CHAPTER 1

The Hodge theorem and Sobolev spaces

We will start by considering a question relating to differential forms and de Rham cohomology on a smooth manifold M. We denote by $\Omega^k(M)$ the space of degree-k differential forms on M. So if we have a local coordinate chart $(x_1, \ldots, x_n)\colon U \to \mathbb{R}^n$ for M then we can express the restriction of any element $\omega \in \Omega^k(M)$ to U as

$$\omega|_U = \sum_{i_1<\cdots<i_k} f_{i_1\ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where the various $f_{i_1\ldots i_k}$ are smooth functions. Recall that there is then an exterior differentiation operator $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ which, in terms of these local coordinate expressions, is given by

$$d\left(\sum_{i_1<\cdots<i_k} f_{i_1\ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) = \sum_j \sum_{i_1<\cdots<i_k} \frac{\partial f_{i_1\ldots i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

(where the dx's appearing on the right can then be put in order by using $dx_k \wedge dx_l = -dx_l \wedge dx_k$). The kth de Rham cohomology[1] of M is then by definition

$$H^k(M) = \frac{\ker\left(d\colon \Omega^k(M) \to \Omega^{k+1}(M)\right)}{\operatorname{Im}\left(d\colon \Omega^{k-1}(M) \to \Omega^k(M)\right)}.$$

So in this definition $H^k(M)$ is a subquotient (*i.e.*, a quotient of a subspace) of the space $\Omega^k(M)$ of all differential forms, and an element of $H^k(M)$ is an equivalence class within the space of closed differential forms (those ω with $d\omega = 0$), and indeed a fairly large one (for $1 \le k \le n$) since d will typically have infinite-dimensional image. In the hopes of making $H^k(M)$ a little more concrete, we could ask:

QUESTION 1.0.1. *Is there a natural way of identifying $H^k(M)$ with a subspace of $\Omega^k(M)$, rather than a subquotient of $\Omega^k(M)$?*

Hodge theory will give an affirmative answer to this question provided that M is compact, oriented, and endowed with a Riemannian metric, *i.e.* a smoothly-varying inner product on the tangent spaces of M (and the word "natural" is interpreted in the category of Riemannian manifolds rather than just smooth ones). Note that any smooth manifold can be given a Riemannian metric, though the choice is not canonical. (If the manifold is embedded in some $\mathbb{R}^N$ then the obvious choice is the restriction of the standard inner product on $\mathbb{R}^N$, though this depends sensitively on the embedding.) In addition various features of $H^k(M)$ for compact M, such as Poincaré duality and the fact that it is finite-dimensional, have nice and simple interpretations in terms of the description that we will provide. There's also a version of all of this for certain complex manifolds, which leads

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[1] As you may know (though it's not directly relevant to this course), $H^k(M)$ is isomorphic to the real-coefficient versions of various other forms of cohomology, such as singular or Čech.

to especially strong statements about their cohomology which are difficult to understand by other means.

In the following section I will give a sketch of how we will answer Question 1.0.1, though it will be clear from the sketch that various strokes of luck will be necessary for it to work out.

1.1. Formulating the strategy

1.1.1. Complements. We follow the usual notation of putting

$$Z^k(M) = \ker(d\colon \Omega^k(M) \to \Omega^{k+1}(M)) \qquad B^k(M) = \mathrm{Im}(d\colon \Omega^{k-1}(M) \to \Omega^k(M)),$$

so (since $d^2 = 0$) we have $B^k(M) \le Z^k(M) \le \Omega^k(M)$, and by definition

$$H^k(M) = \frac{Z^k(M)}{B^k(M)}.$$

From standard constructions of smooth functions on manifolds[2] it's not hard to see that $B^k(M)$ (and hence also $Z^k(M)$ and $\Omega^k(M)$) is infinite-dimensional for $1 \le k \le \dim M$. (If this weren't true then what we're about to do would be much easier.)

The goal is to identify the quotient of vector spaces $\frac{Z^k(M)}{B^k(M)}$ with a subspace of $\Omega^k(M)$; in fact we'll identify it with a subspace of $Z^k(M)$. The standard way of approaching this is dictated by the following easy fact:

PROPOSITION 1.1.1. *Let V be a vector space, and $W \le V$. Suppose that $X \le V$ is another subspace such that $V = W \oplus X$. Then $x \mapsto [x]$ defines an isomorphism of vector spaces $X \cong V/W$.*

PROOF. That $V = W \oplus X$ means that $W \cap X = \{0\}$ and each element of V can be written as $w + x$ where $w \in W$ and $x \in X$. An element of the kernel of the map $\pi|_X \colon X \to V/W$ given by $x \mapsto [x]$ would belong to $W \cap X$ and hence be zero, so $\pi|_X$ is injective. $\pi|_X$ is also surjective, since any element of V/W has the form $[w+x]$ where $w \in W$ and $x \in X$. But $[w+x] = [x] = \pi|_X(x)$. $\square$

So our goal is now to find a *complement* $\mathscr{H}^k(M)$ to the exact forms $B^k(M)$ within the closed forms $Z^k(M)$, *i.e.* we want a subspace $\mathscr{H}^k(M) \le Z^k(M)$ with $Z^k(M) = B^k(M) \oplus \mathscr{H}^k(M)$. Now the existence of such a complement can be proven fairly easily using Zorn's lemma, but this is quite inexplicit—there would be no way of telling whether a given closed form actually belongs to the complement—and is not natural under any reasonable interpretation of that word.

