most $n + 2$ consist of:

- no orbits of index n ;
- two orbits $(1, 0), (0, 1)$ in degree $n + 1$, with actions in the intervals $(\|(1, 0)\|_{\Omega_1}^* - \delta, \|(1, 0)\|_{\Omega_1}^* + \delta)$ and $(\|(0, 1)\|_{\Omega_1}^* - \delta, \|(0, 1)\|_{\Omega_1}^* + \delta)$, respectively; and
- at most one orbit $(1, 1)$ of index $n + 2$, with filtration level greater than $\|(1, 1)\|_{\Omega_1}^* - \delta$.

(In general we would potentially obtain orbits with actions approximately $\|(\alpha, \beta)\|_{\Omega_1 \times \Omega_2}^* = \|\alpha\|_{\Omega_1}^* + \|\beta\|_{\Omega_2}^*$ for arbitrary $\alpha \in \mathbb{N}^2$ and $\beta \in \mathbb{N}^{n-2}$, but our restriction to filtration levels less than or equal to $\|(1, 1)\|_{\Omega_1}^*$, which is assumed to be less than each $\|e_i\|_{\Omega_2}^*$ forces β to be zero.)

So as in the proof of Lemma 2.7, for L in the interval $[\max\{\|(1, 0)\|_{\Omega_1}^*, \|(0, 1)\|_{\Omega_1}^*\} + \delta, \|(1, 1)\|_{\Omega_1}^* - \delta]$ we have $CC_n^L(Z^{\delta, \varepsilon}, \lambda_0) = CC_{n+2}^L(Z^{\delta, \varepsilon}, \lambda_0) = \{0\}$ and $CC_{n+1}^L(Z^{\delta, \varepsilon}, \lambda_0) \cong \mathbb{Q}^2$, and moreover if L_1, L_2 both lie in this interval with $L_1 < L_2$ then the inclusion of complexes $CC_{n+1}^{L_1}(Z^{\delta, \varepsilon}, \lambda_0) \rightarrow CC_{n+1}^{L_2}(Z^{\delta, \varepsilon}, \lambda_0)$ is an isomorphism. So passing to homology shows that, for $\max\{\|(1, 0)\|_{\Omega_1}^*, \|(0, 1)\|_{\Omega_1}^*\} + \delta \leq L_1 < L_2 \leq \|(1, 1)\|_{\Omega_1}^* - \delta$, the inclusion-induced map $\iota_{L_1, L_2} : CH_{n+1}^{L_1}(Z^{\delta, \varepsilon}, \lambda_0) \rightarrow CH_{n+1}^{L_2}(Z^{\delta, \varepsilon}, \lambda_0)$ is an isomorphism of two-dimensional vector spaces. \square

Lemma 2.23. Let X_{Ω_1} be a convex toric domain in \mathbb{C}^2 with the property that $\max\{\|(1, 0)\|_{\Omega_1}^*, \|(0, 1)\|_{\Omega_1}^*\} < \|(1, 1)\|_{\Omega_1}^*$, and let X_{Ω_2} be a convex toric domain in \mathbb{C}^{n-2} such that $\min_{1 \leq i \leq n-2} \|e_i\|_{\Omega_2}^* > \|(1, 1)\|_{\Omega_1}^*$. Then for all small $\eta > 0$,

$$\text{Rank} \left(CH_{n+1}^{\max\{\|(1, 0)\|_{\Omega_1}^*, \|(0, 1)\|_{\Omega_1}^*\} + \eta} \left((X_{\Omega_1} \times X_{\Omega_2})^\circ, \lambda_0 \right) \longrightarrow CH_{n+1}^{\|(1, 1)\|_{\Omega_1}^* - \eta} \left((X_{\Omega_1} \times X_{\Omega_2})^\circ, \lambda_0 \right) \right) = 2.$$

Proof. Given Proposition 2.22, this is proven in exactly the same way as Lemma 2.12. \square

Lemma 2.24. If $E(a_1, a_2) \subset \mathbb{R}^4$ is an ellipsoid with $1 < \frac{a_2}{a_1} \notin \mathbb{Q}$, and if $E(b_1, \dots, b_{n-2}) \subset \mathbb{R}^{2n-4}$ is any ellipsoid, then $\dim CH_{n+1}^L \left((E(a_1, a_2) \times E(b_1, \dots, b_{n-2}))^\circ, \lambda_0 \right) \leq 1$ for all $L < \min\{b_i\}$.

Proof. Given $p > 1$, consider the Hamiltonian

$$H : \mathbb{C}^2 \times \mathbb{C}^{n-2} \rightarrow \mathbb{R} : H(z, w) := \left(\left(\frac{\pi|z_1|^2}{a_1} + \frac{\pi|z_2|^2}{a_2} \right)^p + \left(\sum_{i=1}^{n-2} \frac{\pi|w_i|^2}{b_i} \right)^p \right)^{\frac{1}{p}}$$

A computation shows the Hamiltonian vector field X_H of H obeys $\lambda_0(X_H) = -H$, from which one deduces that the Reeb vector field of λ_0 along the boundary of $Z_p := \{H \leq 1\}$ is equal to $-X_H$. (Here we use the sign convention that defines X_H by $d\lambda_0(X_H, \cdot) = dH$.)

Note that $Z_p \subset E(a_1, a_2) \times E(b_1, \dots, b_{n-2})$, and that (because the ℓ^p norm on \mathbb{R}^2 converges uniformly on compact subsets to the ℓ^∞ norm as $p \rightarrow \infty$), for any $\varepsilon > 0$ we have $(1 - \varepsilon)(E(a_1, a_2) \times E(b_1, \dots, b_{n-2})) \subset Z_p$ for all sufficiently large p .

The Reeb flow on ∂Z_p rotates the w_i coordinates with period $b_i \left(\sum_{i=1}^{n-2} \frac{\pi|w_i|^2}{b_i} \right)^{-(p-1)}$, which is greater than or equal to b_i since, on ∂Z_p , we have $\sum_{i=1}^{n-2} \frac{\pi|w_i|^2}{b_i} \leq 1$. Hence any closed Reeb orbit on ∂Z_p having action less than $\min\{b_i\}$ must have all b_i identically zero.

Because $\frac{a_2}{a_1} \notin \mathbb{Q}$, it is easy to check that any closed Reeb orbit on ∂Z_p must have one or both of z_1, z_2 identically equal to zero. Such an orbit which also has all b_i equal to zero has action ka_1 or ka_2 where $k \in \mathbb{N}$. Moreover the Conley-Zehnder index of such an orbit is given by $2k + 2 \lfloor \frac{ka_1}{a_2} \rfloor + n - 1$ or by $2k + 2 \lfloor \frac{ka_2}{a_1} \rfloor + n - 1$. Indeed the linearized flow splits into the symplectic sum of the linearized flows on $E(a_1, a_2)$ and on $E(b_1, \dots, b_{n-2})$. Thus the Conley-Zehnder index is the sum of the Conley-Zehnder indices of each individual linearized flow.

In particular, there is only one such orbit of Conley-Zehnder index $n + 1$, namely the one which rotates once in the z_1 plane and has all other coordinates equal to zero. It follows that Z_p is arbitrarily well-approximated by non-degenerate star-shaped domains $Z_p^\varepsilon \subset Z_p^\circ$ such that, for $L < \min\{b_i\}$, we have $\dim CH_{n+1}^L(Z_p^\varepsilon, \lambda_0) \leq 1$. By using these Z_p^ε for $p \gg 1$ to approximate $(E(a_1, a_2) \times E(b_1, \dots, b_{n-2}))^\circ$ it is not hard to see (using arguments like the one in the proof of Lemma 2.12) that $\dim CH_{n+1}^L((E(a_1, a_2) \times E(b_1, \dots, b_{n-2}))^\circ, \lambda_0) \leq 1$. \square

Proof of Theorem 2.21. In Case (a), the proof of Theorem 1.8 shows that $\delta_{\text{ell}}(X_\Omega) < \frac{1}{c} \|(1, 1)\|_\Omega^*$. Hence there is a sequence of symplectic embeddings $X_\Omega \hookrightarrow E(a_1, a_2) \hookrightarrow \alpha X_\Omega^\circ$ where (without loss of generality) $1 < \frac{a_2}{a_1} \notin \mathbb{Q}$ and $1 < \alpha < \frac{1}{c} \|(1, 1)\|_\Omega^* = \frac{\|(1, 1)\|_\Omega^*}{\max\{\|(1, 0)\|_\Omega^*, \|(0, 1)\|_\Omega^*\}}$. By taking a product with the identity, this yields symplectic embeddings

$$X_\Omega \times E(b_1, \dots, b_{n-2}) \hookrightarrow E(a_1, a_2) \times E(b_1, \dots, b_{n-2}) \hookrightarrow \alpha (X_\Omega \times E(b_1, \dots, b_{n-2}))^\circ.$$

If the composition of these embeddings were unknotted, then Corollary 2.19 (applied with L slightly smaller than $\|(1, 1)\|_\Omega^*$) and Lemma 2.23 would show

that $\dim CH_{n+1}^L((E(a_1, a_2) \times E(b_1, \dots, b_{n-2}))^\circ, \lambda_0) \geq 2$, a contradiction with Lemma 2.24 provided that we choose $R \geq \|(1, 1)\|_\Omega^*$.

In Case (b), the proof of Theorem 1.8 likewise shows that there is a sequence of symplectic embeddings $P(a, b) \hookrightarrow E(a_1, a_2) \hookrightarrow \alpha P(a, b)^\circ$ where $1 < \frac{a_2}{a_1} \notin \mathbb{Q}$ and $1 < \alpha < \frac{a+b}{b} = \frac{\|(1,1)\|_\Omega^*}{\max\{\|(1,0)\|_\Omega^*, \|(0,1)\|_\Omega^*\}}$. (Here we write $\Omega = [0, a] \times [0, b]$). Then the same argument as in Case (a) applies to show that the product of the composition of these embeddings with the identity on $E(b_1, \dots, b_{n-2})$ will be knotted provided that $b_i \geq R := a + b$ for all i . \square

3. SOME EMBEDDINGS OF FOUR-DIMENSIONAL ELLIPSOIDS

The main goal of this section is to prove Theorem 1.13, which asserts the existence of certain symplectic embeddings to and from four-dimensional ellipsoids. The machinery for constructing (or, perhaps more accurately, ascertaining the existence of) such embeddings has its roots in Taubes-Seiberg-Witten theory and in papers such as [MP94], [M09], [C14] which relate the question of whether certain four-dimensional domains symplectically embed into certain other domains to questions about symplectic ball-packing problems and then to questions about the symplectic cones of blowups of $\mathbb{C}P^2$, which are then converted to elementary problems by results from [LiLi02]. We will presently recall some of these results, rephrasing them in a way suitable for our applications.

In this section we will consider a limited class of toric domains in \mathbb{C}^2 , given as the preimage under the standard moment map $\mu: (w, z) \mapsto (\pi|w|^2, \pi|z|^2)$ of a quadrilateral having a right-angled vertex at the origin and satisfying a couple of other conditions, see Figure 3. More specifically:

Definition 3.1. Let $a, b, x, y \in [0, \infty)$ satisfy the following properties:

- (i) $x \leq a$ and $y \leq b$.
- (ii) If $\frac{x}{a} + \frac{y}{b} < 1$, then $x + y \leq \min\{a, b\}$.
- (iii) If $\frac{x}{a} + \frac{y}{b} > 1$, then $x + y \geq \max\{a, b\}$.

We denote by $T(a, b, x, y)$ the preimage under μ of the quadrilateral in \mathbb{R}^2 having vertices $(0, 0), (a, 0), (x, y), (0, b)$.

Any such set $T(a, b, x, y)$ is said to be a **toric quadrilateral**; it is said to be concave if $\frac{x}{a} + \frac{y}{b} \leq 1$ and convex if $\frac{x}{a} + \frac{y}{b} \geq 1$.

If $a, b, x, y \in \mathbb{Q}$, then $T(a, b, x, y)$ is said to be a **rational toric quadrilateral**.

We allow the possibility that (x, y) lies on the line segment from $(a, 0)$ to $(0, b)$ so that the relevant quadrilateral degenerates to a triangle; indeed in this case $T(a, b, x, y)$ is the ellipsoid $E(a, b)$ (and is both concave and convex).

For any rational concave toric quadrilateral $T(a, b, x, y)$ (and indeed for somewhat more general toric domains), [C14] (generalizing [M09]) explains how to construct the so-called “weight sequence” $w(T(a, b, x, y))$ of $T(a, b, x, y)$, which is a finite unordered sequence of positive numbers. We will rephrase this as follows. Given two unordered sequences of positive numbers $a = [a_1, \dots, a_k], b = [b_1, \dots, b_l]$ we write $a \sqcup b$ for the union with repetitions: $a \sqcup b = [a_1, \dots, a_k, b_1, \dots, b_l]$.

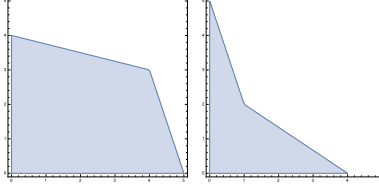


FIGURE 3. The images under μ of the toric quadrilaterals $T(5, 4, 4, 3)$ and $T(4, 5, 1, 2)$.

We will abbreviate the weight sequence $w(E(a, b)) = w(T(a, b, a, 0))$ as $\mathcal{W}(a, b)$. Then for a general rational concave toric quadrilateral the weight sequence is determined recursively by the following prescriptions:

- For any $a \geq 0$, $\mathcal{W}(a, 0) = \mathcal{W}(0, a) = []$ (the empty sequence).
- For $0 < a \leq b$, $\mathcal{W}(a, b) = \mathcal{W}(b, a) = [a] \sqcup \mathcal{W}(a, b - a)$.
- If $\frac{x}{a} + \frac{y}{b} < 1$ (which by our assumptions in Definition 3.1 imply that $x + y \leq \min\{a, b\}$) then

$$w(T(a, b, x, y)) = [x + y] \sqcup \mathcal{W}(a - x - y, y) \sqcup \mathcal{W}(b - x - y, x).$$

For instance,

$$\begin{aligned} w(T(4, 5, 1, 2)) &= [3] \sqcup \mathcal{W}(1, 2) \sqcup \mathcal{W}(2, 1) = [3] \sqcup [1] \sqcup \mathcal{W}(1, 1) \sqcup [1] \sqcup \mathcal{W}(1, 1) \\ &= [3, 1, 1, 1, 1] \sqcup \mathcal{W}(1, 0) \sqcup \mathcal{W}(1, 0) = [3, 1, 1, 1, 1]. \end{aligned}$$

Dually, to a convex toric quadrilateral T is associated a “weight expansion,” which takes the form of a pair $(h(T); \hat{w}(T))$ where $h(T) \in [0, \infty)$ is called the “head” and $\hat{w}(T)$ is a possibly-empty unordered sequence of positive numbers and is called the “negative weight sequence.” For a general rational convex toric quadrilateral the weight expansion is determined as follows:

- If $a \leq b$ then $h(E(a, b)) = h(E(b, a)) = b$ and $\hat{w}(E(a, b)) = \hat{w}(E(b, a)) = \mathcal{W}(b, b - a)$.
- If $\frac{x}{a} + \frac{y}{b} > 1$ (which by our assumptions in Definition 3.1 imply that $x + y \geq \max\{a, b\}$), then $h(T(a, b, x, y)) = x + y$ and $\hat{w}(T(a, b, x, y)) = \mathcal{W}(x + y - a, y) \sqcup \mathcal{W}(x + y - b, x)$.

