MATH 8230, FALL 2009: NOTES FOR THE WEEK OF NOVEMBER 17

1. S^1-actions and the symplectic cut

Let (M, ω) be a symplectic manifold and $H\colon M \to \mathbb{R}$ a smooth function. As usual H generates a Hamiltonian vector field X_H by the prescription $\iota_{X_H}\omega = dH$, and hence (at least under suitable compactness conditions on M or growth conditions on H that will hold in all the cases that we'll consider) a family of diffeomorphisms ϕ_H^t defined by $\frac{d}{dt}(\phi_H^t(m)) = X_H(\phi_H^t(m))$ and $\phi_H^0 = I$ (here and below I denotes the identity). Thus (using that H and hence X_H is independent of t) the Hamiltonian flow of H provides an action of the additive group $\mathbb{R}$ on M. We consider the rare occasion that we have $\phi_H^1 = I$, so that this action descends to an action of the quotient group $\mathbb{R}/\mathbb{Z} = S^1$. In this situation, the action $t \cdot m = \phi_H^t(m)$ is called a *Hamiltonian S^1-action* on M, and the function H is sometimes referred to as the *moment map* for this action.

Recall that an action of a group G on a set M is said to be *free* if, for all $m \in M$, $g \cdot m = m$ only when g is the identity. Now in the setting above the action on all of M can't possibly be free at least if M is closed, since the function H will have critical points p (for instance at the global maximum of M), and if p is a critical point of H we'll clearly have $\phi_H^t(p) = p$ for all t. However, recall that (because $dH(X_H) = \omega(X_H, X_H) = 0$) the level sets $H^{-1}(\{r\})$ for regular values r of H are preserved by the flow ϕ_H^t, so we have an action of S^1 on each regular level set $H^{-1}(\{r\})$, and this action does have a chance at being free.

By the end of the previous lecture, we had proven the following theorem:

Theorem 1.1. *Let $H\colon M \to \mathbb{R}$ be a Hamiltonian whose flow generates an S^1-action as above, and suppose that $r \in \mathbb{R}$ is a regular value of H such that the restriction of the S^1 action to $H^{-1}(\{r\})$ is free. Then the quotient space $M_r := H^{-1}(\{r\})/S^1$ carries a natural smooth manifold structure such that the quotient map $\pi\colon H^{-1}(\{r\}) \to M_r$ is a submersion. Moreover, there is a symplectic form ω_r on M_r, uniquely specified by the property that $\pi^*\omega_r = \omega|_{\pi^{-1}(\{r\})}$.*

The new symplectic manifold (M_r, ω_r) is then called the (symplectic) reduction (or the reduced space) of (M, ω) at r.

Example 1.2. Consider $\mathbb{C}^n = \{(z_1, \ldots, z_n)\} = \{(x_1 + iy_1, \ldots, z_n + iy_n)\}$ with its standard symplectic structure $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$. Define $H\colon \mathbb{C}^n \to \mathbb{R}$ by

$$H(\vec{z}) = \pi\|\vec{z}\|^2 := \pi\sum_{j=1}^n (x_j^2 + y_j^2).$$

Note that

$$X_H(\vec{x} + i\vec{y}) = \sum_{j=1}^n 2\pi\left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right);$$

in other words (identifying the tangent space at a point in our vector space $\mathbb{C}^n$ with the vector space itself, as is customary)

$$X_H(\vec{z}) = -2\pi i\vec{z}.$$

Thus

$$\phi_H^t(\vec{z}) = e^{-2\pi it}\vec{z}.$$

In particular, H generates an S^1-action. We've seen the reduced spaces before: notice that, for $\lambda > 0$,

$$H^{-1}(\pi\lambda^2) = \{\vec{x} + i\vec{y} | \pi \sum (x_j^2 + y_j^2) = \pi\lambda^2\}$$

is just the sphere $S^{2n-1}(\lambda)$ of radius λ in $\mathbb{C}^n$. Evidently the S^1-action is free on each $H^{-1}(\pi\lambda^2)$ for $\lambda > 0$, and as a smooth manifold the quotient