A familiar way of constructing a complement to a subspace W of a vector space V is to use the *orthogonal complement* $W^\perp$. This requires V to be endowed with an inner product $\langle \cdot, \cdot \rangle$ in which case, by definition, $W^\perp = \{x \in V | (\forall w \in W)(\langle w, x \rangle = 0)\}$. As you are undoubtedly aware, if V is finite-dimensional then $V = W \oplus W^\perp$. There are two issues, one much more serious than the other, in applying this to our situation. The less serious issue is that we need to have an inner product on $Z^k(M)$ to try to do this. As we will see later, there is a natural way of constructing such an inner product *provided that M is endowed with a Riemannian metric* (as it always can be, though the resulting inner product depends on the metric). The more serious issue is that the space $Z^k(M)$ that is playing the role of V is not finite-dimensional, so *a priori* there is no guarantee that the orthogonal complement $B^k(M)^\perp$ will actually be a complement. The following general result expresses part of the issue; for context note that an inner product $\langle \cdot, \cdot \rangle$ on a vector space V induces a topology, given by the metric $d(v_1, v_2) = \sqrt{\langle v_1 - v_2, v_1 - v_2 \rangle}$.

PROPOSITION 1.1.2. *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and $W \le V$. Suppose that $V = W \oplus W^\perp$. Then W is closed with respect to the topology on V induced by $\langle \cdot, \cdot \rangle$.*

[2] showing for instance that if $p \in U$ where U is an open subset of M there is a smooth function taking value 1 at p and vanishing identically outside of U

PROOF. The main point is that, denoting by $\bar{W}$ the closure of W with respect to the topology induced by $\langle\cdot,\cdot\rangle$, we have $\bar{W} \subset (W^\perp)^\perp$. Indeed, if $x \in W^\perp$, the function $y \mapsto \langle x, y\rangle$ is continuous (this is an easy corollary of the Cauchy–Schwarz inequality), so $\{y \in V | \langle x, y\rangle = 0\}$ is a closed set that contains W and hence also contains $\bar{W}$. Since this holds for all $x \in W^\perp$ we indeed have $\bar{W} \subset (W^\perp)^\perp$.

If $V = W \oplus W^\perp$, it suffices to show that if $y \in \bar{W}$ is written as $y = w + x$ where $w \in W$ and $x \in W^\perp$ then $x = 0$. But by the previous paragraph we have $\langle y, x\rangle = 0$, and moreover $\langle w, x\rangle = 0$ by the definition of $W^\perp$, so we find that

$$\langle x, x\rangle = \langle w - y, x\rangle = 0$$

and hence that $x = 0$. □

So if the strategy of taking $\mathscr{H}^k(M)$ to be the orthogonal complement of $B^k(M)$ in $Z^k(M)$ is to work (which it eventually will), it would need to be the case that $B^k(M)$ is closed as a subset of $Z^k(M)$, *i.e.* that the image of d is closed inside the kernel of d (or equivalently is closed inside all of $\Omega^k(M)$, at least assuming that $Z^k(M)$ is closed, which it will be). This is not at all easy to show—the reader might want to think through at this point what it means to say that the image of a linear transformation between infinite-dimensional vector spaces is closed to get a sense of where the difficulties lie.

By the way, as you may know, if $(V, \langle\cdot,\cdot\rangle)$ is a Hilbert space (*i.e.* the metric defined by the inner product is complete) then the converse to Proposition 1.1.2 is true, *i.e.* if $W \leq V$ is closed then $W^\perp$ is a complement to W. But this isn't immediately useful to us because it's not so easy to define an inner product on $\Omega^k(M)$ that yields a complete metric space, and the inner product that we will use certainly does not. The difficulty here is that the elements of $\Omega^k(M)$ are infinitely-differentiable. It is possible to get a Hilbert space using certain finitely-differentiable versions of differential forms, but then our exterior derivative operator d wouldn't preserve the differentiability condition. We will manage these sorts of issues later on using Sobolev spaces.