(This is a complete prescription, since by definition any convex toric quadrilateral $T(a, b, x, y)$ has $\frac{x}{a} + \frac{y}{b} \geq 1$, with equality implying that $T(a, b, x, y) = E(a, b)$. A more obviously-consistent phrasing is that the head $h(T)$ is equal to the capacity of the smallest ball containing T , and that the negative weight sequence is the union of the weight sequences of ellipsoids whose interiors are equivalent under the action of translations and $SL_2(\mathbb{Z})$ to the components of $B^4(h(T))^\circ \setminus T$.)

The deep result that we need is:

Theorem 3.2. [C14, Theorem 1.4] Let S_1, \dots, S_k be a rational concave toric quadrilaterals and T be a rational convex toric quadrilateral. Then the following are equivalent:

- (i) For all $\alpha > 1$ there is a symplectic embedding $\coprod_{i=1}^k S_i \hookrightarrow \alpha T$.
- (ii) For all $\alpha > 1$ there is a symplectic embedding

$$\left(\prod_{i=1}^k \prod_{c \in w(S_i)} B^4(c) \right) \sqcup \prod_{c \in \hat{w}(T)} B^4(c) \hookrightarrow B^4(\alpha h(T)).$$

(While [C14, Theorem 1.4] is stated for a single concave toric domain S_1 , the proof—which closely follows the proof for ellipsoids in [M09]—extends without change to a collection of several disjoint such domains, as was already noted when all of the domains are ellipsoids in [M11, Proposition 3.5].)

Let us introduce the following notation. If $[a_1, \dots, a_m]$ is an unordered sequence of nonnegative real numbers, and if t is another nonnegative real number, we will write

$$[a_1, \dots, a_m] \preceq [t]$$

if and only if

$$t \geq \inf \left\{ u : \prod_{i=1}^m B^4(a_i) \text{ symplectically embeds into } B^4(u) \right\}$$

Then Theorem 3.2 can be rephrased as stating that, for concave toric quadrilaterals S_1, \dots, S_k and a convex toric quadrilateral T , the statement that for all $\alpha > 1$ there is a symplectic embedding $\coprod_{i=1}^k S_i \hookrightarrow \alpha T$ is equivalent to the statement that

$$w(S_1) \sqcup \dots \sqcup w(S_k) \sqcup \hat{w}(T) \preceq [h(T)].$$

Remark 3.3. As follows from [MP94] and [LiLiu01], if we denote by H the hyperplane class and E_1, \dots, E_m the exceptional divisors of the manifold X_m obtained by blowing up $\mathbb{C}P^2$ m times, the statement that $[a_1, \dots, a_m] \preceq [t]$ is equivalent to the statement that the Poincaré dual of the class $tH - \sum a_i E_i$ lies in the set $\bar{\mathcal{C}}_K$ given as the closure of the subset of $H^2(X_m; \mathbb{R})$ consisting of the cohomology classes of symplectic forms having associated canonical class Poincaré dual to $-3H + \sum E_i$.

We will often find it useful to combine Theorem 3.2 with the following elementary but somewhat subtle fact. In the special case that b and c are integer multiples of a this has a well-known proof as in [M09, Lemma 2.6]; see also [M11, Lemma 2.6] for a corresponding statement about ECH capacities in a different special case.

Proposition 3.4. Let $a, b, c \in (0, \infty)$. Then for all $\alpha > 1$ there is a symplectic embedding $E(a, b) \coprod E(a, c) \hookrightarrow \alpha E(a, b + c)$.

Proof. For any $v, w > 0$ let us write $\square(v, w) = (0, v) \times (0, w)$ and $\Delta(v, w) = \{(x_1, x_2) \in (0, \infty)^2 \mid \frac{x_1}{v} + \frac{x_2}{w} < 1\}$. Also for $A, B \subset \mathbb{R}^2$ write $A \times_L B$ for the “Lagrangian product” $\{(x_1 + iy_1, x_2 + iy_2) \mid (x_1, x_2) \in A, (y_1, y_2) \in B\} \subset \mathbb{C}^2$. Now the Traynor trick [T95, Corollary 5.3] shows that for all $\gamma < 1$ there is a symplectic embedding $\gamma E(v, w) \hookrightarrow \Delta(v, w) \times_L \square(1, 1)$. Conversely $(x_1 + iy_1, x_2 + iy_2) \mapsto$

$(\sqrt{\frac{x_1}{\pi}}e^{2\pi iy_1}, \sqrt{\frac{x_2}{\pi}}e^{2\pi iy_2})$ defines a symplectic embedding $\Delta(v, w) \times_L \square(1, 1) \rightarrow E(v, w)$. Meanwhile the symplectomorphism of \mathbb{C}^2 given by $(x_1 + iy_1, x_2 + iy_2) \mapsto (v^{-1}x_1 + ivy_1, w^{-1}x_2 + iw y_2)$ maps $\Delta(v, w) \times_L \square(1, 1)$ to $\Delta(1, 1) \times_L \square(v, w)$. Hence:

$$(3.1) \quad \text{For any } v, w > 0 \text{ and any } \gamma < 1, \text{ there are symplectic embeddings} \\ \gamma E(v, w) \hookrightarrow \Delta(1, 1) \times_L \square(v, w) \hookrightarrow E(v, w).$$

The proof readily follows from this: if $\gamma < 1$, we may symplectically embed $E(\gamma a, \gamma b) \hookrightarrow \Delta(1, 1) \times_L \square(a, b) = \{(x_1 + iy_1, x_2 + iy_2) | x_1, x_2, y_1, y_2 > 0, x_1 + x_2 < 1, y_1 < a, y_2 < b\}$, and likewise, by composing an embedding as in (3.1) with a translation in the y_2 direction, we may symplectically embed

$$E(\gamma a, \gamma c) \hookrightarrow \{(x_1 + iy_1, x_2 + iy_2) | x_1, x_2, y_1 > 0, x_1 + x_2 < 1, y_1 < a, b < y_2 < b + c\}.$$

The images of these two embeddings are evidently disjoint, and their union is contained in $\Delta(1, 1) \times_L \square(a, b + c)$, which symplectically embeds into $E(a, b + c)$. We thus obtain, for any $\gamma < 1$, a symplectic embedding $E(\gamma a, \gamma b) \sqcup E(\gamma a, \gamma c) \hookrightarrow E(a, b + c)$; conjugation by a rescaling then gives the embeddings required in the proposition. \square

The following family of embeddings is used in Case (i) of Theorem 1.8; see Figure 2 for more context in a particular instance.

Proposition 3.5. Let $a, b, x, y \in (0, \infty)$ with $x \leq a, y \leq b$, and $a \leq b \leq x + y$. Then for all $\alpha > 1$ there is a symplectic embedding $E(a, x + y) \hookrightarrow \alpha T(a, b, x, y)$.

Proof. It evidently suffices to prove the statement when $a, b, x, y \in \mathbb{Q}$. Then by Theorem 3.2 the statement is equivalent to the statement that $\mathscr{W}(a, x + y) \sqcup \mathscr{W}(x + y - a, y) \sqcup \mathscr{W}(x + y - b, x) \preceq [x + y]$. But another application of Theorem 3.2 shows that this, in turn, is equivalent to the statement that for all $\alpha > 1$ there exists a symplectic embedding of a disjoint union of three ellipsoids:

$$E(a, x + y) \sqcup E(x + y - a, y) \sqcup E(x + y - b, x) \hookrightarrow \alpha B^4(x + y).$$

Since we assume that $a \leq b$ (so $x + y - b \leq x + y - a$), we have symplectic embeddings

$$E(x + y - a, y) \sqcup E(x + y - b, x) \hookrightarrow E(x + y - a, y) \sqcup E(x + y - a, x) \hookrightarrow \sqrt{\alpha} E(x + y - a, x + y)$$

where the first map is the inclusion and the second is given by Proposition 3.4. Combining this with another application of Proposition 3.4 yields:

$$E(a, x + y) \sqcup E(x + y - a, y) \sqcup E(x + y - b, x) \hookrightarrow E(a, x + y) \sqcup \sqrt{\alpha} E(x + y - a, x + y) \\ \subset \sqrt{\alpha} (E(a, x + y) \sqcup E(x + y - a, x + y)) \hookrightarrow \alpha E(x + y, x + y) = \alpha B^4(x + y).$$

\square

Similarly in the concave case, we obtain:

Proposition 3.6. Let $a, b, x, y \in (0, \infty)$ with $x + y \leq a \leq b$. Then for all $\alpha > 1$ there is a symplectic embedding $T(a, b, x, y) \hookrightarrow E(b, x + y)$.

Proof. It again suffices to assume that $a, b, x, y \in \mathbb{Q}$. Theorem 3.2 shows that the proposition is equivalent to the statement that $[x + y] \sqcup \mathcal{W}(a - x - y, y) \sqcup \mathcal{W}(b - x - y, x) \sqcup \mathcal{W}(b - x - y, b) \preceq [b]$, which in turn is equivalent to the existence of a symplectic embedding, for all $\alpha > 1$,

$$B^4(x + y) \sqcup E(a - x - y, y) \sqcup E(b - x - y, x) \sqcup E(b, b - x - y) \hookrightarrow \alpha B^4(b).$$

Proposition 3.4 (together with the inclusion $E(a - x - y, y) \subset E(b - x - y, y)$) gives, for all $\nu > 1$, embeddings $E(a - x - y, y) \sqcup E(b - x - y, x) \hookrightarrow \nu E(b - x - y, x + y)$ and then $B^4(x + y) \sqcup E(b - x - y, x + y) \hookrightarrow \nu E(b, x + y)$, and finally $E(b, x + y) \sqcup E(b, b - x - y) \hookrightarrow \nu E(b, b) = \nu B^4(b)$. Combining these three embeddings (with $\nu = \alpha^{1/3}$) then implies the result. \square

Remark 3.7. Note that the volume of $T(a, b, x, y)$ is $\frac{1}{2}(ay + bx)$, while that of $E(a, x + y)$ is $\frac{1}{2}a(x + y)$. So in the case that $a = b$, the embeddings $E(a, x + y) \rightarrow \alpha T(a, a, x, y)$ (in the convex case) or $T(a, a, x, y) \rightarrow \alpha E(a, x + y)$ (in the concave case) fill all but an arbitrarily small proportion of the volumes of their targets as $\alpha \rightarrow 1$.

Since $P(1, b) = T(1, b, 1, b)$, a special case of Proposition 3.5 is that, for any $\alpha > 1$ and $b \geq 1$, there is a symplectic embedding $E(1, 1 + b) \hookrightarrow \alpha P(1, b)$. The following reproduces this embedding when $1 \leq b < 2$, and improves on it for $b = m + \varepsilon \geq 2$. The case that $\varepsilon = 0$ is well-known; see [CFS17, Remark 1.2(1)].

Proposition 3.8. Let $m \in \mathbb{Z}_+$ and $0 \leq \varepsilon < 1$. Then for all $\alpha > 1$ there is a symplectic embedding $E(1, 2m + \varepsilon) \hookrightarrow \alpha P(1, m + \varepsilon)$.

Proof. By Theorem 3.2 the statement is equivalent to the statement that $\mathcal{W}(1, 2m + \varepsilon) \sqcup \mathcal{W}(m + \varepsilon, m + \varepsilon) \sqcup \mathcal{W}(1, 1) \preceq [m + 1 + \varepsilon]$. From the recursive description of $\mathcal{W}(a, b)$ given earlier we see that $\mathcal{W}(1, 2m + \varepsilon) = \mathcal{W}(1, m) \sqcup \mathcal{W}(1, m + \varepsilon)$, so this is equivalent to the existence, for all $\alpha > 1$, of a symplectic embedding

$$E(1, m) \sqcup E(1, m + \varepsilon) \sqcup E(m + \varepsilon, m + \varepsilon) \sqcup E(1, 1) \hookrightarrow \alpha B^4(m + 1 + \varepsilon).$$

But by Proposition 3.4 there are symplectic embeddings

$$E(1, m) \sqcup E(1, 1) \hookrightarrow \sqrt{\alpha} E(1, m + 1) \subset \sqrt{\alpha} E(1, m + 1 + \varepsilon)$$

and

$$E(1, m + \varepsilon) \sqcup E(m + \varepsilon, m + \varepsilon) \hookrightarrow \sqrt{\alpha} E(m + \varepsilon, m + 1 + \varepsilon),$$

and then another application of Proposition 3.4 gives a symplectic embedding $\sqrt{\alpha} E(m + \varepsilon, m + 1 + \varepsilon) \sqcup \sqrt{\alpha} E(1, m + 1 + \varepsilon) \hookrightarrow \alpha E(m + 1 + \varepsilon, m + 1 + \varepsilon) = \alpha B^4(m + 1 + \varepsilon)$, from which the result is immediate. \square

The embeddings in Propositions 3.5, 3.6, and 3.8 will give rise to many of the knotted embeddings described in the introduction. Some of our other knotted embeddings require a somewhat less straightforward application of Theorem 3.2 and Proposition 3.4. The key additional (and standard) ingredient is the use of *Cremona moves*, based on [LiLi02, Proof of Lemma 3.4]. As in Remark 3.3 we regard the question of whether $[a_1, \dots, a_m] \preceq [t]$ as equivalent to the

question of whether the Poincaré dual of $tH - \sum a_i E_i$ lies in the closure $\bar{\mathcal{C}}_K$ of the appropriate connected component of the symplectic cone of the m -fold blowup X_m of $\mathbb{C}P^2$. Since $[a_1, \dots, a_m] \preceq [t]$ if and only if $[a_1, \dots, a_m, 0] \preceq [t]$ we may without loss of generality assume that $m \geq 3$. Then X_m contains a sphere in the class $H - E_1 - E_2 - E_3$ of self-intersection -2 and Chern number zero; the cohomological action of a Dehn-Seidel twist in this sphere preserves $\bar{\mathcal{C}}_K$ and maps the Poincaré dual of $tH - \sum a_i E_i$ to the Poincaré dual of $(2t - a_1 - a_2 - a_3)H - (t - a_2 - a_3)E_1 - (t - a_1 - a_3)E_2 - (t - a_1 - a_2)E_3 - \sum_{i=4}^m a_i E_i$. So we have:

Proposition 3.9. [LiLi02] Assume that $t \geq \max\{a_1 + a_2, a_1 + a_3, a_2 + a_3\}$. Then

$$[a_1, a_2, a_3, a_4, \dots, a_m] \preceq [t] \quad \text{if and only if}$$

$$[t - a_2 - a_3, t - a_1 - a_3, t - a_1 - a_2, a_4, \dots, a_m] \preceq [2t - a_1 - a_2 - a_3]$$

The following will help us construct the knotted polydisks from Case (iv) of Theorem 1.8.