$$M_{\pi\lambda^2} = \frac{H^{-1}(\pi\lambda^2)}{S^1} = \frac{\{(z_1, \ldots, z_n) | \|\vec{z}\| = \lambda\}}{\vec{z} \sim e^{-2\pi i t}\vec{z}}$$

is clearly diffeomorphic to $\mathbb{C}P^{n-1}$. Thus $\omega_{\pi\lambda^2}$ is a symplectic form on $\mathbb{C}P^{n-1}$; it turns out that one has $\omega_{\pi\lambda^2} = \lambda^2\omega_{FS}$ where ω_{FS} is the Fubini-Study form that we discussed earlier in the course (i.e., the one induced by the form $\frac{i}{2}\partial\bar{\partial}(\|\vec{z}\|^2)$ on $\mathbb{C}^n \setminus \{0\}$).

Example 1.3. Now let $\tilde{M} = \mathbb{C}^{n+1}$ with its standard symplectic structure; write a general element as $(w, \vec{z})$ with $w \in \mathbb{C}, \vec{z} \in \mathbb{C}^n$. Define $\tilde{H}(w, \vec{z}) = -\pi|w|^2 + \pi\|\vec{z}\|^2$. Evidently $\tilde{H}$ induces the flow

$$\phi^t_{\tilde{H}}(w, \vec{z}) = (e^{2\pi i t}w, e^{-2\pi i t}\vec{z}),$$

which in particular is an S^1-action. Choose $\epsilon > 0$. Now $\tilde{H}^{-1}(\{\epsilon\}) = A_\epsilon \coprod B_\epsilon$, where

$$A_\epsilon = \{(0, \vec{z}) | \pi\|\vec{z}\|^2 = \epsilon\}$$

and

$$B_\epsilon = \{(w, \vec{z}) | w \in \mathbb{C} \setminus \{0\}, \pi\|\vec{z}\|^2 = \epsilon + \pi|w|^2\}.$$

Since any element of $H^{-1}(\epsilon)$ has $\vec{z} \neq 0$, S^1 acts freely on $H^{-1}(\epsilon)$, so we can consider the reduced space $(\tilde{M}_\epsilon, \omega_\epsilon)$. The S^1 action preserves both of the sets A_ϵ and B_ϵ, so $M_\epsilon = (A_\epsilon/S^1) \coprod (B_\epsilon/S^1)$. As should be clear from the previous example, A_ϵ/S^1 is $\mathbb{C}P^{n-1}$, and the induced symplectic structure on it is $(\epsilon/\pi)\omega_{FS}$. Meanwhile consider B_ϵ/S^1. If $(w, \vec{z}) \in B_\epsilon$ (so in particular $w \neq 0$), the equivalence class of $(w, \vec{z})$ under the S^1-action contains a unique point whose first coordinate is a positive real number (specifically, this point is $(|w|, \frac{w}{|w|}\vec{z})$). (In the language typically used in the study of group actions, we could say that the set S of elements of B_ϵ with positive real first coordinate comprises a *global slice* for the S^1-action). Conversely, if $\vec{z} \in \mathbb{C}^n$ has $\pi\|\vec{z}\|^2 > \epsilon$, then the positive number s with $\pi s^2 = \pi|\vec{z}|^2 - \epsilon$ has the property that $(s, \vec{z}) \in S$. Consequently $(s, \vec{z}) \mapsto \vec{z}$ defines a diffeomorphism from S to $\{\vec{z} \in \mathbb{C}^n | \pi\|\vec{z}\|^2 > \epsilon\}$. But because S is a global slice, the map $B_\epsilon/S^1 \to S$ sending an orbit to the unique element of S contained in the orbit is a diffeomorphism, and so B_ϵ/S^1 is diffeomorphic to $\{\vec{z} \in \mathbb{C}^n | \pi\|\vec{z}\|^2 > \epsilon\}$ (an explicit diffeomorphism is $[w, \vec{z}] \mapsto \frac{w}{|w|}\vec{z}$).

Summing up, the reduced space $\tilde{M}_\epsilon$ consists mostly of a piece naturally diffeomorphic to $\{\vec{z} \in \mathbb{C}^n | \pi\|\vec{z}\|^2 > \epsilon\}$, with the rest of $\tilde{M}_\epsilon$ being a real-codimension-two piece obtained by collapsing $\{\vec{z} \in \mathbb{C}^n | \pi\|\vec{z}\|^2 = \epsilon\}$ to $\mathbb{C}P^{n-1}$ via the Hopf map $S^{2n-1} \to \mathbb{C}P^{n-1}$. We'll come back to this example later, but first we'll put it in a more general framework.