1.1.2. Adjoints. Since the plan is to consider the orthogonal complement to $B^k(M) = \mathrm{Im}(d\colon \Omega^{k-1}(M) \to Z^k(M) \subset \Omega^k(M))$ inside $Z^k(M)$, let us think more about how we can describe this complement. Abstracting the problem a bit, we have a linear map $A\colon U \to V$ where U and V carry an inner product, and we plan to consider the orthogonal complement of $A(U)$ in V. Now if U and V are finite-dimensional, there is a familiar description of this orthogonal complement: we can consider the adjoint $A^*\colon V \to U$ characterized by the relationship

$$\langle Au, v\rangle = \langle u, A^* v\rangle$$

(and given in terms of matrix representatives by matrix transposition), and then it will hold that $A(U)^\perp = \ker(A^*)$. The inclusion $\geq$ here is obvious: if $A^*v = 0$ then clearly $\langle Au, v\rangle = \langle u, A^*v\rangle = 0$, and then the reverse inclusion follows by a dimension count.

Of course in our setting the relevant vector spaces are infinite-dimensional, so one can't use a dimension count to infer that $\ker(A^*) = A(U)^\perp$. But this will turn out to be true in our case.

So the plan will be to let $\mathscr{H}^k(M)$ be the kernel of the adjoint of $d\colon \Omega^{k-1}(M) \to Z^k(M)$ with respect to the inner products that we will define later. Note that we're taking the codomain to be $Z^k(M)$ here (so that the adjoint maps $Z^k(M) \to \Omega^{k-1}(M)$); it is more natural to define an adjoint $d^*\colon \Omega^k(M) \to \Omega^{k-1}(M)$ to $d\colon \Omega^{k-1}(M) \to \Omega^k(M)$, and then the map of which $\mathscr{H}^k(M)$ is the kernel is $d^*|_{Z^k(M)}$. Said differently, we will set

$$\mathscr{H}^k(M) = \ker(d) \cap \ker(d^*)$$

where $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ is the exterior derivative and $d^*\colon \Omega^k(M) \to \Omega^{k-1}(M)$ is the adjoint of the exterior derivative $\Omega^{k-1}(M) \to \Omega^k(M)$.

We can usefully rephrase this as follows:

PROPOSITION 1.1.3. *If d and d^* are as above, then $\omega \in \ker(d) \cap \ker(d^*)$ if and only if $(d^*d + dd^*)\omega = 0$.*

PROOF. The forward implication is trivial. For the backward implication, if $(d^*d + dd^*)\omega = 0$ then

$$0 = \langle (d^*d + dd^*)\omega, \omega\rangle = \langle d\omega, d\omega\rangle + \langle d^*\omega, d^*\omega\rangle$$

by the defining property of the adjoint, and this equality forces $d\omega = 0$ and $d^*\omega = 0$. □

It remains of course to actually construct the adjoint (at this point we have neither a construction nor a general result implying its existence), which does not even make sense until we say what the inner product on differential forms is.

1.1.3. d^* on forms on $\mathbb{R}^n$. Before doing this in general it is probably instructive to work out the local behavior for differential forms on $\mathbb{R}^n$ with compact support. Of course (since we assume compact support) such forms can be transplanted to arbitrary smooth manifolds by working in a coordinate chart and extending by zero outside of the coordinate chart; conversely any differential form on a smooth manifold M can be written as a sum of forms supported in coordinate charts by using a partition of unity. We write $\Omega^k_c(\mathbb{R}^n)$ for the space of compactly supported k-forms on $\mathbb{R}^n$.

We introduce some notation to reduce the number of indices that we need to write: we will generically use I (or sometimes J) to denote a k-tuple (for some k of elements of $\{1, \dots, n\}$ with $i_1 < \dots < i_k$, and we will write $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$. So a general element of $\Omega^k_c(\mathbb{R}^n)$ is given by

$$\sum_I f_I dx_I$$

where the f_I are compactly supported smooth functions. We use the following inner product on $\Omega^k_c(\mathbb{R}^n)$:

$$\left\langle \sum_I f_I dx_I, \sum_J g_J dx_J \right\rangle = \sum_I \int_{\mathbb{R}^n} f_I g_I dx_1 \wedge \dots \wedge dx_n. \tag{1}$$

EXAMPLE 1.1.4. *Let's work out the adjoint of d on $\Omega^k_c(\mathbb{R}^2)$ for $k = 1, 2$ (which are the only values of k for which it could be nonzero). Recall that, on 0- and 1-forms, d is given by*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$d(Pdx + Qdy) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

So $d^*\colon \Omega^1_c(\mathbb{R}^n) \to \Omega^0_c(\mathbb{R}^n)$ should satisfy, for any $Adx + Bdy \in \Omega^1_c(\mathbb{R}^2)$ and $f \in \Omega^0_c(\mathbb{R}^2)$,

$$\begin{aligned}\int_{\mathbb{R}^2} f\, d^*(Adx + Bdy) dx \wedge dy = \langle f, d^*(Adx + Bdy)\rangle = \langle df, Adx + Bdy\rangle = \left\langle \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy, Adx + Bdy \right\rangle \\ = \int_{\mathbb{R}^2} \left(\frac{\partial f}{\partial x}A + \frac{\partial f}{\partial y}B\right) dx \wedge dy = -\int_{\mathbb{R}^2} f \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}\right) dx \wedge dy\end{aligned}$$

where we have integrated by parts in the final step.