Proposition 3.10. Let $a \leq y \leq b \leq 2a$. Then for all $\alpha > 1$ there is a symplectic embedding

$$E\left(\frac{a+b}{3}, 2a+y\right) \hookrightarrow \alpha T(a, b, a, y).$$

Proof. As usual assuming that $a, b, y \in \mathbb{Q}$, by Theorem 3.2 the proposition is equivalent to the statement that

$$(3.2) \quad \mathcal{W}\left(\frac{a+b}{3}, 2a+y\right) \sqcup \mathcal{W}(y, y) \sqcup \mathcal{W}(a+y-b, a) \preceq [a+y].$$

Since $a \leq y$ and $b \leq 2a$ we have $a+b \leq 2a+y$, in view of which

$$\mathcal{W}\left(\frac{a+b}{3}, 2a+y\right) = \left[\frac{a+b}{3}, \frac{a+b}{3}, \frac{a+b}{3}\right] \sqcup \mathcal{W}\left(\frac{a+b}{3}, a+y-b\right).$$

Meanwhile of course $\mathcal{W}(y, y) = [y]$, and (since $y \leq b$) $\mathcal{W}(a+y-b, a) = [a+y-b] \sqcup \mathcal{W}(a+y-b, b-y)$. So (3.2) amounts to the statement that

$$\left[y, a+y-b, \frac{a+b}{3}, \frac{a+b}{3}, \frac{a+b}{3}\right] \sqcup \mathcal{W}\left(\frac{a+b}{3}, a+y-b\right) \sqcup \mathcal{W}(b-y, a+y-b) \preceq [a+y].$$

Applying Proposition 3.9 and reordering the sequence in brackets shows that this is equivalent to the statement that

$$\left[\frac{a+b}{3}, \frac{a+b}{3}, \frac{2b-a}{3}, \frac{2a-b}{3}, b-y\right] \sqcup \mathcal{W}\left(\frac{a+b}{3}, a+y-b\right) \sqcup \mathcal{W}(b-y, a+y-b) \preceq \left[\frac{2a+2b}{3}\right].$$

Then another application of Proposition 3.9 shows that this last statement (and hence also (3.2)) is equivalent to the statement that

$$\left[\frac{2a-b}{3}, \frac{2a-b}{3}, 0, \frac{2a-b}{3}, b-y\right] \sqcup \mathcal{W}\left(\frac{a+b}{3}, a+y-b\right) \sqcup \mathcal{W}(b-y, a+y-b) \preceq [a].$$

The left hand side above can be rewritten as

$$[0] \sqcup \mathcal{W}\left(\frac{2a-b}{3}, 2a-b\right) \sqcup \mathcal{W}\left(\frac{a+b}{3}, a+y-b\right) \sqcup \mathcal{W}(b-y, a+y-b) \sqcup \mathcal{W}(b-y, b-y).$$

So by Theorem 3.2 it suffices to show that for all $\alpha > 1$ there is a symplectic embedding

$$(3.3) \quad E\left(\frac{2a-b}{3}, 2a-b\right) \sqcup E\left(\frac{a+b}{3}, a+y-b\right) \sqcup E(b-y, a+y-b) \sqcup E(b-y, b-y) \hookrightarrow \alpha E(a, a).$$

We now repeatedly use Proposition 3.4, obtaining for any $\nu > 1$ symplectic embeddings:

- $E(b-y, a+y-b) \sqcup E(b-y, b-y) \hookrightarrow \nu E(b-y, a)$;
- $E\left(\frac{2a-b}{3}, 2a-b\right) \sqcup E\left(\frac{a+b}{3}, a+y-b\right) \subset E\left(\frac{2a-b}{3}, a+y-b\right) \sqcup E\left(\frac{a+b}{3}, a+y-b\right) \hookrightarrow \nu E(a, a+y-b)$ (since $a \leq y$);
- $E(a, a+y-b) \sqcup E(b-y, a) \cong E(a+y-b, a) \sqcup E(b-y, a) \hookrightarrow \nu E(a, a)$.

Combining these embeddings (with $\nu = \sqrt{\alpha}$) yields the embedding (3.3) and hence proves the proposition. \square

3.1. Proof of Theorem 1.13. We begin with the following easy observation, using the terminology and notation from Section 1.

Proposition 3.11. Let $\Omega \subset [0, \infty)^n$ be any star-shaped domain such that X_Ω contains the origin in its interior. Then

$$(3.4) \quad \delta_{\text{ell}}(X_\Omega) \leq \inf \left\{ \left\| \left(\frac{1}{a_1}, \dots, \frac{1}{a_n} \right) \right\|_{\Omega}^* \mid \text{There is a symplectic embedding } E(a_1, \dots, a_n) \hookrightarrow X_\Omega^\circ \right\}$$

and

$$(3.5) \quad \delta_{\text{ell}}(X_\Omega) \leq \frac{1}{\sup \left\{ \left[\left(\frac{1}{a_1}, \dots, \frac{1}{a_n} \right) \right]_{\Omega} \mid \text{There is a symplectic embedding } X_\Omega \hookrightarrow E(a_1, \dots, a_n)^\circ \right\}}.$$

Proof. We first prove (3.4). Suppose that there is a symplectic embedding $E(a_1, \dots, a_n) \hookrightarrow X_\Omega^\circ$ and let $\alpha = \left\| \left(\frac{1}{a_1}, \dots, \frac{1}{a_n} \right) \right\|_{\Omega}^*$. So by definition, each point $(x_1, \dots, x_n) \in \Omega$ obeys $\sum_i \frac{x_i}{a_i} \leq \alpha$. But $\alpha E(a_1, \dots, a_n)$ is precisely the preimage under μ of $\{(x_1, \dots, x_n) \in [0, \infty)^n \mid \sum_i \frac{x_i}{a_i} \leq \alpha\}$, while $X_\Omega = \mu^{-1}(\Omega)$. So we have $E(a_1, \dots, a_n) \hookrightarrow X_\Omega^\circ$ and $X_\Omega \subset \alpha E(a_1, \dots, a_n)$, and hence for $E = \alpha E(a_1, \dots, a_n)$ there are symplectic embeddings $X_\Omega \hookrightarrow E \hookrightarrow \alpha X_\Omega^\circ$. Thus $\delta_{\text{ell}}(X_\Omega) \leq \alpha$. Since (a_1, \dots, a_n) was arbitrary subject to the assumption that there is a symplectic embedding $E(a_1, \dots, a_n) \hookrightarrow X_\Omega^\circ$, this proves (3.4).

Similarly, suppose that there is a symplectic embedding $X_\Omega \hookrightarrow E(a_1, \dots, a_n)^\circ$ and let $\nu = \left[\left(\frac{1}{a_1}, \dots, \frac{1}{a_n} \right) \right]_{\Omega}$. Then for each $(x_1, \dots, x_n) \in [0, \infty)^2 \setminus \Omega$ we have $\sum_i \frac{x_i}{a_i} \geq \nu$.

So since Ω is closed it then follows that $\{(y_1, \dots, y_n) \in [0, \infty)^n \mid \sum_i \frac{y_i}{a_i} \leq \nu\} \subset \Omega$. Taking preimages under μ then shows that $\nu E(a_1, \dots, a_n) \subset X_\Omega$, and hence $E(a_1, \dots, a_n) \hookrightarrow \nu^{-1} X_\Omega$. Thus $\delta_{\text{ell}}(X_\Omega) \leq \frac{1}{\nu}$, which implies (3.5) since ν was arbitrary subject to the assumption that there is a symplectic embedding $X_\Omega \hookrightarrow E(a_1, \dots, a_n)^\circ$. \square

The proof of Theorem 1.13 now follows almost immediately based on Propositions 3.5, 3.6, and 3.10. For part (a), the hypotheses that $\hat{\Omega}$ is convex and that $(a, 0), (0, b), (x, y) \in \Omega$ imply that also $(0, 0), (0, y) \in \Omega$. Since $(0, y) \in \Omega$ and since the right-hand-side of the desired inequality is independent of b , there is no loss of generality in assuming that $b \geq y$, while the hypothesis of the theorem gives inequalities $x \leq a \leq b \leq x + y$. The fact that Ω is a convex region containing $(a, 0), (0, 0), (0, b), (x, y)$ implies that the quadrilateral with these points as its vertices is contained in Ω , and hence that $T(a, b, x, y) \subset X_\Omega$. So for any $\alpha > 1$ Proposition 3.5 gives a symplectic embedding $E(\alpha^{-1}a, \alpha^{-1}(x+y)) \hookrightarrow X_\Omega^\circ$, whence (3.4) yields Theorem 1.13 (a).

Similarly in part (b), by hypothesis we have $(a, 0), (x, y), (0, b) \in \overline{[0, \infty)^2 \setminus \Omega}$, and moreover $[0, \infty)^2 \setminus \Omega$ (and hence also its closure) is convex. Since Ω is bounded, it follows that $\overline{[0, \infty)^2 \setminus \Omega}$ contains all points of form $t\vec{v}$ where $t \geq 1$ and \vec{v} lies on the line segment from $(a, 0)$ to (x, y) or the line segment from (x, y) to $(0, b)$. The preimage under μ of the set of all such points is $\mathbb{R}^4 \setminus T(a, b, x, y)^\circ$, while the preimage under μ of $\overline{[0, \infty)^2 \setminus \Omega}$ is $\mathbb{R}^4 \setminus X_\Omega^\circ$, so this shows that $X_\Omega^\circ \subset T(a, b, x, y)^\circ$ and hence (recalling our convention that “domains” are closures of open subsets) that $X_\Omega \subset T(a, b, x, y)$. Thus part (b) of Theorem 1.13 follows from Proposition 3.6 and (3.5).

Part (c) of Theorem 1.13 is an immediate application of Proposition 3.10 (applied to $P(a, b) = T(a, b, a, b)$) together with (3.4).

3.2. An explicit construction. The embeddings from Propositions 3.5, 3.6, and 3.10 that underlie Theorem 1.13 are obtained by very indirect methods and are difficult to understand concretely. We will now explain a more direct construction that, for instance, leads to an explicit formula for a knotted embedding $P(1, 1) \rightarrow \alpha P(1, 1)^\circ$ for any $\alpha \in \left(\frac{1}{2-\sqrt{2}}, 2\right)$.

The key ingredient is a toric structure on the complement of the antidiagonal in $S^2 \times S^2$ that appears (at least implicitly) in [EP09, Example 1.22], [FOOO12], [OU16, Section 2]. View S^2 as the unit sphere in \mathbb{R}^3 and let $A = \{(v, w) \in S^2 \times S^2 \mid w = -v\}$ be the antidiagonal. Define functions $F_1, F_2: S^2 \times S^2 \rightarrow \mathbb{R}$ by

$$F_1(v, w) = v_3 + w_3 \quad F_2(v, w) = \|v + w\|.$$

Now F_2 fails to be smooth along $A = F_2^{-1}(\{0\})$, but on $S^2 \times S^2 \setminus A$ the Hamiltonian flows of the functions F_1 and F_2 induce S^1 -actions that commute with each other and are rather simple to understand: F_1 induces simultaneous rotation of the factors about the z -axis, and F_2 induces the flow which rotates the pair $(v, w) \in S^2 \times S^2 \setminus A$ about an axis in the direction of $v + w$. In formulas:

(3.6)

$$\begin{aligned} & \phi_{F_1}^t \left((v_1, v_2, v_3), (w_1, w_2, w_3) \right) \\ &= \left(((\cos t)v_1 - (\sin t)v_2, (\sin t)v_1 + (\cos t)v_2, v_3), ((\cos t)w_1 - (\sin t)w_2, (\sin t)w_1 + (\cos t)w_2, w_3) \right) \end{aligned}$$

and

(3.7)

$$\phi_{F_2}^t(v, w) = \left(\frac{v+w}{2} + (\cos t) \frac{v-w}{2} + (\sin t) \frac{w \times v}{\|v+w\|}, \frac{v+w}{2} + (\cos t) \frac{w-v}{2} + (\sin t) \frac{v \times w}{\|v+w\|} \right).$$

Define

$$J: S^2 \times S^2 \rightarrow \mathbb{R}^2 \quad \text{by} \quad J(v, w) = (2 - \|v+w\|, \|v+w\| - v_3 - w_3),$$

i.e. $J = (2 - F_2, -F_1 + F_2)$. Then J is smooth away from A , and its restriction to $S^2 \times S^2 \setminus A$ is the moment map for a Hamiltonian T^2 -action.⁵ It is not hard to see that J has image equal to $\Delta := \{(x, y) \in [0, \infty)^2 \mid x/2 + y/4 \leq 1\}$, and that the preimage of $\{x/2 + y/4 = 1\}$ is equal to $Q := \{(v, w) \in S^2 \times S^2 \mid v_3 + w_3 = -\|v+w\|\}$. (In other words, Q is the locus of pairs $(v, w) \in S^2 \times S^2$ such that $v+w$ is on the nonpositive z axis.)

Proposition 3.12. Let $\Delta^\circ = \{(x, y) \in [0, \infty)^2 \mid \frac{x}{2} + \frac{y}{4} < 1\}$ and define $s: \Delta^\circ \rightarrow S^2 \times S^2$ by

$$s(x, y) = \left(\left(\sqrt{x \left(1 - \frac{x}{4}\right)}, \sqrt{y \left(1 - \frac{x}{2} - \frac{y}{4}\right)}, 1 - \frac{x+y}{2} \right), \left(-\sqrt{x \left(1 - \frac{x}{4}\right)}, \sqrt{y \left(1 - \frac{x}{2} - \frac{y}{4}\right)}, 1 - \frac{x+y}{2} \right) \right).$$

Then, writing $E(4\pi, 8\pi)^\circ = \{(w, z) \in \mathbb{C}^2 \mid \frac{|w|^2}{4} + \frac{|z|^2}{8} < 1\}$, the map

$$\Phi(|z_1|e^{i\theta}, |z_2|e^{i\varphi}) = \phi_{F_1}^\varphi \left(\phi_{F_2}^{\theta-\varphi} \left(s \left(\frac{|z_1|^2}{2}, \frac{|z_2|^2}{2} \right) \right) \right)$$

defines a symplectomorphism $\Phi: E(4\pi, 8\pi)^\circ \rightarrow S^2 \times S^2 \setminus Q$ which satisfies $J \circ \Phi(z_1, z_2) = \left(\frac{|z_1|^2}{2}, \frac{|z_2|^2}{2} \right)$.