1.1. **The symplectic cut.** The following can be thought of as a generalization of the previous two examples. Start with a symplectic manifold (M, ω) and a smooth function $H\colon M \to \mathbb{R}$ so that the Hamiltonian flow $\{\phi^t_H\}$ of H is an S^1-action. Let ϵ be any regular value of H with the property that S^1 acts freely on $H^{-1}(\epsilon)$, which in particular makes it possible to define the reduced space $(M_\epsilon, \omega_\epsilon)$.

Now define $\tilde{M} = \mathbb{C} \times M$, $\tilde{\omega} = ((dx \wedge dy) \oplus \omega)$, and $\tilde{H}\colon \tilde{M} \to \mathbb{R}$ by

$$\tilde{H}(w, m) = -\pi|w|^2 + H(m).$$

$\tilde{H}$ then has Hamiltonian flow given by

$$\phi^t_{\tilde{H}}(w, m) = (e^{2\pi i t}w, \phi^t_H(m));$$

in particular the flow is an S^1-action. With ϵ as in the previous paragraph, consider $\tilde{H}^{-1}(\epsilon)$. Just as in the previous example, it's natural to split this up into two parts, separately preserved by the S^1-action: $\tilde{H}^{-1}(\epsilon) = A_\epsilon \coprod B_\epsilon$ where

$$A_\epsilon = \{(0, m) | m \in H^{-1}(\epsilon)\}$$

and

$$B_\epsilon = \{(w, m) | H(m) > \epsilon \text{ and } \pi|w|^2 = H(m) - \epsilon\}.$$

Since the S^1-action on $H^{-1}(\epsilon)$ is assumed free, the action of S^1 on A_ϵ is free, while since B_ϵ consists only of points with nonvanishing w-coordinate the S^1-action on B_ϵ is free. Hence the action on all of $\tilde{H}^{-1}(\epsilon)$ is free, and we can consider the reduced space $(\tilde{M}_\epsilon, \tilde{\omega}_\epsilon)$. (Sometimes $\tilde{M}_\epsilon$ is written $M_{\geq\epsilon}$). As should be clear, A_ϵ/S^1 is just a copy of the original reduced space $M_\epsilon = H^{-1}(\epsilon)/S^1$. Meanwhile, each S^1-orbit in B_ϵ has a unique point whose w-coordinate is a positive real number, so we can use the pairs (s, m) where $s > 0$ as a global slice; this identifies B_ϵ/S^1 with $M_{>\epsilon} := \{m \in M | H(m) > 0\}$ (an explicit diffeomorphism is $[(re^{2\pi it}, m)] \mapsto \phi_H^{-t}(m)$). Hence

$$\tilde{M}_\epsilon = M_\epsilon \coprod M_{>\epsilon}$$

is, at least set-theoretically, the union of the reduced space of M at ϵ (which has real codimension two) with the open set $\{m | H(m) > \epsilon\} \subset M$. In fact, a homework exercise shows that the symplectic form $\tilde{\omega}_\epsilon$ induced by the reduction procedure on $M_{>\epsilon}$ is the same as the (restriction of the) original symplectic form ω. Meanwhile, it should be clear that $\tilde{\omega}_\epsilon$ restricts to M_ϵ as the reduced symplectic form ω_ϵ.

Topologically, the construction can be described as follows. Starting from M, first remove all points with $H < \epsilon$; this leaves $H^{-1}([\epsilon, \infty))$, which (since ϵ is a regular value) is a manifold with boundary $H^{-1}(\{\epsilon\})$. But the boundary carries an S^1-action, and we "collapse" the boundary down to the reduced space $H^{-1}(\{\epsilon\})/S^1$, leaving a union of the open subset $M_{>\epsilon} = \{H > \epsilon\}$ of M and this collapsed boundary. From the last two sentences it's not even immediately obvious that one gets a smooth manifold, but our general machinery shows that in fact the result is smooth (and without boundary), consisting of a (symplectomorphic) copy of $M_{>\epsilon}$ and a codimension-two symplectic submanifold symplectomorphic to the original reduced space M_ϵ.