This holds for every function $f \in \Omega^0_c(\mathbb{R}^2)$ if and only if we define $d^*(Adx + Bdy) = -\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}\right)$. Note that under the obvious identification of one-forms on $\mathbb{R}^2$ with vector fields, d corresponds to the scalar curl and $-d^*$ corresponds to the divergence.

Let us also compute d^* on $\Omega^2_c(\mathbb{R}^2)$. We have, for any $Pdx+Qdy \in \Omega^1_c(\mathbb{R}^2)$ and $g dx\wedge dy \in \Omega^2_c(\mathbb{R}^2)$,

$$\langle d(Pdx+Qdy), g dx\wedge dy\rangle = \int_{\mathbb{R}^2}\left(-\frac{\partial P}{\partial y}g + \frac{\partial Q}{\partial x}g\right)dx\wedge dy = \int_{\mathbb{R}^2}\left(P\frac{\partial g}{\partial y} - Q\frac{\partial g}{\partial x}\right)dx\wedge dy$$
$$= \left\langle Pdx+Qdy, \frac{\partial g}{\partial y}dx - \frac{\partial g}{\partial x}dy\right\rangle.$$

So we can (and must) take

$$d^*(g dx\wedge dy) = \frac{\partial g}{\partial y}dx - \frac{\partial g}{\partial x}dy$$

for $g dx\wedge dy \in \Omega^2_c(\mathbb{R}^2)$.

In view of Proposition 1.1.3 let us now compute d^*d+dd^* on each $\Omega^k_c(\mathbb{R}^2)$. For $k=0$, we have:

$$(d^*d+dd^*)f = d^*\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) = -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right);$$

for $k=1$:

$$\begin{aligned}(d^*d+dd^*)(Pdx+Qdy) &= d^*\left(\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy\right) - d\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\\
&= \left(\frac{\partial^2 Q}{\partial x\partial y} - \frac{\partial^2 P}{\partial y^2}\right)dx + \left(-\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 P}{\partial x\partial y}\right)dy - \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 Q}{\partial x\partial y}\right)dx - \left(\frac{\partial^2 P}{\partial x\partial y} + \frac{\partial^2 Q}{\partial y^2}\right)dy\\
&= -\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right)dx - \left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2}\right)dy.\end{aligned}$$

Finally, for $k=2$,

$$(d^*d+dd^*)(g dx\wedge dy) = d\left(\frac{\partial g}{\partial y}dx - \frac{\partial g}{\partial x}dy\right) = -\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right)dx\wedge dy.$$

In other words, at least in this case, the operator d^*d+dd^* (sometimes called the "Hodge Laplacian") acts by the negative of the standard Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ on each function component f_I of the differential form $\sum_I f_I dx_I$. This situation persists in more general cases, though when the Riemannian metric used to define the inner product on $\Omega^k(M)$ has curvature there are additional terms that appear when the Hodge Laplacian is written out in coordinates.

Although with sufficient patience we could perform a similar computation to that in Example 1.1.4 for $\Omega^k_c(\mathbb{R}^n)$ for any values of k and n, it is better to proceed by rephrasing the definition of the inner product (1) in the following way. We will use the "Hodge star" operator $\star\colon \Omega^k_c(\mathbb{R}^n) \to \Omega^{n-k}_c(\mathbb{R}^n)$ which we define presently. For any k-tuple $I = (i_1,\dots,i_k) \in \{1,\dots,n\}^k$ such that $i_1 < \dots < i_k$, let us write I° for the complementary increasing $(n-k)$-tuple $(j_1,\dots,j_{n-k}) \in \{1,\dots,n\}^{n-k}$, such that $\{j_1,\dots,j_{n-k}\} = \{1,\dots,n\}\setminus\{i_1,\dots,i_k\}$ and $j_1 < \dots < j_{n-k}$. Also let $\epsilon(I)$ denote the sign of the permutation that sends $1,\dots,k,k+1,\dots,n$, respectively, to $i_1,\dots,i_k,j_1,\dots,j_{n-k}$. The Hodge star operator $\star\colon \Omega^k_c(\mathbb{R}^n) \to \Omega^{n-k}_c(\mathbb{R}^n)$ may be defined by the formula

$$\star\left(\sum_I f_I dx_I\right) = \sum_I \epsilon(I) f_I dx_{I^\circ}. \tag{2}$$

Note in particular that if $I = (i_1,\dots,i_k)$ and $J = (j_1,\dots,j_k)$ are both in increasing order, then

$$dx_I \wedge \star dx_J = \begin{cases} dx_1\wedge\dots\wedge dx_n & \text{if } I = J\\ 0 & \text{otherwise.}\end{cases}$$