Proof. First we observe that s indeed takes values in $S^2 \times S^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$, which follows by computing

$$\begin{aligned} & x \left(1 - \frac{x}{4}\right) + y \left(1 - \frac{x}{2} - \frac{y}{4}\right) + \left(1 - \frac{x+y}{2}\right)^2 \\ &= x + y - \frac{x^2 + y^2}{4} - \frac{xy}{2} + 1 - x - y + \frac{x^2 + 2xy + y^2}{4} = 1. \end{aligned}$$

Given $(x, y) \in \Delta^\circ$, if we write $(v, w) = s(x, y)$, then

$$\|v+w\|^2 = 4y \left(1 - \frac{x}{2} - \frac{y}{4}\right) + (2-x-y)^2 = x^2 - 4x + 4 = (2-x)^2,$$

so (since $x < 2$)

$$J(s(x, y)) = (2 - \|v+w\|, -v_3 - w_3 + \|v+w\|) = (x, x+y-2+2-x) = (x, y).$$

⁵Here we view T^2 as $(\mathbb{R}/2\pi\mathbb{Z})^2$. On the other hand the map $\mu(w, z) = (\pi|w|^2, \pi|z|^2)$ that we have considered elsewhere is the moment map for a Hamiltonian $(\mathbb{R}/\mathbb{Z})^2$ -action; to get a $(\mathbb{R}/2\pi\mathbb{Z})^2$ -action one would take $\frac{\mu}{2\pi}$.

In particular, the image of s is contained in $S^2 \times S^2 \setminus Q = J^{-1}(\Delta^\circ)$, and it intersects each fiber of $J|_{J^{-1}(\Delta^\circ)}$ just once.

Moreover, since the image of s is contained in $\{(v, Rv) \mid v \in S^2\}$ where R is the reflection through the $v_2 v_3$ -plane and hence is antisymplectic, we see that $s^* \Omega = 0$ where Ω is the standard product symplectic form on $S^2 \times S^2$. Thus $s: \Delta^\circ \rightarrow J^{-1}(\Delta^\circ)$ is a Lagrangian right inverse to the moment map J .

Write $\psi_{(\theta, \varphi)}^1(z_1, z_2) = (e^{-i\theta} z_1, e^{-i\varphi} z_2)$ for the standard T^2 -action on $E(4\pi, 8\pi)^\circ$ (with moment map $\frac{\mu}{2\pi}$ having image equal to Δ° ; the negative signs in front of θ and φ arise because our convention for Hamiltonian vector fields is $\omega_0(X_H, \cdot) = dH$). Likewise write $\psi_{(\theta, \varphi)}^2 = \phi_{F_1}^{-\varphi} \circ \phi_{F_2}^{\varphi - \theta}$ for the T^2 -action on $S^2 \times S^2 \setminus Q$ induced by the moment map J . Our map Φ maps the Lagrangian section of $\frac{\mu}{2\pi}$ given by the nonnegative real locus of $E(4\pi, 8\pi)^\circ$ to the Lagrangian section of $J|_{S^2 \times S^2 \setminus Q}$ given by the image of s , and Φ obeys $J \circ \Phi = \frac{\mu}{2\pi}$ and, for all $(\theta, \varphi) \in T^2$, $\Phi \circ \psi_{(\theta, \varphi)}^1 = \psi_{(\theta, \varphi)}^2 \circ \Phi$. These facts are easily seen to imply that Φ is a symplectomorphism, as it identifies action-angle coordinates on $E(4\pi, 8\pi)^\circ$ with action-angle coordinates on $S^2 \times S^2 \setminus Q$. The last statement is immediate from the formula for Φ and the facts that $J \circ s$ is the identity and that J is preserved under the Hamiltonian flows of F_1 and F_2 . \square

Remark 3.13. With sufficient effort, one can derive the following equivalent formula for the map $\Phi: E(4\pi, 8\pi)^\circ \rightarrow S^2 \times S^2$ from Proposition 3.12: regarding S^2 as the unit sphere in $\mathbb{C} \times \mathbb{R}$, we have

$$(3.8) \quad \Phi(w, z) = (\Gamma(w, z), \Gamma(-w, z)) \quad \text{where}$$

$$\Gamma(w, z) = \left(\frac{\sqrt{8 - |w|^2} ((8 - 2|w|^2 - |z|^2)w + \bar{w}z^2)}{8(4 - |w|^2)} + \frac{iz}{4} \sqrt{8 - 2|w|^2 - |z|^2}, \right. \\ \left. 1 - \frac{|w|^2 + |z|^2}{4} - \frac{\sqrt{(8 - |w|^2)(8 - 2|w|^2 - |z|^2)}}{4(4 - |w|^2)} \operatorname{Im}(w\bar{z}) \right).$$

Since $E(4\pi, 8\pi)^\circ$ is precisely the locus where $2|w|^2 + |z|^2 < 8$, this makes clear that Φ is a smooth (indeed even real-analytic) map despite the appearance of square roots in the formula for s in Proposition 3.12.

Now if $D(4\pi)$ denotes the open disk of area 4π (so radius 2) in \mathbb{C} , there is a symplectomorphism $\sigma: S^2 \setminus \{(0, -1)\} \rightarrow D(4\pi)$ defined by

$$(3.9) \quad \sigma(z, v_3) = \sqrt{\frac{2}{1 + v_3}} z$$

where as in Remark 3.13 we regard S^2 as the unit sphere in $\mathbb{C} \times \mathbb{R}$.

So if we let $\mathcal{S} = (\{(0, -1)\} \times S^2) \cup (S^2 \times \{(0, -1)\})$ then $\sigma \times \sigma$ defines a symplectomorphism $S^2 \times S^2 \setminus \mathcal{S} \cong P(4\pi, 4\pi)^\circ = D(4\pi) \times D(4\pi)$.

For $v = (z, v_3) \in S^2 \subset \mathbb{C} \times \mathbb{R}$, we have

$$\|v + (0, -1)\|^2 = |z|^2 + v_3^2 - 2v_3 + 1 = 2 - 2v_3$$

and hence

$$J(v, (0, -1)) = J((0, -1), v) = (2 - \sqrt{2 - 2v_3}, \sqrt{2 - 2v_3} + (1 - v_3)).$$

Thus

$$J(\mathcal{J}) \subset \{(x, y) \in \mathbb{R}^2 \mid (2-x)^2 = 2(x+y)-4\} = \{(x, y) \in \mathbb{R}^2 \mid y = \frac{x^2}{2} - 3x + 4\}.$$

Since $\frac{\mu}{2\pi} = J \circ \Phi$, we have $\frac{\mu}{2\pi}(\Phi^{-1}(\mathcal{J})) = J(\mathcal{J})$. From this we obtain the following:

Proposition 3.14. Suppose that X_Ω is a convex toric domain where $\Omega \subset \{(2\pi x, 2\pi y) \in [0, \infty)^2 \mid y < \frac{x^2}{2} - 3x + 4\}$. Then there is an ellipsoid E such that $X_\Omega \subset E^\circ$ and such that the map Φ from Proposition 3.12 maps E to a subset of $S^2 \times S^2 \setminus \mathcal{J}$. Hence $(\sigma \times \sigma) \circ \Phi|_E$ is a symplectic embedding from E to $P(4\pi, 4\pi)^\circ$.

Proof. The sets $\frac{1}{2\pi}\Omega$ and $S := \{(x, y) \in [0, \infty)^2 \mid y \geq \frac{x^2}{2} - 3x + 4\}$ are disjoint, closed, convex subsets of \mathbb{R}^2 , and the first of these sets is compact, so the hyperplane separation theorem shows that they must be separated by a line ℓ , which passes through the first quadrant since both sets are contained in the first quadrant. This line ℓ must have negative slope, since S intersects all lines with positive slope and also intersects all horizontal or vertical lines that pass through the first quadrant. So we can write the separating line as $\ell = \{(x, y) \in \mathbb{R}^2 \mid \frac{x}{a} + \frac{y}{b} = 1\}$ with $a, b > 0$, and then it will hold that $\frac{1}{2\pi}\Omega \subset \{\frac{x}{a} + \frac{y}{b} < 1\}$ and $S \subset \{\frac{x}{a} + \frac{y}{b} > 1\}$. The first inclusion shows that $X_\Omega \subset E(2\pi a, 2\pi b)^\circ$. Meanwhile since $(2, 0), (0, 4) \in S \subset \{\frac{x}{a} + \frac{y}{b} > 1\}$, we have $a < 2$ and $b < 4$. So $E(2\pi a, 2\pi b)$ is contained in the domain of the map Φ from Proposition 3.12, and by the discussion before the proposition the fact that $\ell \cap S = \emptyset$ implies that $E(2\pi a, 2\pi b) \cap \Phi^{-1}(\mathcal{J}) = \emptyset$. Hence the proposition holds with $E = E(2\pi a, 2\pi b)$. \square

Corollary 3.15. Suppose that X_Ω is a convex toric domain with $\Omega \subset \{(2\pi x, 2\pi y) \in [0, \infty)^2 \mid y < \frac{x^2}{2} - 3x + 4\}$, and that we have $P(4\pi, 4\pi) \subset \alpha X_\Omega$ for some $\alpha < \delta_{\text{ell}}^u(X_\Omega)$. Then $(\sigma \times \sigma) \circ \Phi|_{X_\Omega} : X_\Omega \hookrightarrow P(4\pi, 4\pi)^\circ \subset \alpha X_\Omega^\circ$ defines a knotted embedding $X_\Omega \hookrightarrow \alpha X_\Omega^\circ$.

Proof. By Proposition 3.14 we have an ellipsoid E and a sequence $X_\Omega \hookrightarrow E^\circ \hookrightarrow P(4\pi, 4\pi)^\circ \subset \alpha X_\Omega^\circ$ where the first map is the inclusion and the second map is $(\sigma \times \sigma) \circ \Phi|_E$. So the corollary follows directly from the assumption that $\alpha < \delta_{\text{ell}}^u(X_\Omega)$ and the definition of δ_{ell}^u . \square

We emphasize that this embedding $(\sigma \times \sigma) \circ \Phi$ is completely explicit: σ is defined in (3.9) and Φ is defined in Proposition 3.12 based partly on the formulas (3.6) and (3.7), or even more explicitly is given by (3.8).

Example 3.16. For instance, Ω could be taken to be a square $[0, 2\pi c]^2$ with c smaller than the smallest root of the polynomial $\frac{x^2}{2} - 4x + 4$, namely $4 - 2\sqrt{2}$

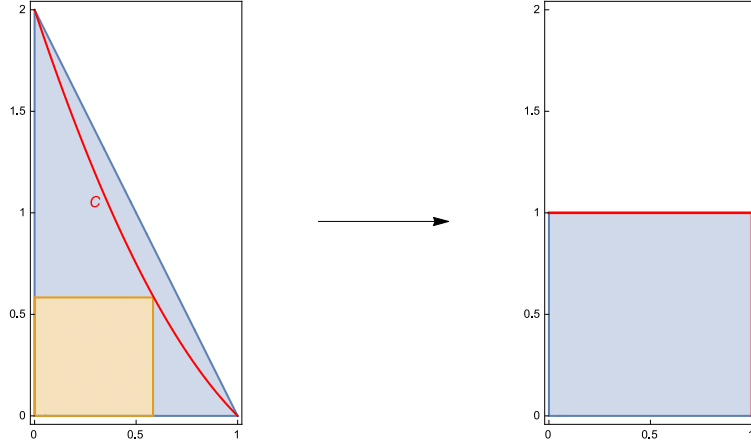


FIGURE 4. After appropriate rescalings, the map Φ from Proposition 3.12 sends the interior of the ellipsoid $E(1, 2)$ to a product of spheres of area 1, with the preimage of $(S^2 \times \{(0, 0, -1)\}) \cup (\{(0, 0, -1)\} \times S^2)$ contained in $\mu^{-1}(C)$ where C is the red curve at left. Consequently the preimage under μ of any domain lying below C , such as the small square at left, is embedded into the polydisk $P(1, 1)^\circ$ by a rescaling of $(\sigma \times \sigma) \circ \Phi$. This gives an explicit knotted embedding $P(c, c) \hookrightarrow P(1, 1)^\circ$ for $1/2 < c < 2 - \sqrt{2}$.

(see Figure 3.2). So we obtain an embedding $(\sigma \times \sigma) \circ \Phi: P(2\pi c, 2\pi c) \hookrightarrow P(4\pi, 4\pi)^\circ = \frac{2}{c}P(2\pi c, 2\pi c)^\circ$, which is knotted provided that $\frac{2}{c} < \delta_{\text{ell}}^u(P(2\pi c, 2\pi c))$. By Theorem 1.12 we have $\delta_{\text{ell}}^u(P(a, a)) \geq 2$ for any a , so our embedding is knotted provided that $1 < c < 4 - 2\sqrt{2}$. So after conjugating by appropriate rescalings our explicit embedding $(\sigma \times \sigma) \circ \Phi$ defines a knotted embedding $P(a, a) \hookrightarrow \alpha P(a, a)^\circ$ provided that $2 > \alpha > \frac{1}{2 - \sqrt{2}} \approx 1.71$. For comparison, our less explicit construction based on Proposition 3.5 (leading to the bound $\delta_{\text{ell}}(P(a, a)) \leq 3/2$ from Theorem 1.13) gives knotted embeddings $P(a, a) \hookrightarrow \alpha P(a, a)^\circ$ whenever $2 > \alpha > 1.5$.

Choosing the scaling so that the codomain is $P(4\pi, 4\pi)^\circ$, the image of this embedding $\alpha^{-1}P(4\pi, 4\pi) \hookrightarrow P(4\pi, 4\pi)^\circ$ is not hard to describe explicitly as a subset of $P(4\pi, 4\pi)^\circ$: it is given as the region

$$\{(z_1, z_2) \in P(4\pi, 4\pi)^\circ \mid G_2(z_1, z_2) \geq 2 - 2/\alpha, -G_1(z_1, z_2) + G_2(z_1, z_2) \leq 2/\alpha\},$$

where $G_i = F_i \circ (\sigma \times \sigma)^{-1}$, i.e.,

$$G_1(z_1, z_2) = 2 - \frac{|z_1|^2 + |z_2|^2}{2}$$

and

$$G_2(z_1, z_2)^2 = \left(\sqrt{1 - \frac{|z_1|^2}{4}} \operatorname{Re}(z_1) + \sqrt{1 - \frac{|z_2|^2}{4}} \operatorname{Re}(z_2) \right)^2 \\ + \left(\sqrt{1 - \frac{|z_1|^2}{4}} \operatorname{Im}(z_1) + \sqrt{1 - \frac{|z_2|^2}{4}} \operatorname{Im}(z_2) \right)^2 + \left(2 - \frac{|z_1|^2 + |z_2|^2}{2} \right)^2.$$

Corollary 3.15 also applies to some other convex toric domains besides the cube $P(a, a)$, though it is not as broadly applicable as Theorem 1.8. For example the reader may check that, in Corollary 3.15, for appropriate α one can take X_Ω equal to a polydisk $P(1, a)$ with $1 \leq a \leq 1.2$, or to an appropriately rescaled ℓ^p ball as in Theorem 1.8 for $p \geq 10$.