For reasons that should be apparent, the result $(M_{\geq\epsilon}, \tilde{\omega})$ is called the symplectic cut of M at ϵ.

Note that if we'd instead put $\tilde{H}(w, m) = \pi|w|^2 + H(m)$ we'd have gotten an entirely analagous manifold $M_{\leq\epsilon}$ consisting of $\{H < \epsilon\}$ together with a copy of the reduced space M_ϵ.

2. Blowing up

Consider again the manifold $\tilde{M}_\epsilon$ produced in Example 1.3. From the perspective of the symplectic cut, we've started with $\mathbb{C}^n$, removed the open ball $B(\lambda)$ of radius $\lambda = (\epsilon/\pi)^{1/2}$ to obtain a manifold $(\mathbb{C}^n \setminus B(\lambda))$ with boundary $S^{2n-1}(\lambda)$, and then collapsed $S^{2n-1}(\lambda)$ to a copy of $\mathbb{C}P^{n-1}$. Let $\pi\colon S^{2n-1}(\lambda) \to \mathbb{C}P^{n-1}$ be the quotient map. Also, let $\rho\colon \mathbb{C}^n \setminus B(\lambda) \to S^{2n-1}(\lambda)$ be the retraction $\vec{z} \mapsto \frac{\lambda}{\|\vec{z}\|}\vec{z}$ to $S^{2n-1}(\lambda)$, and consider the map

$$\pi \circ \rho\colon \mathbb{C}^n \setminus B(\lambda) \to \mathbb{C}P^{n-1}.$$

This map rescales an arbitrary $\vec{z} \in \mathbb{C}^n \setminus B(\lambda)$ to lie on the sphere of radius λ, and then sends the result to the unique complex line through the origin on which the rescaled point lies. But of course the original point lies on the same complex line (indeed even the same real line) as does the rescaled point, so a more concise way of describing $\pi \circ \rho$ is that it sends a point in $\mathbb{C}^n \setminus B(\lambda)$ to the unique complex line through the origin on which the point lies.

Thus the preimage of an element $\ell \in \mathbb{C}P^{n-1}$ (viewed as a complex line) under $\pi \circ \rho$ is the set of all points $\vec{z}$ on the complex line ℓ having $\|\vec{z}\| \geq \lambda$. When we pass to the quotient $\tilde{M}_\epsilon$, those points on ℓ with $\|\vec{z}\| = \lambda$ all get identified (while all the other points of ℓ are unaffected). As a result, choosing some increasing diffeomorphism $f\colon [0, \infty) \to [\lambda, \infty)$, we can identify the image of $(\pi \circ \rho)^{-1}(\ell)$ in the quotient $\tilde{M}_\epsilon$ with ℓ itself—a point of magnitude $\vec{w} \in \ell$ gets identified with $\frac{f^{-1}(\|\vec{w}\|)}{\|\vec{w}\|}\vec{w}$ when $\vec{w} \neq 0$, while $\vec{0}$ gets identified with the contracted S^1-orbit $\pi(\{\|\vec{z}\| = \lambda\} \cap \ell)$. Moreover, this can be done smoothly as the line ℓ varies. This proves (or at least sketches a proof of):

Proposition 2.1. *Where $\tilde{M}_\epsilon$ is the reduced space constructed in Example 1.3, there is a diffeomorphism $\Phi\colon \tilde{M}_\epsilon \to \tilde{\mathbb{C}}^n$, where $\tilde{\mathbb{C}}^n$ is the total space of the* tautological line bundle *over $\mathbb{C}P^{n-1}$,* i.e.*,*

$$\tilde{\mathbb{C}}^n = \{(\vec{z}, \ell) \in \mathbb{C}^n | \vec{z} \in \ell\}.$$

The map $\pi \circ \rho\colon \mathbb{C}^n \setminus B(\lambda) \to \mathbb{C}P^{n-1}$, which sends a point to the complex line through the origin on which it lies, descends to a map $\bar{\pi}\colon \tilde{M}_\epsilon \to \mathbb{C}P^{n-1}$, and where $\pi_2\colon \tilde{\mathbb{C}}^n \to \mathbb{C}P^{n-1}$ is the projection onto the second factor we have

$$\pi_2 \circ \Phi = \bar{\pi}.$$

Further the image of $\mathbb{C}P^{n-1} \cong A_\epsilon/S^1 \subset \tilde{M}_\epsilon$ under Φ is the zero section of the bundle $\pi_2\colon \tilde{\mathbb{C}}^n \to \mathbb{C}P^{n-1}$.