In view of this, the definition (1) of the inner product on compactly supported k-forms on $\mathbb{R}^n$ is easily seen to be equivalent to saying that, for any $\omega, \theta \in \Omega^k_c(\mathbb{R}^n)$,

$$\langle \omega, \theta \rangle = \int_{\mathbb{R}^n} \omega \wedge \star\theta. \tag{3}$$

While the definition of $\star$ appears rather coordinate-dependent, we will see later that its definition extends to the compactly supported forms on any oriented Riemannian manifold, allowing us to define an inner product on such forms just as in (3). Before doing this, let us work out the adjoint of d with respect to this inner product. To do this, first note that for $I = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$, the product of the signs $\epsilon(I)$ and $\epsilon(I^\circ)$ is equal to the sign of the permutation which maps $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$ to $(j_1, \dots, j_{n-k}, i_1, \dots, i_k)$, and this latter sign is just $(-1)^{k(n-k)}$. Consequently the definition of $\star$ yields

$$\star\star = (-1)^{k(n-k)} \quad \text{as a map } \Omega^k_c(\mathbb{R}^n) \to \Omega^k_c(\mathbb{R}^n).$$

If $\omega \in \Omega^{k-1}_c(\mathbb{R}^n)$ and $\theta \in \Omega^k_c(\mathbb{R}^n)$, we then find by the Leibniz rule and Stokes' theorem that

$$\begin{aligned}\langle d\omega, \theta \rangle &= \int_{\mathbb{R}^n} (d\omega) \wedge \star\theta = \int_{\mathbb{R}^n} \big(d(\omega \wedge \star\theta) - (-1)^{k-1} \omega \wedge d(\star\theta)\big) \\ &= (-1)^k \int_{\mathbb{R}^n} \omega \wedge d(\star\theta) = (-1)^{k+(k-1)(n-k+1)} \int_{\mathbb{R}^n} \omega \wedge \star(\star d \star \theta) = \langle \omega, (-1)^{(k-1)n+1} \star d \star \theta \rangle.\end{aligned}$$

So the adjoint of d with respect to the inner product (3) is the map $d^*\colon \Omega^k_c(\mathbb{R}^n) \to \Omega^{k-1}_c(\mathbb{R}^n)$ given by

$$d^* = (-1)^{(k-1)n+1} \star d \star.$$

In particular (since $\star$ is its own inverse up to sign), d^* is, up to sign, the conjugate of d by the $\star$ operator. For instance, on $\mathbb{R}^3$, $\star$ sets up an isomorphism between 1-forms and 2-forms, and since d acts on 1-forms like the curl operator on vector fields, it follows that d^* acts on 2-forms like the curl operator on vector fields.

1.1.4. Hodge star on a general oriented manifold. The inner product on forms given in (3) would have an obvious generalization to compactly-supported differential forms on an arbitrary n-dimensional oriented Riemannian manifold M if only we knew how to define the Hodge star operator $\star\colon \Omega^k_c(M) \to \Omega^{n-k}_c(M)$. We will now make this construction. Recall that an element of $\Omega^k_c(M)$ is a (smooth and compactly supported) choice, for every $p \in M$, of element ω_p from the $\binom{n}{k}$-dimensional vector space $\Lambda^k(T_pM)^*$. Here for any n-dimensional vector space V, $\Lambda^k V^*$ is the vector space of k-linear, alternating functions $V^k \to \mathbb{R}$, and has basis given by the wedge products $e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}$ $(1 \le i_1 < \cdots < i_k \le n)$ for any basis $\{e_1, \dots, e_n\}$ for T_pM with dual basis $\{e^*_1, \dots, e^*_n\}$.[3]

In the case of $\mathbb{R}^n$, with its standard basis $\{e_1, \dots, e_n\}$, the standard differential forms $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ have value $e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}$ at each point p, and the Hodge star operator was induced by the linear map $\Lambda^k(\mathbb{R}^n)^* \to \Lambda^k(\mathbb{R}^{n-k})^*$ that sends $e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}$ to $\epsilon(I) e^*_{j_1} \wedge \cdots \wedge e^*_{j_{n-k}}$ where $I = (i_1, \dots, i_k)$ and $I^\circ = (j_1, \dots, j_{n-k})$. The key, non-obvious, fact is that we would obtain the same map if we used any other oriented, orthonormal basis for $\mathbb{R}^n$ in place of $\{e_1, \dots, e_n\}$ in the construction. One could try to prove this by a computation involving the change of basis matrix, but this quickly gets ugly; instead we will find a manifestly basis-independent formulation of $\star$. (See also [**W**, p. 79, Ex. 13] for a different approach.)

[3] I'm using the same approach to the alternating algebra that I have in previous courses, see *e.g.* [**U2**, Section 4.1]; in particular in this formulation the wedge product is defined in [**U2**, p. 23, (7)]. The approach in, for instance, [**W**, Chapter 2] is of course ultimately equivalent but makes more use of tensor products, universal mapping properties, etc.