Remark 3.17. By construction, the embedding Φ from Proposition 3.12 maps the torus $T_{\sqrt{2}} := \{(w, z) \in \mathbb{C}^2 \mid |w| = |z| = \sqrt{2}\}$ to the Lagrangian torus in $S^2 \times S^2$ that is denoted K in [EP09, Example 1.22], and which can be identified with the Chekanov-Schlenk twist torus Θ , see [CS10],[OU16]. Since, as shown in [EP09], there is no symplectomorphism mapping K to the Clifford torus in $S^2 \times S^2$ (i.e., to the image of $T_{\sqrt{2}}$ under the standard embedding $(\sigma \times \sigma)^{-1}$ of $P(4\pi, 4\pi)^\circ$ into $S^2 \times S^2$), one easily infers independently of our other results that $(\sigma \times \sigma) \circ \Phi: P(2\pi c, 2\pi c) \hookrightarrow P(4\pi, 4\pi)^\circ$ must not be isotopic to the inclusion by a compactly supported Hamiltonian isotopy for $1 < c < 4 - 2\sqrt{2}$ (for such a Hamiltonian isotopy could be extended to $S^2 \times S^2$, giving a symplectomorphism that would send K to the Clifford torus). However this argument based on Lagrangian tori does not seem to adapt to yield the full result that $(\sigma \times \sigma) \circ \Phi$ is knotted in the stronger sense of Definition 1.7.

By the way, if $c < 1$, our embedding $(\sigma \times \sigma) \circ \Phi: P(2\pi c, 2\pi c) \hookrightarrow P(4\pi, 4\pi)^\circ$ is unknotted. Indeed in this case the ball $B^4(4\pi c)$ is contained both in $P(4\pi, 4\pi)^\circ$ and in $E(4\pi, 8\pi) \setminus \Phi^{-1}(\mathcal{A})$, and so both $(\sigma \times \sigma) \circ \Phi|_{P(2\pi c, 2\pi c)}$ and the inclusion $P(2\pi c, 2\pi c) \hookrightarrow P(4\pi, 4\pi)^\circ$ extend to embeddings $B^4(4\pi c, 4\pi c) \hookrightarrow P(4\pi, 4\pi)^\circ$; these two embeddings of the ball are symplectically isotopic by [C14, Proposition 1.5]. Thus a transition between knottedness and unknottedness occurs at the value $c = 1$, which is precisely the first value for which $P(2\pi c, 2\pi c)$ contains the torus $T_{\sqrt{2}}$ mentioned at the start of the remark.

Remark 3.18. A similar construction to that in Proposition 3.12, using results from [OU16, Section 3], allows one to construct a symplectic embedding of $E(3\pi, 12\pi)^\circ$ into $\mathbb{C}P^2$ where the symplectic form on $\mathbb{C}P^2$ is normalized to give area 6π to a complex projective line, such that the torus $T_{\sqrt{2}}$ is sent to the $\mathbb{C}P^2$ version of the Chekanov-Schlenk twist torus Θ . Combining this with a symplectomorphism from the complement of a line in $\mathbb{C}P^2$ to a ball and restricting to $P(2\pi c, 2\pi c)$ for c slightly larger than 1, we obtain a symplectic embedding $P(2\pi c, 2\pi c) \hookrightarrow B^4(6\pi)^\circ$ which cannot be Hamiltonian isotopic to the inclusion because Θ is not Hamiltonian isotopic to the Clifford torus. It is less clear whether this embedding $P(2\pi c, 2\pi c) \hookrightarrow B^4(6\pi)^\circ$ is knotted in the sense of Definition 1.7; the symplectic-homology-based methods in the present paper seem

ill-equipped to address this because the filtered positive S^1 -equivariant symplectic homology of $B^4(6\pi)$ does not have as rich a structure as that of the domains X that appear in Theorem 1.8.

4. MORE KNOTTED POLYDISKS

The lower bounds on δ_{ell}^u that are used to show that our embeddings $X \hookrightarrow \alpha X^\circ$ are knotted are generally based on showing that, for suitable k, L , the maps $\Phi^L: CH_k^L(\alpha X^\circ, \lambda_0) \rightarrow CH_k^L(X^\circ, \lambda_0)$ have sufficiently large image and then appealing to Corollary 2.19. One can in principle use Corollary 2.19 to prove the knottedness of embeddings $X \hookrightarrow V$ for more general star-shaped open subsets V which are not dilates of X° ; the main difficulty in this case is that one can no longer simply appeal to Lemma 2.11 in order to estimate the rank of $\Phi^L: CH_k^L(V, \lambda_0) \rightarrow CH_k^L(X^\circ, \lambda_0)$.

In this section we carry this procedure out when X and V are four-dimensional polydisks $P(a_0, b_0), P(a_1, b_1)^\circ$, typically with $\frac{b_0}{a_0} \neq \frac{b_1}{a_1}$.

A polydisk $P(a, b)$ is the toric domain associated to the rectangle $R_{a,b} = [0, a] \times [0, b]$, which has $\|(x, y)\|_{R_{a,b}}^* = ax + by$, so by Lemma 2.12 we see that $\dim CH_3^L(P(a, b)^\circ, \lambda_0) = 2$ whenever $\max\{a, b\} < L < a + b$, and that

(4.1)

$$\iota_{L_1, L_2}: CH_3^{L_1}(P(a, b)^\circ, \lambda_0) \rightarrow CH_3^{L_2}(P(a, b)^\circ, \lambda_0) \text{ is an isomorphism for } \max\{a, b\} < L_1 < L_2 < a + b.$$

Lemma 4.1. Assume that $a \leq b < b'$ and that $b' < L < a + b$. Then the transfer map $\Phi^L: CH_3^L(P(a, b')^\circ, \lambda_0) \rightarrow CH_3^L(P(a, b)^\circ, \lambda_0)$ is an isomorphism.

Proof. Choose $\delta > 0$ such that $(1 + \delta)L < a + b$ and such that, for some $N \in \mathbb{N}$, $(1 + \delta)^N b = b'$. Consider any $c \in [b, b']$. Lemma 2.11 gives a commutative diagram

$$\begin{array}{ccc} & CH_3^L((1 + \delta)^{-1}P(a, c)^\circ, \lambda_0) & \\ & \nearrow \Phi^L & \downarrow \cong \\ CH_3^L(P(a, c)^\circ, \lambda_0) & & \\ & \searrow \iota_{L, (1+\delta)L} & \\ & CH_3^{(1+\delta)L}(P(a, c)^\circ, \lambda_0) & \end{array}$$

where $\iota_{L, (1+\delta)L}$ is an isomorphism by Lemma 2.12 since our assumptions give $a \leq b \leq c \leq b' < L < (1 + \delta)L < a + b \leq a + c$. Thus $\Phi^L: CH_3^L(P(a, c)^\circ, \lambda_0) \rightarrow CH_3^L(P((1 + \delta)^{-1}a, (1 + \delta)^{-1}c)^\circ, \lambda_0)$ is an isomorphism. But this latter map factors as a composition

$$CH_3^L(P(a, c)^\circ, \lambda_0) \xrightarrow{\Phi^L} CH_3^L(P(a, (1 + \delta)^{-1}c)^\circ, \lambda_0) \xrightarrow{\Phi^L} CH_3^L(P((1 + \delta)^{-1}a, (1 + \delta)^{-1}c)^\circ, \lambda_0),$$

so since all three vector spaces above have dimension two it follows that, again for any $c \in [b, b']$, $\Phi^L: CH_3^L(P(a, c)^\circ, \lambda_0) \rightarrow CH_3^L(P(a, (1 + \delta)^{-1}c), \lambda_0)$ is an isomorphism.

Since $(1 + \delta)^{-N}b' = b$, we may apply this successively with $c = b', (1 + \delta)^{-1}b', \dots, (1 + \delta)^{-(N-1)}b'$ and appeal to the functoriality of Φ^L to see that $\Phi^L: CH_3^L(P(a, b')^\circ, \lambda_0) \rightarrow CH_3^L(P(a, b)^\circ, \lambda_0)$ is an isomorphism. \square

A similar argument gives:

Lemma 4.2. Assume that $a < a' \leq b$ and that $b < L < a + b$. Then the transfer map $\Phi^L: CH_3^L(P(a', b)^\circ, \lambda_0) \rightarrow CH_3^L(P(a, b)^\circ, \lambda_0)$ is an isomorphism.

Proof. Analogously to the proof of Lemma 4.1, choose $\delta > 0$ such that $(1 + \delta)L < a + b$ and, for some $N \in \mathbb{N}$, $(1 + \delta)^N a = a'$. Using Lemma 2.11 and (4.1), we find that for all $c \in [a, a']$ the transfer map $\Phi^L: CH_3^L(P(c, b)^\circ, \lambda_0) \rightarrow CH_3^L(P((1 + \delta)^{-1}c, (1 + \delta)^{-1}b)^\circ, \lambda_0)$ is an isomorphism. Since this map factors through $CH_3^L(P(1 + \delta)^{-1}c, b)^\circ, \lambda_0)$, we deduce by dimensional considerations that $\Phi^L: CH_3^L(P(c, b)^\circ, \lambda_0) \rightarrow CH_3^L(P((1 + \delta)^{-1}c, b)^\circ, \lambda_0)$ is an isomorphism for all $c \in [a, a']$. Just as in the proof of Lemma 4.1, iterating this for $c = a', (1 + \delta)^{-1}a', \dots, (1 + \delta)^{-(N-1)}a'$ yields the result. \square

Proposition 4.3. Assume that $a_1 \leq b_1$, that $a_0 \leq a_1$ and that $b_0 \leq b_1$. For any L with $b_1 < L < a_0 + b_0$, the transfer map $\Phi^L: CH_3^L(P(a_1, b_1)^\circ, \lambda_0) \rightarrow CH_3^L(P(a_0, b_0)^\circ, \lambda_0)$ is an isomorphism (and so in particular has rank two).

Proof. If $a_0 \leq b_0$ and $\frac{b_1}{a_1} \geq \frac{b_0}{a_0}$, then we can factor $\Phi^L: CH_3^L(P(a_1, b_1)^\circ) \rightarrow CH_3^L(P(a_0, b_0)^\circ)$ as a composition

$$CH_3^L(P(a_1, b_1)^\circ) \xrightarrow{\Phi^L} CH_3^L\left(P\left(a_1, \frac{a_1 b_0}{a_0}\right)^\circ\right) \xrightarrow{\Phi^L} CH_3^L(P(a_0, b_0)^\circ)$$

where the first map is an isomorphism by Lemma 4.1 and the second is an isomorphism by (4.1) and Lemma 2.11 (which identifies the map with the inclusion-induced map $\iota_{\frac{a_0}{a_1}L, L}^{\frac{a_0}{a_1}L}: CH_3^{\frac{a_0}{a_1}L}(P(a_0, b_0)^\circ, \lambda_0) \rightarrow CH_3^L(P(a_0, b_0)^\circ, \lambda_0)$).

Similarly, if $a_0 \leq b_0$ and $\frac{b_1}{a_1} \leq \frac{b_0}{a_0}$, then we can factor $\Phi^L: CH_3^L(P(a_1, b_1)^\circ, \lambda_0) \rightarrow CH_3^L(P(a_0, b_0)^\circ, \lambda_0)$ as a composition

$$CH_3^L(P(a_1, b_1)^\circ, \lambda_0) \xrightarrow{\Phi^L} CH_3^L\left(P\left(\frac{a_0 b_1}{b_0}, b_1\right)^\circ, \lambda_0\right) \xrightarrow{\Phi^L} CH_3^L(P(a_0, b_0)^\circ, \lambda_0)$$

where the first map is an isomorphism by Lemma 4.2 and the second is an isomorphism by (4.1) and Lemma 2.11.

We have now proven the result whenever $a_0 \leq b_0$. If instead $a_0 > b_0$, then the hypotheses imply that $P(a_0, b_0) \subset P(a_0, a_0) \subset P(a_1, b_1)$, and that $a_0 \leq b_1 < L < a_0 + b_0 < 2a_0$. We can then factor the map in question as

$$CH_3^L(P(a_1, b_1)^\circ, \lambda_0) \xrightarrow{\Phi^L} CH_3^L(P(a_0, a_0)^\circ, \lambda_0) \xrightarrow{\Phi^L} CH_3^L(P(a_0, b_0)^\circ, \lambda_0)$$

where the first map is an isomorphism by a case of the present corollary that we have already proven, and the second is an isomorphism by Lemma 4.2 (after conjugating by a symplectomorphism that switches the factors of \mathbb{C}^2). \square

Corollary 4.4. If $E \subset \mathbb{R}^4$ is an ellipsoid, if $g: E \rightarrow P(c, d)^\circ$ is a symplectic embedding where $c \leq d$, and if $P(a, b) \subset E \cap P(c, d)^\circ$, then $g|_{P(a, b)}$ is knotted provided that $d < a + b$.

Proof. If $g|_{P(a, b)}$ were unknotted, we could apply Corollary 2.19 with f equal to the inclusion, with $k = 3$, and with L equal to any number with $d < L < a + b$. This would yield $\text{Rank}(\Phi^L: CH_3^L(P(c, d)^\circ, \lambda_0) \rightarrow CH_3^L(P(a, b)^\circ, \lambda_0)) \leq 1$, in contradiction with Proposition 4.3. \square

Remark 4.5. In the case that $\max\{a, b\} < c$, so that $P(c, d)^\circ$ contains both $P(a, b)$ and $P(b, a)$, then one example of a symplectic embedding $P(a, b) \hookrightarrow P(c, d)^\circ$ is $\sigma: (z_1, z_2) \mapsto (z_2, z_1)$, which has image equal to $P(b, a)$. In [FHW94, Theorem 4] it is shown that, when $c = d < a + b$, this embedding is not Hamiltonian isotopic to the inclusion within $P(c, c)^\circ$. However our definition of knottedness is such that (when $c = d$) this embedding would be considered unknotted, because the symplectomorphism of $P(c, c)^\circ$ which swaps the factors maps $P(a, b)$ to $P(b, a)$ (and we do not require our ambient symplectomorphisms to be induced by Hamiltonian isotopies supported in the codomain). Likewise when $a = b$ but $c \neq d$, σ is unknotted according to our definition because we take knottedness to depend only on the image of the embedding.

In the situation that both $a \neq b$ and $c < d$ (and still $\max\{a, b\} \leq c$ and $d < a + b$) it can be shown that the above embedding $\sigma: P(a, b) \rightarrow P(c, d)$ with image $P(b, a)$ is knotted. More specifically, by using arguments like those in [FHW94, Section 3.3] one can show that for $a < L_1 < b$ and $c < L_2 < d$ the inclusion-induced map $SH_3^{[L_1, L_2]}(P(c, d)^\circ) \rightarrow SH^{[L_1, L_2]}(P(a, b)^\circ)$ on action-window symplectic homology vanishes, while the inclusion-induced map $SH_3^{[L_1, L_2]}(P(c, d)^\circ) \rightarrow SH^{[L_1, L_2]}(P(b, a)^\circ)$ is nontrivial, which is sufficient to show that $P(a, b)$ cannot be mapped to $P(b, a)$ by a symplectomorphism of $P(c, d)^\circ$; we omit the details.