As alluded to above, $\tilde{\mathbb{C}}^n$ is known as the tautological line bundle on $\mathbb{C}P^n$—it is indeed a (complex) line bundle (so a real two-dimensional vector bundle), with the fiber over a given line $\ell \in \mathbb{C}P^{n-1}$ consisting simply of all the points on ℓ. While the projection $\pi_2\colon \tilde{\mathbb{C}}^n \to \mathbb{C}P^{n-1}$ onto the second factor is a bundle projection, one can instead consider the projection $\pi_1\colon \tilde{\mathbb{C}}^n \to \mathbb{C}^n$ onto the first factor. If $\vec{z} \in \mathbb{C}^n$, then $\pi_1^{-1}(\{\vec{z}\})$ consists of those complex lines through the origin on which $\vec{z}$ lies. Of course, if $\vec{z}$ is nonzero, then it lies on just one line through the origin, while if $\vec{z} = 0$, it lies on every line through the origin; hence all but one of the preimages of π_1 is a singleton, with the one exception being a copy of $\mathbb{C}P^{n-1}$. Thus, $\tilde{\mathbb{C}}^n$ can be thought of as a copy of $\mathbb{C}^n$, except with the origin replaced by the set of all complex lines through the origin.

2.1. **Complex blowups.** Suppose that M is a complex manifold of complex dimension n—recall in particular that this means that the transition maps for its defining charts are *holomorphic*. Let $p \in M$. One can then form the (complex) *blowup* $\tilde{M}$ of M at p as follows (filling in the details is an optional exercise). Take a (holomorphic) chart $\phi\colon U \to \phi(U)$ around p, say with $\phi(p) = \vec{0}$. Let

$$\widetilde{\phi(U)} = \{(\vec{z}, \ell) \in \tilde{\mathbb{C}}^n | \vec{z} \in \phi(U)\}.$$

Now $\phi^{-1} \circ \pi_1$ biholomorphically identifies $\widetilde{\phi(U)} \setminus \{\vec{z} = 0\}$ with $U \setminus \{p\}$. Hence

$$\tilde{M} = \frac{(M \setminus \{p\}) \cup \widetilde{\phi(U)}}{(\phi^{-1} \circ \pi_1)(\vec{z}, \ell) \sim (\vec{z}, \ell)}$$

inherits a complex manifold structure. The effect of doing this, roughly speaking, is to replace p by the space of complex lines through $\vec{0} = \phi(p)$ in $\mathbb{C}^n$; equivalently, this replaces p with the set of complex lines in T_pM (of course this latter is a copy of $\mathbb{C}P^{n-1}$). The construction is independent of the chart, since the transition functions between two charts ϕ_1, ϕ_2 are holomorphic, and so the linearization of $\phi_2 \circ \phi_1^{-1}$ maps complex lines through the origin in $T_{\phi_1(p)}\mathbb{C}^n$ to complex lines through the origin in $T_{\phi_2(p)}\mathbb{C}^n$.

Blowing up a complex manifold thus replaces a point with a copy of $\mathbb{C}P^{n-1}$ which is conventionally known as the *exceptional divisor*. At the level of smooth oriented manifolds, this is equivalent to replacing the manifold M by the connected sum of $M \# \overline{\mathbb{C}P^n}$, where $\overline{\mathbb{C}P^n}$ means $\mathbb{C}P^n$ with its noncomplex orientation.