Let us consider an oriented, n-dimensional inner product space $(V, \langle\cdot,\cdot\rangle, \mathfrak{o})$. From V we can construct the vector spaces $\Lambda^k V^*$ mentioned above; we can also construct $\Lambda^k V$, which has basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k} | i_1 < \cdots < i_k\}$ for any basis $\{e_1, \ldots, e_n\}$ for V; for consistency with how we defined $\Lambda^k V^*$ we will say that we are just defining $\Lambda^k V$ as $\Lambda^k (V^*)^*$ using the canonical identification of V with $(V^*)^*$ (though there are other formulations). Note that a vector space isomorphism $A \colon V \to W$ induces a vector space isomorphism $\Lambda^k A \colon \Lambda^k V \to \Lambda^k W$ via $\Lambda^k A(e_{i_1} \wedge \cdots \wedge e_{i_k}) = (Ae_{i_1}) \wedge \cdots \wedge (Ae_{i_k})$.[4]

The inner product $\langle\cdot,\cdot\rangle$ on V induces an isomorphism

$$\ell \colon V \to V^* \quad \text{via} \quad \ell(x)(y) = \langle x, y\rangle. \tag{4}$$

The orientation $\mathfrak{o}$ on V, together with the inner product, defines a canonical generator $\omega_V = e_1 \wedge \cdots \wedge e_n$ for $\Lambda^n V$ where $e_1 \wedge \cdots \wedge e_n$ is an *oriented, orthonormal* basis for V. This element ω_V is independent of the choice of such a basis because for any other basis $\{f_1, \ldots, f_n\}$ the standard properties of wedge products show that $f_1 \wedge \cdots \wedge f_n = (\det P) e_1 \wedge \cdots \wedge e_n$ where P is the change of basis matrix from $\{e_1, \ldots, e_n\}$ to $\{f_1, \ldots, f_n\}$ and if both $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ are orthonormal and oriented then $P \in SO(n)$ and so $\det P = 1$.

Our map $\star \colon \Lambda^k V^* \to \Lambda^{n-k} V^*$ will be the following composition:

$$\Lambda^k V^* \xrightarrow{(\Lambda^k \ell)^{-1}} \Lambda^k V \xrightarrow{\phi} (\Lambda^{n-k} V)^* \xrightarrow{\iota^{-1}} \Lambda^{n-k} V^* \tag{5}$$

where:

- $\Lambda^k \ell \colon \Lambda^k V \to \Lambda^k V^*$ is the isomorphism induced by the isomorphism $\ell \colon V \to V^*$ from (4).
- $\phi \colon \Lambda^k V \to (\Lambda^{n-k} V)^*$ is defined by, for $\alpha \in \Lambda^k V$ and $\beta \in \Lambda^{n-k} V$, setting $(\phi\alpha)(\beta)$ equal to the number t such that $\alpha \wedge \beta = t\omega_V$, where ω_V is the canonical generator for $\Lambda^n V$ associated to the inner product and orientation.
- The isomorphism $\iota \colon \Lambda^{n-k} V^* \to (\Lambda^{n-k} V)^*$ is determined as follows. Let $\{e_1, \ldots, e_n\}$ be a basis for V. Then for $\theta \in \Lambda^{n-k} V^*$ we define $\iota\theta \in (\Lambda^{n-k} V)^*$ by extending linearly from

$$(\iota\theta)(e_{i_1} \wedge \cdots \wedge e_{i_{n-k}}) = \theta(e_{i_1}, \ldots, e_{i_{n-k}}) \tag{6}$$

(we make the definition initially just for $i_1 < \ldots < i_{n-k}$, but since both sides are antisymmetric in the e_i the identity continues to hold for any $(n-k)$-tuple $(i_1, \ldots, i_{n-k})$). This map ι is clearly linear and injective, so by a dimension count it is a linear isomorphism. It's not hard to see that ι is independent of the choice of basis $\{e_1, \ldots, e_n\}$ used in its definition: in fact if $f_1, \ldots, f_{n-k}$ are any elements of V given in terms of the basis $\{e_1, \ldots, e_n\}$ by $f_j = \sum_i P_{i,j} e_i$, then

$$f_1 \wedge \cdots \wedge f_{n-k} = \sum_{i_1, \ldots, i_k} P_{i_1,1} \cdots P_{i_{n-k},n-k} e_{i_1} \wedge \cdots \wedge e_{i_{n-k}},$$

and so with $\iota\theta$ defined by (6) we will have

$$(\iota\theta)(f_1 \wedge \cdots \wedge f_{n-k}) = \sum_{i_1, \ldots, i_{n-k}} P_{i_1,1} \cdots P_{i_{n-k},n-k} \theta(e_{i_1}, \ldots, e_{i_{n-k}}) = \theta(f_1, \ldots, f_{n-k}).$$

So the prescription (6) yields an isomorphism $\iota \colon \Lambda^{n-k} V^* \to (\Lambda^{n-k} V)^*$ that is independent of the basis $\{e_1, \ldots, e_n\}$ used in the formula, and in fact (6) holds for arbitrary elements $e_1, \ldots, e_{n-k}$, not necessarily coming from a basis for V.