However because Proposition 4.3 shows that, for $d < L < a + b$, the map $\Phi^L: CH_3^L(P(c, d)^\circ, \lambda_0) \rightarrow CH_3^L(P(b, a)^\circ, \lambda_0)$ has rank two, the embeddings described by Corollary 4.4 (for which Φ^L has rank one) have different knot types from σ . (In other words, the image of such an embedding is not taken by a symplectomorphism of $P(c, d)^\circ$ to either one of $P(a, b)$ or $P(b, a)$.) In particular this comment applies to the embeddings in Corollaries 4.6 and 4.7 in each case that the target contains the image of the domain under σ .

Corollary 4.6. Let $m \in \mathbb{Z}_+$ and $0 \leq \varepsilon < 1$. If $a + \frac{b}{2m+\varepsilon} < 1$ and $0 \leq a \leq b < m + \varepsilon < a + b$ then there is a knotted embedding of $P(a, b)$ into $P(1, m + \varepsilon)^\circ$.

Proof. Choose μ such that $a + \frac{b}{2m+\varepsilon} < \mu < 1$; we then have $P(a, b) \subset (\mu E(1, 2m + \varepsilon)) \cap P(1, m + \varepsilon)^\circ$. Proposition 3.8 moreover gives a symplectic embedding $\mu E(1, 2m + \varepsilon) \hookrightarrow P(1, m + \varepsilon)^\circ$. The conclusion then follows from Corollary 4.4. \square

We conclude by restating and proving Theorem 1.10:

Corollary 4.7. Given any $y \geq 1$, there exist polydisks $P(a, b)$ and $P(c, d)$ and knotted embeddings of $P(a, b)$ into $P(1, y)^\circ$ and of $P(1, y)$ into $P(c, d)^\circ$.

Proof. For a knotted embedding $P(a, b) \hookrightarrow P(1, y)^\circ$, write $y = m + \varepsilon$ where $m \in \mathbb{Z}_+$ and $0 \leq \varepsilon < 1$. We can then set $a = \frac{1}{2}$ and $b = m + \frac{3\varepsilon-1}{4}$ and apply Corollary 4.6.

For a knotted embedding $P(1, y) \hookrightarrow P(c, d)^\circ$, write $y = 2k + \delta$ where $k \in \mathbb{Z}_+$ and $-1 \leq \delta < 1$. If $\delta \geq 0$, then Corollary 4.6 gives a knotted embedding of $P(\mu, \mu y)$ into $P(1, k + \frac{\delta}{2})^\circ$ for any μ with

$$\frac{1}{2} \frac{2k + \delta}{2k + \delta + 1} < \mu < \frac{4k + \delta}{8k + 3\delta},$$

and so conjugating by a rescaling by μ gives the desired embedding (with $c = \frac{1}{\mu}$, $d = \frac{1}{\mu} (k + \frac{\delta}{2}) = \frac{y}{2\mu}$). If instead $-1 \leq \delta < 0$, then for $1 + \frac{\delta}{4k} < \alpha < 1 + \frac{1+\delta}{2k}$ Corollary 4.6 (with $m = k, \varepsilon = 0$) gives a knotted embedding of $\frac{1}{2\alpha} P(1, y)$ into $P(1, k)^\circ$, and so again conjugating by a rescaling gives the desired embedding with $c = 2\alpha, d = 2\alpha k$. \square

APPENDIX A. PROOF OF LEMMA 2.1

The purpose of this appendix is to prove Lemma 2.1. This lemma seems to be widely expected, but some care is required to give a full proof. A related statement is proven in [GG16] for a slightly different version of S^1 -equivariant symplectic homology; the main difference between our result and theirs is that they construct a filtered complex after choosing a certain action interval and prove that their complex computes the filtered S^1 -equivariant symplectic homology associated to this action interval, whereas we construct a single complex that works simultaneously for all action intervals. One can in fact show based on arguments similar to those below that the filtration on the complex constructed in [GG16] in the case of the action interval $(0, \infty)$ does have filtered homologies that recover their version of filtered CH in arbitrary action intervals, but since this is not explicitly proven in [GG16] we give a detailed proof in our case.

The main ingredient is an algebraic lemma concerning filtered complexes which shows that, up to isomorphism, the images of inclusion-induced maps between the filtered parts of the complexes can be recovered from the filtered homology of a new chain complex whose underlying vector space is the E^1 term of the spectral sequence associated to the original filtered complex. This lemma is proven in the following section, and in the subsequent section we apply this together with results from [G15],[GH17] to complete the proof of Lemma 2.1. We assume that the reader is familiar with positive S^1 -equivariant symplectic homology and we use the notation from [GH17].

A.1. A lemma on filtered complexes. In this section we consider a \mathbb{Z} -graded chain complex (C_*, ∂) of vector spaces over a field K equipped with a filtration

$$\{0\} = F_0 C_* \subset F_1 C_* \subset \cdots \subset F_r C_* \subset \cdots \subset C_*$$

(where each $F_r C_*$ is a subcomplex of C_*) that is bounded below by zero and exhausting (i.e. $F_\infty C_* := \cup_r F_r C_*$ is equal to C_*). We extend the above filtration by \mathbb{N} to a filtration by \mathbb{Z} by setting $F_i C_* = \{0\}$ for $i < 0$.

Recall that the associated graded complex of (C_*, ∂) , denoted $\mathcal{G}(C_*)$, is the direct sum of quotient complexes $\bigoplus_{p \geq 1} \frac{F_p C_*}{F_{p-1} C_*}$, equipped with obvious boundary operator induced from ∂ . The homology $H_*(\mathcal{G}(C_*))$ evidently splits as a direct sum

$$H_k(\mathcal{G}(C_*)) = \bigoplus_{p \geq 1} H_k\left(\frac{F_p C_*}{F_{p-1} C_*}\right).$$

The following is the main algebraic input needed for Lemma 2.1:

Lemma A.1. With notation and assumptions as above, there is a chain complex (D_*, δ) equipped with a filtration

$$\{0\} = F_0 D_* \subset F_1 D_* \subset \cdots \subset F_r D_* \subset \cdots \subset D_*$$

where for each r, k

$$(A.1) \quad F_r D_k = \bigoplus_{1 \leq p \leq r} H_k\left(\frac{F_p C_*}{F_{p-1} C_*}\right)$$

and $F_\infty D_* := \cup_r F_r D_* = D_*$, such that the boundary operator δ on D_* strictly lowers filtration in the sense that $\delta(F_r D_*) \subset F_{r-1} D_*$, and such that for $1 \leq s \leq t \leq \infty$ there exists an isomorphism of vector spaces

$$\text{Im}(H_k(F_s C_*, \partial) \rightarrow H_k(F_t C_*, \partial)) \cong \text{Im}(H_k(F_s D_*, \delta) \rightarrow H_k(F_t D_*, \delta))$$

where the maps on both sides are induced by inclusion of filtered subcomplexes.

The proof of Lemma A.1 will occupy the rest of this section. To begin, let us recall from [Wei94, Section 5.4] some ingredients in the construction of the spectral sequence associated to the filtration on (C_*, ∂) .

For $p \in \mathbb{Z}$ write $\eta_p: F_p C_* \rightarrow \frac{F_p C_*}{F_{p-1} C_*}$ for the natural projection, and for $p, q, r \in \mathbb{Z}$ define:

$$A_{p,q}^r = \{x \in F_p C_{p+q} \mid \partial x \in F_{p-r} C_{p+q-1}\},$$

$$\hat{Z}_{p,q}^r = \eta_p(A_{p,q}^r), \quad \hat{B}_{p,q}^r = \eta_p(\partial(A_{p+r-1,q-r+2}^{r-1})).$$

For any $r \geq 1$ one then has inclusions

$$\{0\} = \hat{B}_{p,q}^0 \subset \hat{B}_{p,q}^1 \subset \cdots \subset \hat{B}_{p,q}^r \subset \hat{B}_{p,q}^{r+1} \subset \hat{Z}_{p,q}^{r+1} \subset \hat{Z}_{p,q}^r \subset \cdots \subset \hat{Z}_{p,q}^0 = \frac{F_p C_{p+q}}{F_{p-1} C_{p+q}}.$$

We also write

$$\hat{B}_{p,q}^\infty = \cup_{r=1}^\infty \hat{B}_{p,q}^r = \cup_{r=1}^\infty \eta_p(\partial(A_{p+r-1,q-r+2}^{r-1})).$$

Note also that since we assume that $F_i C_* = \{0\}$ for $i \leq 0$, we have

$$\hat{Z}_{p,q}^r = \hat{Z}_{p,q}^p = \eta_p(\ker \partial|_{F_p C_{p+q}}) \text{ for } r \geq p.$$

Accordingly if we let $\hat{Z}_{p,q}^\infty = \hat{Z}_{p,q}^p$ then we will have

$$\hat{Z}_{p,q}^\infty = \bigcap_{r=1}^\infty \hat{Z}_{p,q}^r.$$

As is standard, we write

$$E_{p,q}^r = \frac{\hat{Z}_{p,q}^r}{\hat{B}_{p,q}^r}$$

for $r \in \mathbb{N} \cup \{\infty\}$. For the case that $r = 1$, notice that $\hat{Z}_{p,q}^1$ is equal to the set of degree- $(p+q)$ cycles in the quotient complex $\frac{F_p C_*}{F_{p-1} C_*}$ and that $\hat{B}_{p,q}^1$ is equal to the set of degree- $(p+q)$ boundaries in $\frac{F_p C_*}{F_{p-1} C_*}$; thus

$$(A.2) \quad E_{p,q}^1 = H_{p+q} \left(\frac{F_p C_*}{F_{p-1} C_*} \right).$$

The following is standard and easily-checked:

Proposition A.2. (cf. [Wei94, Construction 5.4.6]) For each p, q, r , the boundary operator ∂ induces a map

$$\hat{\partial}_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$$

such that

$$\ker(\hat{\partial}_{p,q}^r) = \pi(\hat{Z}_{p,q}^{r+1}) \quad \text{and} \quad \text{Im}(\hat{\partial}_{p,q}^r) = \pi(\hat{B}_{p,q}^{r+1}),$$

where $\pi : \hat{Z}_{p,q}^r \rightarrow \frac{\hat{Z}_{p,q}^r}{\hat{B}_{p,q}^r}$ is the quotient projection.

We also have the following fact concerning the maps $H_k(F_p C_*, \partial) \rightarrow H_k(F_t C_*, \partial)$ for $p \leq t$ induced by inclusion of filtered subcomplexes; this is a slight extension of the familiar fact that the spectral sequence of a suitable filtered complex converges to the associated graded of the homology.

Proposition A.3. Let $1 \leq p \leq t \leq \infty$ with $p < \infty$. Then there is an isomorphism

$$\frac{\text{Im}(H_k(F_p C_*, \partial) \rightarrow H_k(F_t C_*, \partial))}{\text{Im}(H_k(F_{p-1} C_*, \partial) \rightarrow H_k(F_t C_*, \partial))} \cong \frac{\hat{Z}_{p,k-p}^\infty}{\hat{B}_{p,k-p}^{t-p+1}}.$$

(Here for the case $t = \infty$ we interpret $F_\infty C_*$ as C_* and $\hat{B}_{p,k-p}^{\infty-p+1}$ as $\hat{B}_{p,k-p}^\infty$.)

Proof. There is an obvious surjective map

$$\phi : \ker(\partial|_{F_p C_k}) \rightarrow \frac{\text{Im}(H_k(F_p C_*, \partial) \rightarrow H_k(F_t C_*, \partial))}{\text{Im}(H_k(F_{p-1} C_*, \partial) \rightarrow H_k(F_t C_*, \partial))}$$

given by including $\ker(\partial|_{F_p C_k})$ into $\ker(\partial|_{F_t C_k})$, then taking homology classes, and then projecting. We see that $x \in \ker(\phi)$ if and only if there is $y \in \ker(\partial|_{F_{p-1} C_k})$ such that x and y represent the same homology class in $H_k(F_t C_*, \partial)$; this holds if and only if we can write $x = y + \partial z$ with $z \in F_t C_{k+1}$, and in this case we

would have $z \in A_{t,k-t+1}^{t-p}$ since $\partial z = x - y \in F_p C_k$. Thus $\ker(\phi) = \ker(\partial|_{F_{p-1}C_k}) + \partial(A_{t,k-t+1}^{t-p})$ and hence

$$(A.3) \quad \frac{\operatorname{Im}(H_k(F_p C_*, \partial) \rightarrow H_k(F_t C_*, \partial))}{\operatorname{Im}(H_k(F_{p-1} C_*, \partial) \rightarrow H_k(F_t C_*, \partial))} \cong \frac{\ker(\partial|_{F_p C_k})}{\ker(\partial|_{F_{p-1} C_k}) + \partial(A_{t,k-t+1}^{t-p})}.$$

(The above discussion implicitly assumes that $t < \infty$, but since $\cup_{s=1}^{\infty} F_s C_* = C_*$ the reasoning is equally valid for $t = \infty$ provided that we interpret the notation $A_{\infty,k-\infty+1}^{t-p}$ as $\cup_{p \leq s \in \mathbb{N}} A_{s,k-s+1}^{s-p}$, as we will continue to do below).

On the other hand the projection $\eta_p: F_p C_k \rightarrow \frac{F_p C_k}{F_{p-1} C_k}$ sends $\ker(\partial|_{F_p C_k})$ to $\hat{Z}_{p,k-p}^{\infty}$ and sends $\ker(\partial|_{F_{p-1} C_k}) + \partial(A_{t,k-t+1}^{t-p})$ to $\hat{B}_{p,k-p}^{t-p+1}$, and it is easy to check that the resulting map

$$\eta: \frac{\ker(\partial|_{F_p C_k})}{\ker(\partial|_{F_{p-1} C_k}) + \partial(A_{t,k-t+1}^{t-p})} \rightarrow \frac{\hat{Z}_{p,k-p}^{\infty}}{\hat{B}_{p,k-p}^{t-p+1}}$$

is an isomorphism. Combining this isomorphism with (A.3) proves the proposition. \square

For $1 \leq r \leq \infty$ let

$$B_{p,q}^r = \frac{\hat{B}_{p,q}^r}{\hat{B}_{p,q}^1}, \quad Z_{p,q}^r = \frac{\hat{Z}_{p,q}^r}{\hat{B}_{p,q}^1},$$

so for $r < p$ we have a chain of inclusions

$$\{0\} = B_{p,q}^1 \subset \cdots \subset B_{p,q}^r \subset B_{p,q}^{r+1} \subset \cdots \subset B_{p,q}^{\infty} \subset Z_{p,q}^{\infty} = Z_{p,q}^p \subset Z_{p,q}^{r+1} \subset Z_{p,q}^r \subset \cdots \subset Z_{p,q}^1 = E_{p,q}^1.$$

Projecting away $\hat{B}_{p,q}^1$ induces isomorphisms $E_{p,q}^r \cong \frac{Z_{p,q}^r}{B_{p,q}^r}$. For each $p, q \in \mathbb{Z}$ and $r \geq 1$ let us choose:

- A complement $H_{p,q}^r$ to the subspace $B_{p,q}^r$ within the vector space $Z_{p,q}^r$, and
- A complement $M_{p,q}^r$ to the subspace $Z_{p,q}^{r+1}$ within the vector space $Z_{p,q}^r$.