Example 2.2. Let $M = \mathbb{C}P^2$ and $p = [1 : 0 : 0]$. The blowup $\tilde{M}$ of M at p then contains an exceptional divisor E, which in this case is a copy of $\mathbb{C}P^1$. As noted above, E is identified with the space of complex lines through the origin in T_pM. In this special case, another way of viewing it is as the space of lines in $\mathbb{C}P^2$ which pass through p—indeed any complex line through the origin in T_pM is easily seen to be tangent to a unique line through p in $\mathbb{C}P^2$. (Here a "line" in $\mathbb{C}P^2$ means, as usual in projective geometry, the image under the projection $\mathbb{C}^3 \setminus \{\vec{0}\}$ of a two-complex-dimensional subspace of

$\mathbb{C}^3$). We have $\tilde{M} = (M \setminus \{p\}) \cup E$. Define a map $\pi\colon \tilde{M} \to E$ by setting $\pi(\ell) = \ell$ when $\ell \in E$ and, for $q \in M \setminus \{p\} = \tilde{M} \setminus E$, setting $\pi(q)$ equal to the unique line ℓ which passes through both q and p. Then $\pi^{-1}(\ell)$ is, essentially, a copy of ℓ: its intersection with $M \setminus \{p\}$ is $\ell \setminus \{p\}$, and the "missing point" p is filled in by the single point $\pi^{-1}(\ell) \cap E$. Now the complex projective lines are, as real manifolds, copies of S^2, and E is also a copy of S^2. Thus (modulo a few additional details) $\pi\colon \tilde{M} \to E$ is a fiber bundle over S^2 whose fibers are diffeomorphic to S^2.

In particular, $\tilde{M}$ is diffeomorphic to some smooth manifold which admits the structure of an S^2-bundle over S^2. It's possible to show (for instance using that the orientation-preserving diffeomorphism group of S^2 retracts to $SO(3)$ and that $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$) that there are only two such smooth manifolds, namely $S^2 \times S^2$ and another manifold sometimes denoted $S^2 \tilde{\times} S^2$. I claim now that $\tilde{M}$ is *not* diffeomorphic to $S^2 \times S^2$. Indeed, consider any complex projective line L in $\mathbb{C}P^2$ that does *not* intersect p. When we blow up $\mathbb{C}P^2$ to form M, L is unaffected, and so gives rise to a copy of S^2, which I'll still denote L, in $\tilde{M}$. Now if I perturb L to a slightly different complex projective line L' in $\mathbb{C}P^2$, L' also won't intersect p, so will also give rise to a complex projective line (still denoted L') in $\tilde{M}$. L and L' are homologous in either manifold, and moreover they intersect exactly once, with positive orientation (because any two complex projective lines in $\mathbb{C}P^2$ intersect once, as you can easily check if you've never seen it before). This proves, then, that the homology class $[L]$ of L in $H_2(\tilde{M};\mathbb{Z})$ has self-intersection 1. But it's easy to see that in $S^2 \times S^2$ every homology class has even self-intersection, so $\tilde{M}$ can't be diffeomorphic to $S^2 \times S^2$. So it must be diffeomorphic to $S^2 \tilde{\times} S^2$.

Incidentally, another way of showing this is to prove very generally that the exceptional divisor in a blown-up complex 2-manifold (or indeed in a blown-up symplectic 4-manifold) *always* has self-intersection number -1; in view of this $S^2 \times S^2$ can't be the blowup up *any* manifold.

2.2. **Symplectic blowups.** Now let (M, ω) be a $2n$-dimensional symplectic manifold and let $p \in M$. Darboux's theorem tells us that there is some neighborhood U of p which can be symplectically identified with a standard ball $(B^{2n}(R), \omega_0)$ of some radius $R > 0$ in $\mathbb{R}^{2n}$. $B^{2n}(R)$ admits the circle action $t \cdot \vec{z} = e^{-2\pi i t}\vec{z}$ induced by the Hamiltonian $H(\vec{z}) = \pi\|\vec{z}\|^2$. If $\epsilon < \pi R^2$ let us then consider the symplectic cut $B^{2n}(R)_{\geq\epsilon}$. Where $\pi r^2 = \epsilon$, as noted earlier this has the effect of cutting out $B^{2n}(r)$ from $B^{2n}(R)$; collapsing the resulting boundary $S^{2n-1}(r)$ to $\mathbb{C}P^{n-1}$ via the Hopf map; and leaving the rest of $B^{2n}(R)$ alone. The induced symplectic form $\tilde{\omega}_\epsilon$ restricts to the newly-created $\mathbb{C}P^{n-1}$ as $r^2\omega_{FS}$, and restricts to $B^{2n}(R) \setminus \overline{B^{2n}(r)}$ as ω_0. By our earlier discussion, $B^{2n}(R)_{\geq\epsilon}$ is diffeomorphic to a neighborhood of the zero section (more specifically, a disc bundle) of the tautological line bundle over $\mathbb{C}P^{n-1}$.