We can now prove the following basic result about the star operator:

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[4]To make clear that this is well-defined and basis-independent, if we're regarding $\Lambda^k V$ as consisting of k-linear alternating functions on V^*, then for $\theta \in \Lambda^k V$ and $w_1^*, \ldots, w_k^* \in W^*$ we will have $\big((\Lambda^k A)(\theta)\big)(w_1^*, \ldots, w_k^*) = \theta(A^* w_1^*, \ldots, A^* w_k^*)$ where $A^* \colon W^* \to V^*$ is the adjoint of A.

PROPOSITION 1.1.5. *Let $(V, \langle\cdot,\cdot\rangle, \mathfrak{o})$ be an n-dimensional, oriented inner product space. Then for $1 \le k \le n-1$ there is a unique linear map $\star\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$ such that, for any oriented, orthonormal basis $\{e_1, \ldots, e_n\}$ for V, we have*

$$\star(e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}) = \epsilon(I) e^*_{j_1} \wedge \cdots \wedge e^*_{j_{n-k}} \tag{7}$$

for any $I = (i_1, \ldots, i_k)$ with $i_1 < \cdots < i_k$, where $I^\circ = (j_1, \ldots, j_{n-k})$ and $j_1 < \cdots < j_{n-k}$.

REMARK 1.1.6. We have excluded the somewhat-degenerate cases $k = 0, n$ from the proposition. Recall that $\Lambda^0 V^* = \mathbb{R}$, while $\Lambda^n V^*$ is one-dimensional and generated by $\mathrm{vol}_V := e^*_1 \wedge \cdots \wedge e^*_n$ for any choice of oriented orthonormal basis $\{e_1, \ldots, e_n\}$ for V. (As with the corresponding generator ω_V for $\Lambda^n V$, different choices of oriented orthonormal basis would be related by an element $P \in SO(n)$, so changing the basis would have the effect of multiplying vol_V by $\det(P^T)^{-1} = 1$; thus vol_V is canonically determined by the orientation and inner product.) So we shall define $\star\colon \Lambda^0 V^* \to \Lambda^n V^*$ by $\star(t) = t\,\mathrm{vol}_V$, and $\star\colon \Lambda^n V^* \to \Lambda^0 V^*$ by $\star(t\,\mathrm{vol}_V) = t$.

PROOF. The linearity of $\star$ and the prescription (7) for even a single oriented orthonormal basis $\{e_1, \ldots, e_n\}$ suffice to uniquely determine $\star$ if it exists, so we just need to prove existence. More specifically, we will show that the composition (5) satisfies the required property. So let $\{e_1, \ldots, e_n\}$ be an oriented orthonormal basis for V, and let $I = (i_1, \ldots, i_k)$ and $I^\circ = (j_1, \ldots, j_{n-k})$ with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{n-k}$, and let us consider the effect of (5) on the element $e^*_{i_1} \wedge \cdots \wedge e^*_{i_k} \in \Lambda^k V^*$.

Since the e_i form an orthonormal basis, the isomorphism $\ell\colon V \to V^*$ sends each e_i to the dual basis element e^*_i, so $(\Lambda^k \ell)^{-1}$ sends $e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}$ to $e_{i_1} \wedge \cdots \wedge e_{i_k}$.

To determine the value $\phi(e_{i_1} \wedge \cdots \wedge e_{i_k}) \in (\Lambda^{n-k} V)^*$, since $\Lambda^{n-k} V$ has basis given by $e_{a_1} \wedge \cdots \wedge e_{a_{n-k}}$ with $a_1 < \cdots < a_{n-k}$ it suffices to find each $\phi(e_{i_1} \wedge \cdots \wedge e_{i_k})(e_{a_1} \wedge \cdots \wedge e_{a_{n-k}})$. Now if $(a_1, \ldots, a_{n-k}) \neq I^\circ$, then the pigeonhole principle (and the fact that the i_r and a_r are ordered) shows that one of the a_r is equal to one of the i_s and hence that $e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{a_1} \wedge \cdots \wedge e_{a_{n-k}} = 0$. Meanwhile for $(a_1, \ldots, a_{n-k}) = I^\circ = (j_1, \ldots, j_{n-k})$, we have (by the definition of the sign $\epsilon(I)$)

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{n-k}} = \epsilon(I) e_1 \wedge \cdots \wedge e_n.$$