Given these choices, the projection $Z_{p,q}^r \rightarrow E_{p,q}^r$ restricts to $H_{p,q}^r$ as an isomorphism, so the maps $\hat{\partial}_{p,q}^r$ from Proposition A.2 induce maps

$$\partial_{p,q}^r: H_{p,q}^r \rightarrow H_{p-r,q+r-1}^r$$

with

$$\ker \partial_{p,q}^r = Z_{p,q}^{r+1} \cap H_{p,q}^r, \quad \operatorname{Im} \partial_{p+r,q-r+1}^r = B_{p,q}^{r+1} \cap H_{p,q}^r.$$

(In particular, since $Z_{p,q}^{r+1} = Z_{p,q}^r$ for $r \geq p$, we have $\partial_{p,q}^r = 0$ for $r \geq p$).

For any $r \geq 2$ the various direct sum decompositions $Z_{p,q}^j = Z_{p,q}^j \oplus M_{p,q}^{j-1}$ yield a direct sum decomposition

$$\begin{aligned} E_{p,q}^1 &= Z_{p,q}^1 = Z_{p,q}^r \oplus M_{p,q}^{r-1} \oplus \cdots \oplus M_{p,q}^1 \\ &= H_{p,q}^r \oplus B_{p,q}^r \oplus M_{p,q}^{r-1} \oplus \cdots \oplus M_{p,q}^1. \end{aligned}$$

(For $r = 1$ we have $B_{p,q}^1 = \{0\}$ and $H_{p,q}^1 = E_{p,q}^1$ and the above direct sum decomposition degenerates to $E_{p,q}^1 = H_{p,q}^1$).

We accordingly extend our map $\partial_{p,q}^r : H_{p,q}^r \rightarrow H_{p,q}^r$ to a linear map (still denoted $\partial_{p,q}^r$) defined on all of $E_{p,q}^1$ by setting it equal to zero on the summands $B_{p,q}^r, M_{p,q}^{r-1}, \dots, M_{p,q}^1$. We also regard the codomain of $\partial_{p,q}^r$ as $E_{p-r,q+r-1}^1$ rather than the subspace $H_{p-r,q+r-1}^r$. With this extended definition, we have

$$\ker \partial_{p,q}^r = (Z_{p,q}^{r+1} \cap H_{p,q}^r) \oplus B_{p,q}^r \oplus M_{p,q}^{r-1} \oplus \dots \oplus M_{p,q}^1 = Z_{p,q}^{r+1} \oplus M_{p,q}^{r-1} \oplus \dots \oplus M_{p,q}^1,$$

where we have used that $B_{p,q}^r \subset Z_{p,q}^{r+1} \subset Z_{p,q}^r = H_{p,q}^r \oplus B_{p,q}^r$, so that $(Z_{p,q}^{r+1} \cap H_{p,q}^r) \oplus B_{p,q}^r = Z_{p,q}^{r+1}$. Since we have a direct sum decomposition

$$E_{p,q}^1 = Z_{p,q}^{r+1} \oplus M_{p,q}^r \oplus M_{p,q}^{r-1} \dots \oplus M_{p,q}^1,$$

it follows that:

Corollary A.4. The maps $\partial_{p,q}^r : E_{p,q}^1 \rightarrow E_{p-r,q+r-1}^1$ restrict as isomorphisms $M_{p,q}^r \rightarrow B_{p-r,q+r-1}^{r+1} \cap H_{p-r,q+r-1}^r$, and vanish identically on the complementary subspace $Z_{p,q}^{r+1} \oplus M_{p,q}^{r-1} \oplus \dots \oplus M_{p,q}^1$ to $M_{p,q}^r$ in $E_{p,q}^1$.

In particular, since for $j > r$ we have $M_{p,q}^j \subset Z_{p,q}^j \subset Z_{p,q}^{r+1} \subset \ker(\partial_{p,q}^r)$, this shows that $\partial_{p,q}^r$ vanishes on $M_{p,q}^j$ for $j \neq r$, while it maps $M_{p,q}^r$ isomorphically to $B_{p-r,q+r-1}^{r+1} \cap H_{p-r,q+r-1}^r$.

Now for any p, q let us write

$$\partial_{p,q} = \sum_{r \geq 1} \partial_{p,q}^r : E_{p,q}^1 \rightarrow \bigoplus_{r \geq 1} H_{p-r,q+r-1}^r \subset E_{p-r,q+r-1}^1.$$

(This has just finitely many nonzero terms since $\partial_{p,q}^r = 0$ for $r \geq p$.) Also define, for $k \in \mathbb{Z}$,

$$D_k = \bigoplus_{p+q=k} E_{p,q}^1,$$

and define $\delta_k : D_k \rightarrow D_{k-1}$ as the map which restricts to $\partial_{p,q}$ on the respective summands $E_{p,q}^1$. Each D_k has a filtration given by

$$F_s D_k = \bigoplus_{p+q=k, p \leq s} E_{p,q}^1,$$

which is consistent with (A.1) by (A.2). By definition, the map δ_k respects this filtration, and indeed satisfies the stronger property $\delta_k(F_s D_k) \subset F_{s-1} D_{k-1}$.

We will now compute the kernel and image of δ_k . For a general element $x = \sum_p x_p \in D_k$ where each $x_p \in E_{p,k-p}^1$, the component of $\delta_k x$ in the summand $E_{m,k-1-m}^1 \subset D_{k-1}$ is equal to

$$\sum_r \partial_{m+r,k-m-r}^r x_{m+r}.$$

Now $\partial_{m+r,k-m-r}^r x_{m+r}$ lies in the subspace $B_{m,k-1-m}^{r+1} \cap H_{m,k-1-m}^r$ of $E_{m,k-1-m}^1$. But these latter subspaces are independent as r varies: indeed given finitely many elements $y_r \in B_{m,k-1-m}^{r+1} \cap H_{m,k-1-m}^r$ that are not all zero, if r_{\max} is chosen maximal subject to the property that $y_{r_{\max}} \neq 0$ then the fact that $0 \neq y_{r_{\max}} \in H_{m,k-1-m}^{r_{\max}}$ while for all $s < r_{\max}$ we have $y_s \in B_{m,k-1-m}^{s+1} \subset B_{m,k-1-m}^{r_{\max}}$ would imply that $\sum y_r \neq 0$ since $H_{m,k-1-m}^{r_{\max}}$ is complementary to $B_{m,k-1-m}^{r_{\max}}$.

The independence of these subspaces implies that, for $x_p \in E_{p,k-p}^1$, the component of $\delta_k \left(\sum_p x_p \right)$ in $E_{m,k-m-1}^1$ is zero only if each $\partial_{m+r,k-m-r}^r x_{m+r}$ separately vanishes. Thus:

$$(A.4) \quad \sum_p x_p \in \ker \delta_k \Leftrightarrow (\forall p, r)(\partial_{p,k-p}^r x_p = 0).$$

Now fixing p and recalling that $Z_{p,k-p}^p = Z_{p,k-p}^\infty$ and $\partial_{p,k-p}^r = 0$ for $r \geq p$, note that we have

$$E_{p,k-p}^1 = Z_{p,k-p}^\infty \oplus M_{p,k-p}^{p-1} \oplus \cdots \oplus M_{p,k-p}^1.$$

Moreover, for $r < p$, $\partial_{p,k-p}^r$ vanishes on $Z_{p,k-p}^{r+1} \supset Z_{p,k-p}^\infty$ and on each $M_{p,k-p}^j$ for $j \neq r$ while restricting injectively to $M_{p,k-p}^r$. Hence $\partial_{p,k-p}^r x_p = 0$ for all r if and only if $x_p \in Z_{p,k-p}^\infty$. In combination with (A.4) this shows:

Proposition A.5.

$$\ker(\delta_k : D_k \rightarrow D_{k-1}) = \bigoplus_p Z_{p,k-p}^\infty$$

and, for each $s \in \mathbb{N}$,

$$\ker(\delta_k|_{F_s D_k}) = \bigoplus_{p \leq s} Z_{p,k-p}^\infty.$$

Next we will show:

Proposition A.6.

$$\text{Im}(\delta_k : D_k \rightarrow D_{k-1}) = \bigoplus_p B_{p,k-1-p}^\infty$$

and, for $s \in \mathbb{N}$,

$$\text{Im}(\delta_k|_{F_s D_k}) = \bigoplus_{p < s} B_{p,k-1-p}^{s-p+1}.$$

Proof. As noted earlier the summand of $\delta_k \left(\sum_p x_p \right)$ in $E_{m,k-1-m}^1$ is $\sum_r \partial_{m+r,k-m-r}^r x_{m+r}$, which is a sum of terms in the mutually independent subspaces $B_{m,k-1-m}^{r+1} \cap H_{m,k-1-m}^r$. Note that, for fixed k, m and any $t \in \mathbb{N}$,

$$(A.5) \quad \bigoplus_{1 \leq r \leq t} \left(B_{m,k-1-m}^{r+1} \cap H_{m,k-1-m}^r \right) = B_{m,k-1-m}^{t+1} :$$

indeed using the inclusions

$$B_{m,k-1-m}^r \subset B_{m,k-1-m}^{r+1} \subset Z_{m,k-1-m}^r = H_{m,k-1-m}^r \oplus B_{m,k-1-m}^r$$

we see that $B_{m,k-1-m}^{r+1} = (B_{m,k-1-m}^{r+1} \cap H_{m,k-1-m}^r) \oplus B_{m,k-1-m}^r$; applying this inductively starting from $B_{m,k-1-m}^1 = \{0\}$ yields (A.5). The same reasoning shows that $\bigoplus_{r=1}^{\infty} (B_{m,k-1-m}^{r+1} \cap H_{m,k-1-m}^r) = B_{m,k-1-m}^{\infty}$. Thus to prove the proposition it suffices to show that, given $p \in \mathbb{N}$ and elements $y_r \in B_{p-r,k+r-p-1}^{r+1} \cap H_{p-r,k+r-p-1}^r$ for $1 \leq r < p$, we can find a single $x \in E_{p,k-p}^1$ with $\partial_{p,k-p}^r x = y_r$ for each r . But this is an easy consequence of Corollary A.4: using the decomposition $E_{p,k-p}^1 = Z_{p,k-p}^{\infty} \oplus M_{p,k-p}^{p-1} \oplus \cdots \oplus M_{p,k-p}^1$ we can take x to be an element with trivial component in $Z_{p,k-p}^{\infty}$ and with component in each respective $M_{p,k-p}^r$ equal to a preimage of y_r under $\partial_{p,k-p}^r$. \square

Corollary A.7. Let $D_* = \bigoplus_k D_k$ and $\delta = \bigoplus_k \delta_k$. Then (D_*, δ) is a filtered chain complex whose total homology is given by

$$H_k(D_*, \delta) = \frac{\bigoplus_p Z_{p,k-p}^{\infty}}{\bigoplus_p B_{p,k-p}^{\infty}}.$$

Moreover, for $s \in \mathbb{N}, t \in \mathbb{N} \cup \{\infty\}$ with $s \leq t$ we have

$$\text{Im}(H_k(F_s D_*, \delta) \rightarrow H_k(F_t D_*, \delta)) = \frac{\bigoplus_{p \leq s} Z_{p,k-p}^{\infty}}{\bigoplus_{p \leq s} B_{p,k-p}^{t-p+1}}.$$

Proof. That (D_*, δ) is a chain complex simply results from Propositions A.5 and A.6 and the fact that $B_{p,q}^{\infty} \subset Z_{p,q}^{\infty}$; the computation of $H_k(D_*, \delta)$ likewise follows immediately. The computation of $\text{Im}(H_k(F_s D_*, \delta) \rightarrow H_k(F_t D_*, \delta))$ also follows because this image is essentially by definition equal to the quotient of $\ker(\delta|_{F_s D_k})$ by $\text{Im}(\delta|_{F_t D_{k+1}}) \cap F_s D_k$. (For the case that $t = s$, it perhaps also bears noting that $B_{s,k-s}^1 = \{0\}$, so that $\bigoplus_{p < s} B_{p,k-p}^{s-p+1} = \bigoplus_{p \leq s} B_{p,k-p}^{s-p+1}$.) \square

Lemma A.1 now follows almost immediately from Corollary A.7 and Proposition A.3. Indeed, projecting away $\hat{B}_{p,k-p}^1$ gives isomorphisms $\frac{\hat{Z}_{p,k-p}^r}{\hat{B}_{p,k-p}^r} \cong \frac{Z_{p,k-p}^r}{B_{p,k-p}^r}$ so Corollary A.7 and Proposition A.3 show that we have, whenever $s \in \mathbb{N}$ and $1 \leq s \leq t \leq \infty$,

$$\text{Im}(H_k(F_s D_*, \delta) \rightarrow H_k(F_t D_*, \delta)) \cong \bigoplus_{p=1}^s \frac{\text{Im}(H_k(F_p C_*, \partial) \rightarrow H_k(F_t C_*, \partial))}{\text{Im}(H_k(F_{p-1} C_*, \partial) \rightarrow H_k(F_t C_*, \partial))}.$$

Since $F_0 C_* = \{0\}$, we can then iteratively choose complements to $\text{Im}(H_k(F_{p-1} C_*, \partial) \rightarrow H_k(F_t C_*, \partial))$ in $\text{Im}(H_k(F_p C_*, \partial) \rightarrow H_k(F_t C_*, \partial))$ to obtain an isomorphism $\text{Im}(H_k(F_s D_*, \delta) \rightarrow H_k(F_t D_*, \delta)) \cong \text{Im}(H_k(F_s C_*, \partial) \rightarrow H_k(F_t C_*, \partial))$. Moreover in the case that $t = \infty$, as s varies this can be done in such a way that if $s < s'$ then the isomorphism $\text{Im}(H_k(F_{s'} D_*, \delta) \rightarrow H_k(D_*, \delta)) \cong \text{Im}(H_k(F_{s'} C_*, \partial) \rightarrow H_k(C_*, \partial))$ restricts to $\text{Im}(H_k(F_s D_*, \delta) \rightarrow H_k(D_*, \delta))$ as the already-chosen isomorphism

$\text{Im}(H_k(F_s D_*, \delta) \rightarrow H_k(D_*, \delta)) \cong \text{Im}(H_k(F_s C_*, \partial) \rightarrow H_k(C_*, \partial))$; hence by taking the union over s we obtain an isomorphism $H_k(D_*, \delta) \cong H_k(C_*, \delta)$ (corresponding to the case $s = t = \infty$ in Lemma A.1).

Since we have already seen that our complex (D_*, δ) satisfies the other required properties, this completes the proof of Lemma A.1.