Hence
$$\tilde{M} = \frac{(M \setminus \phi^{-1}(\overline{B^{2n}(r)})) \coprod B^{2n}(R)_{\geq\epsilon}}{q \sim \phi(q) \text{ if } q \in U \setminus \overline{B^{2n}(r)}}$$
inherits a symplectic form $\bar{\omega}_r$, equal to ω_0 on $M \setminus \phi^{-1}(\overline{B^{2n}(r)})$ and to $\tilde{\omega}_\epsilon$ on $B^{2n}(R)_{\geq\epsilon}$ (since these two forms coincide on the intersection, $\bar{\omega}_r$ is well-defined). So we've proven, essentially, that blowing up a point is a valid operation in the symplectic category:

Theorem 2.3. *Let (M, ω) be a symplectic manifold, and let $p \in M$ be such that p has a neighborhood symplectomorphic to a standard ball of radius R. Then for any r with $0 < r < R$ the following holds. The smooth manifold $\tilde{M}$ formed by cutting out a ball B_r of radius r around p and gluing in a disc bundle in the tautological line bundle over $\mathbb{C}P^{n-1}$ admits a symplectic form $\bar{\omega}_r$ which coincides with ω away from B_r and which restricts to $\mathbb{C}P^{n-1}$ as $r^2\omega_{FS}$.*

In particular, this shows that the blowup of a Kähler manifold (which we already knew to make sense because of the complex structure on the manifold) carries a symplectic structure; closer investigation of the argument would reveal that this symplectic structure is in fact Kähler (*i.e.*, the natural complex structure on the blowup is compatible with it).

You may have noticed that the word "blowup" isn't too compatible with the details of the construction in the symplectic category—we've taken out an entire ball from the manifold, and collapsed its boundary to something with codimension 2, so the volume has decreased by the volume of the removed ball. In the complex category the terminology makes a bit more sense, since there we take the point of view that we're removing just a single point, not a ball, and replacing it by a higher-dimensional submanifold.

The blowup construction can be reversed (either in the complex or in the symplectic world) in the following sense. Suppose you find an embedded copy E of $\mathbb{C}P^{n-1}$ in a real-dimension-$2n$ complex or symplectic manifold, with a neighborhood of E isomorphic (i.e. biholomorphic or symplectomorphic, as appropriate) to a disc bundle in the tautological line bundle over $\mathbb{C}P^{n-1}$. Then this disc bundle neighborhood can be cut out and replaced by a standard ball—this process is called "blowing down." In the symplectic case, if the initial symplectic form restricts to E as $r^2\omega_{FS}$, then the effect will be to delete E and to replace it by a standard ball of radius r.

It may seem like it would be difficult to find submanifolds E that can be blown down in this way. However, particularly in real dimension 4, this is not so much the case. In the symplectic setting, obvious necessary conditions on E are that is should be a symplectic submanifold diffeomorphic to S^2, with self-intersection -1 (the last condition resulting from the fact that the zero section in the tautological line bundle has self-intersection -1). But it's not too hard to show that these conditions are in fact sufficient—symplectic structures on S^2 are classified by the cohomology class of the form, so E with the restricted symplectic structure is symplectomorphic to $\mathbb{C}P^1$ with the form $r^2\omega_{FS}$ for some $r > 0$, and then Weinstein's symplectic neighborhood theorem (together perhaps with an exercise based on what we did earlier in the course with symplectic vector bundles over S^2) shows that a neighborhood of E will be symplectomorphic to a neighborhood of the zero section in the tautological bundle as long as E has self-intersection -1. A somewhat similar statement holds in the complex category due to a result called Castelnuovo contractibility criterion, which ensures that any genus zero complex curve in a complex surface having self-intersection -1 can be blown down.