A.2. Construction of $CC_*(X, \lambda)$. Since we assume that the Reeb flow on the boundary of (X, λ) is nondegenerate, the set of actions (equivalently, periods) of the Reeb orbits on ∂X is discrete; of course every element of this set is positive, so let us denote by $T_1 < T_2 < \dots < T_r < \dots$ the numbers which arise as actions of Reeb orbits on ∂X . Also write $T_0 = 0$. By [GH17, Proposition 3.1], the maps $\iota_{L_1, L_2}: CH^{L_1}(X, \lambda) \rightarrow CH^{L_2}(X, \lambda)$ give a directed system (i.e. $\iota_{L_2, L_3} \circ \iota_{L_1, L_2} = \iota_{L_1, L_3}$), and ι_{L_1, L_2} is an isomorphism if the interval $(L_1, L_2]$ does not contain any of the actions T_i . So in particular if $L \leq L'$ with $L \in [T_i, T_{i+1})$, $L' \in [T_j, T_{j+1})$ then there is a commutative diagram

$$(A.6) \quad \begin{array}{ccc} CH^L(X, \lambda) & \xrightarrow{\iota_{L, L'}} & CH^{L'}(X, \lambda) \\ \iota_{T_i, L} \uparrow \cong & & \cong \uparrow \iota_{T_j, L'} \\ CH^{T_i}(X, \lambda) & \xrightarrow{\iota_{T_i, T_j}} & CH^{T_j}(X, \lambda) \end{array}$$

where both vertical arrows are isomorphisms. So to understand the maps ι_{L_1, L_2} it suffices to understand the maps ι_{T_i, T_j} .

By definition ([GH17, Definition 6.1]), we have

$$CH^L(X, \lambda) = \varinjlim_{N, H} HF^{S^1, N, +, \leq L}(H, J)$$

where the direct limit is taken over parametrized Hamiltonians $H: S^1 \times \hat{X} \times S^{2N+1} \rightarrow \mathbb{R}$ on the Liouville completion \hat{X} of X that satisfy a certain admissibility condition, with the structure maps being given by parametrized versions of continuation maps associated to pairs $(N, H), (N', H')$ with $N \leq N', H \leq H'|_{S^1 \times \hat{X} \times S^{2N+1}}$. Here $HF^{S^1, N, +, \leq L}(H, J)$ is the homology of the subcomplex (which for brevity we will denote by $C(N, H)^L$) generated by orbits of symplectic action at most L of the positive equivariant Floer complex $C(N, H)^\infty := \frac{CF^{S^1, N, +}(N, H)}{CF^{S^1, N, +, \leq \varepsilon}(N, H)}$ where $0 < \varepsilon \ll T_1$. The maps $\iota_{L_1, L_2}: CH^{L_1}(X, \lambda) \rightarrow CH^{L_2}(X, \lambda)$ are by definition the maps induced on the direct limit by the maps $HF^{S^1, N, +, \leq L_1}(N, H) \rightarrow HF^{S^1, N, +, \leq L_2}(N, H)$ given by the inclusion of subcomplexes $C(N, H)^{L_1} \hookrightarrow C(N, H)^{L_2}$.

Suppose that $\{(N_i, H_i)\}_{i=1}^\infty$ is any cofinal, linearly ordered subset of the partially ordered set of pairs (N, H) used to define $CH^L(X, \lambda)$. We can then form the direct limit of the chain complexes $C(N_i, H_i)^\infty$, using as structure maps the compositions of chain level continuation maps $C(N_i, H_i)^\infty \rightarrow C(N_{i+1}, H_{i+1})^\infty$.

Denote this direct limit by \underline{C} . Since the continuation maps preserve the filtration by symplectic action, for any $L \in \mathbb{R}$ we likewise have a direct limit $\underline{C}^L = \varinjlim_i C(N_i, H_i)^L$, and the \underline{C}^L form an \mathbb{R} -valued filtration of \underline{C} .

Let us coarsen this \mathbb{R} -filtration to an \mathbb{N} filtration by, for each $p \in \mathbb{N}$, choosing T'_p with $T_p < T'_p < T_{p+1}$, and letting

$$F_p \underline{C} = \underline{C}^{T'_p}$$

(Recall our notation that $T_0 = 0$ and the T_p for $p > 0$ are the distinct actions of Reeb orbits along ∂X , in increasing order.) As in [GH17, Remark 5.6], for i sufficiently large every generator of $C(N_i, H_i)$ will have filtration level greater than T'_0 , so that $C(N_i, H_i)^{T'_0} = \{0\}$ for i sufficiently large and so $F_0 \underline{C} = \{0\}$. The fact that $\cup_p C(N_i, H_i)^{T'_p} = C(N_i, H_i)$ for each i implies that likewise $\cup_p F_p \underline{C} = \underline{C}$. All of our complexes are \mathbb{Z} -graded because of the assumption that $c_1(TX)|_{\tau_2(X)} = 0$. Thus Lemma A.1 applies to the filtered complex \underline{C} , producing a filtered complex (D_*, δ) with

$$F_r D_* = \bigoplus_{1 \leq p \leq r} H_* \left(\frac{\varinjlim_i C(N_i, H_i)^{T'_p}}{\varinjlim_i C(N_i, H_i)^{T'_{p-1}}} \right)$$

such that for each $k \in \mathbb{Z}, s \leq t$ we have

$$\mathrm{Im} \left(H_k(F_s D_*, \delta) \rightarrow H_k(F_t D_*, \delta) \right) \cong \mathrm{Im} \left(H_k(\underline{C}^s) \rightarrow H_k(\underline{C}^t) \right);$$

note that (for finite t) the right-hand side is precisely the image of ι_{T_s, T_t} in grading k . Also, since \varinjlim is an exact functor, we have

$$H_* \left(\frac{\varinjlim_i C(N_i, H_i)^{T'_p}}{\varinjlim_i C(N_i, H_i)^{T'_{p-1}}} \right) \cong \varinjlim_i H_* \left(\frac{C(N_i, H_i)^{T'_p}}{C(N_i, H_i)^{T'_{p-1}}} \right).$$

Thus we have a filtered complex (D_*, δ) whose r -filtered part is

$$F_r D_* = \varinjlim_i \bigoplus_{1 \leq p \leq r} H_* \left(\frac{C(N_i, H_i)^{T'_p}}{C(N_i, H_i)^{T'_{p-1}}} \right)$$

and such that, for $1 \leq s \leq t < \infty$,

(A.7)

$$\mathrm{Im} \left(\iota_{T'_s, T'_t} : CH_k^{T'_s}(X, \lambda) \rightarrow CH_k^{T'_t}(X, \lambda) \right) \cong \mathrm{Im} \left(H_k(F_s D_*, \delta) \rightarrow H_k(F_t D_*, \delta) \right).$$

The foregoing discussion applies to an arbitrary cofinal linearly ordered subset $\{(N_i, H_i)\}_{i=1}^\infty$ of the set of admissible pairs (N, H) . For a particular choice of such a cofinal subset consisting of Hamiltonians as described in [G15, Section 3.1] and [GH17, Remark 5.15], the homologies $H_* \left(\frac{C(N_i, H_i)^{T'_p}}{C(N_i, H_i)^{T'_{p-1}}} \right)$ are computed

in [G15, Section 3.2],[GH17, Section 6.7]. Namely, the space $H_* \left(\frac{C(N_i, H_i)^{T'_p}}{C(N_i, H_i)^{T'_{p-1}}} \right)$ is generated by elements $\tilde{\gamma}$ and $u^{N_i} \otimes \hat{\gamma}$ as γ ranges over good Reeb orbits having action equal to T_p ; writing CZ for the Conley–Zehnder index, the grading of $\tilde{\gamma}$ is $CZ(\gamma)$ and that of $u^{N_i} \otimes \hat{\gamma}$ is $CZ(\gamma) + 2N_i + 1$. The continuation maps $H_* \left(\frac{C(N_i, H_i)^{T'_p}}{C(N_i, H_i)^{T'_{p-1}}} \right) \rightarrow H_* \left(\frac{C(N_{i+1}, H_{i+1})^{T'_p}}{C(N_{i+1}, H_{i+1})^{T'_{p-1}}} \right)$ moreover map $\tilde{\gamma}$ to $\tilde{\gamma}$ and $u^{N_i} \otimes \hat{\gamma}$ to zero, as one can see based on [BO13a, Remark 3.7]. Thus in any given degree k the direct limit $\varinjlim_i H_k \left(\frac{C(N_i, H_i)^{T'_p}}{C(N_i, H_i)^{T'_{p-1}}} \right)$ has basis in bijection with the good Reeb orbits on ∂X of action T_p and Conley–Zehnder index k .

So the \mathbb{N} -filtered complex (D_*, δ) produced by Lemma A.1 has the property that $F_r D_k$ is the span of a set of generators in bijection with the good Reeb orbits on ∂X of Conley–Zehnder index k and action at most T_r . The complex $CC_*(X, \lambda)$ promised in Lemma 2.1 is then given by converting (D_*, δ) into an \mathbb{R} -filtered complex by taking the L -filtered part $CC_*^L(X, \lambda)$ to be equal $F_r D_*$ where r is maximal subject to the condition that $T_r \leq L$. In particular we have equalities $CC_*^L(X, \lambda) = CC_*^{L'}(X, \lambda)$ whenever $L, L' \in [T_r, T_{r+1})$. Since δ strictly decreases the \mathbb{N} -filtration on D_* , it likewise strictly decreases this \mathbb{R} -filtration.

By (A.7), we have isomorphisms

$$\mathrm{Im} \left(\iota_{T'_s, T'_t} : CH_k^{T'_s}(X, \lambda) \rightarrow CH_k^{T'_t}(X, \lambda) \right) \cong \mathrm{Im} \left(H_k(CC_*^{T'_s}(X, \lambda)) \rightarrow H_k(CC_*^{T'_t}(X, \lambda)) \right)$$

for $s \leq t$, and then by applying (A.6) we obtain a similar isomorphism with T'_s, T'_t replaced by arbitrary L, L' with $L \leq L'$. The special case that $L = L'$ shows that $CH_k^L(X, \lambda)$ is isomorphic to $H_k(CC_*^L(X, \lambda))$ since in this case the relevant inclusion-induced map is the identity. This completes the proof that the filtered complex $CC_*(X, \lambda) = \cup_L CC_*^L(X, \lambda)$ with boundary operator δ satisfies the properties required by Lemma 2.1.

REFERENCES

- [BCE07] F. Bourgeois, K. Cieliebak, and T. Ekholm. *A note on Reeb dynamics on the tight 3-sphere*. J. Mod. Dyn. **1** (2007), no. 4, 597–613.
- [BO10] F. Bourgeois and A. Oancea. *Fredholm theory and transversality for the parametrized and for the S^1 -equivariant symplectic action*. J. Eur. Math. Soc. (JEMS) **12** (2010), no. 5, 1181–1229.
- [BO13a] F. Bourgeois and A. Oancea. *The Gysin exact sequence for S^1 -equivariant symplectic homology*. J. Topol. Anal. **5** (2013), no. 4, 361–407.
- [BO13b] F. Bourgeois and A. Oancea. *The index of Floer moduli problems for parametrized action functionals*. Geom. Dedicata **165** (2013), 5–24.
- [BO16] F. Bourgeois and A. Oancea. *S^1 -equivariant symplectic homology and linearized contact homology*. Int. Math. Res. Notices (2016), article ID rrw029, 1–89.
- [CS10] Y. Chekanov and F. Schlenk. *Notes on monotone Lagrangian twist tori*. Electron. Res. Announc. Math. Sci. **17** (2010), 104–121.
- [CCGF+14] K. Choi, D. Cristofaro-Gardiner, D. Frenkel, M. Hutchings, and V. Ramos. *Symplectic embeddings into four-dimensional concave toric domains*. J. Topol. **7** (2014), 1054–1076.

- [C14] D. Cristofaro-Gardiner. *Symplectic embeddings from concave toric domains into convex ones*. arXiv:1409.4378.
- [CFS17] D. Cristofaro-Gardiner, D. Frenkel, and F. Schlenk. *Symplectic embeddings of four-dimensional ellipsoids into integral polydisks*. *Algebr. Geom. Topol.* **17** (2017), 1189–1260.
- [EP09] M. Entov and L. Polterovich. *Rigid subsets of symplectic manifolds*. *Compos. Math.* **145** (2009), no. 3, 773–826.
- [FOOO12] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Toric degeneration and nondisplaceable Lagrangian tori in $S^2 \times S^2$* . *Int. Math. Res. Not. IMRN* **2012**, no. 13, 2942–2993.
- [FHW94] A. Floer, H. Hofer, and K. Wysocki. *Applications of symplectic homology I*. *Math. Z.* **217** (1994), 577–606.
- [FM15] D. Frenkel and D. Müller. *Symplectic embeddings of 4-dimensional ellipsoids into cubes*. *J. Symplectic Geom.* **13** (2015), no. 4, 765–847.
- [GG16] V. Ginzburg and B. Gürel. *Lusternik-Schnirelmann theorem and closed Reeb orbits*, arXiv:1601.03092.
- [G15] J. Gutt. *The positive S^1 -equivariant symplectic homology as an invariant for some contact manifolds*, arXiv:1503.01443, to appear in *J. Symplectic Geom.*
- [GH17] J. Gutt and M. Hutchings. *Symplectic capacities from S^1 -equivariant symplectic homology*, arXiv:1707.06514.
- [H13] R. Hind. *Symplectic folding and nonisotopic polydisks*. *Algebr. Geom. Topol.* **13** (2013), no. 4, 2171–2192.
- [LiLi02] B.-H. Li and T.-J. Li. *Symplectic genus, minimal genus and diffeomorphisms*. *Asian J. Math.* **6** (2002), no. 1, 123–144.
- [LiLiu01] T.-J. Li and A.-K. Liu. *Uniqueness of symplectic canonical class, surface cone and symplectic cone of 4-manifolds with $b^+ = 1$* . *J. Differential Geom.* **58** (2001), no. 2, 331–370.
- [M91] D. McDuff. *Blow ups and symplectic embeddings in dimension 4*. *Topology* **30** (1991), no. 3, 409–421.
- [M09] D. McDuff. *Symplectic embeddings of 4-dimensional ellipsoids*. *J. Topol.* **2** (2009), no. 1, 1–22.
- [M11] D. McDuff. *The Hofer conjecture on embedding symplectic ellipsoids*. *J. Differential Geom.* **88** (2011), no. 3, 519–532.
- [MP94] D. McDuff and L. Polterovich. *Symplectic packings and algebraic geometry*. *Invent. Math.* **115** (1994), no. 3, 405–434.
- [MS12] D. McDuff and F. Schlenk. *The embedding capacity of 4-dimensional symplectic ellipsoids*. *Ann. Math. (2)* **175** (2012), 1191–1282.
- [OU16] J. Oakley and M. Usher. *On certain Lagrangian submanifolds of $S^2 \times S^2$ and CP^n* . *Algebr. Geom. Topol.* **16** (2016), no. 1, 149–209.
- [S] F. Schlenk. *Embedding problems in symplectic geometry*. de Gruyter, 2005.
- [T95] L. Traynor. *Symplectic packing constructions*. *J. Differential Geom.* **42** (1995), no. 2, 411–429.
- [Vit99] C. Viterbo. *Functors and computations in Floer homology with applications. I*. *Geom. Funct. Anal.* **9** (1999), no. 5, 985–1033.
- [Wei94] C. Weibel. *An introduction to homological algebra*. Camb. Univ. Press, 1994.

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