So for any ordered $(n-k)$-tuple $(a_1, \ldots, a_{n-k})$ we have

$$\Big(\phi \circ (\Lambda^k \ell)^{-1}(e^*_{i_1} \wedge \cdots \wedge e^*_{i_k})\Big)(e_{a_1} \wedge \cdots \wedge e_{a_{n-k}}) = \begin{cases} \epsilon(I) & (a_1, \ldots, a_{n-k}) = I^\circ \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

But using the basis $\{e_1, \ldots, e_n\}$ to compute the isomorphism ι via (6), it is clear that (8) still holds if its left hand side is replaced by $\iota(\epsilon(I) e^*_{j_1} \wedge \cdots \wedge e^*_{j_{n-k}})(e_{a_1} \wedge \cdots \wedge e_{a_k})$. So we have

$$\Big(\phi \circ (\Lambda^k \ell)^{-1}(e^*_{i_1} \wedge \cdots \wedge e^*_{i_k})\Big) = \iota(\epsilon(I) e^*_{j_1} \wedge \cdots \wedge e^*_{j_{n-k}}),$$

which proves that our map $\star = \iota^{-1} \circ \phi \circ (\Lambda^k \ell)^{-1}$ indeed satisfies the property (7). □

With Proposition 1.1.5 in hand we can readily generalize the constructions of Section 1.1.3. First we explicitly define the notion of a Riemannian manifold:

DEFINITION 1.1.7. A Riemannian manifold is a pair (M, g) where M is a smooth manifold and a Riemannian metric g. In other words, g is a choice, for all $p \in M$, of an inner product g_p on the real vector space $T_p M$, satisfying the smoothness condition that, for any two (smooth) vector fields X, Y on M, the function $g(X, Y)\colon M \to \mathbb{R}$ defined by $p \mapsto g_p(X_p, Y_p)$ is a smooth function.

The smoothness condition is equivalent to the statement that, in any local coordinate chart $(x_1, \ldots, x_n)\colon U \to \mathbb{R}^n$ for M, the inner products $g_{ij}(p) = g_p\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ are smooth functions of p. Thus in local coordinates the Riemannian metric g is represented by a symmetric, positive-definite matrix $(g_{ij}(p))$ that varies smoothly from point to point. One should not expect in general to be able to choose coordinates in which the matrix is (even locally) constant—the curvature of the metric (which we do not define here) gives an obstruction to doing so.

Let (M,g) be an oriented n-dimensional Riemannian manifold. The orientation of M together with the Riemannian metric g endow each T_pM with the structure of an oriented inner product space. This allows us to define a Hodge star operator $\Omega_c^k(M) \to \Omega_c^{n-k}(M)$ for any $0 \le k \le n$ simply by $(\star\omega)_p = \star(\omega_p)$ where the $\star$ on the right hand side is given by Proposition 1.1.5 or Remark 1.1.6. The same discussion as at the end of Section 1.1.2 shows that

$$\star\star = (-1)^{k(n-k)} \colon \Omega_c^k(M) \to \Omega_c^k(M).$$

Just as in the case of $\mathbb{R}^n$ we can define an inner product $\langle\cdot,\cdot\rangle$ on $\Omega_c^k(M)$ by setting

$$\langle\omega,\theta\rangle = \int_M \omega\wedge\star\theta. \tag{9}$$

EXERCISE 1.1.8. *Prove that (9) is indeed an inner product.*

Furthermore, we can again use Stokes' theorem[5] to find, for $\omega\in\Omega_c^{k-1}(M)$ and $\theta\in\Omega_c^k(M)$ that

$$\begin{aligned}\langle d\omega,\theta\rangle &= \int_M \left(d(\omega\wedge\star\theta) - (-1)^{k-1}\omega\wedge d(\star\theta)\right)\\ &= (-1)^k\int \omega\wedge(-1)^{(n-k+1)(k-1)}\star\star d\star\theta = \langle\omega,(-1)^{n(k-1)+1}\star d\star\theta\rangle\end{aligned}$$

in view of which

$$d^* = (-1)^{n(k-1)+1}\star d\star\colon \Omega^k(M)\to\Omega^{k-1}(M) \tag{10}$$

satisfies the adjoint relation $\langle d\omega,\theta\rangle = \langle\omega,d^*\theta\rangle$.

EXERCISE 1.1.9. *Prove that $d^*\circ d^* = 0$, and that the map $\star$ induces an isomorphism*

$$\frac{\ker(d^*\colon \Omega^k(M)\to\Omega^{k-1}(M))}{\operatorname{Im}(d^*\colon \Omega^{k+1}(M)\to\Omega^k(M))} \cong H^{n-k}(M).$$

As suggested in Section 1.1.2 we will let

$$\mathscr{H}^k(M) = \ker(d)\cap\ker(d^*) = \ker(d^*d + dd^*), \tag{11}$$

where the second equality follows from Proposition 1.1.3, and prove:

THEOREM 1.1.10 (Hodge theorem, first version). *Let (M,g) be a compact, oriented Riemannian manifold, define d^* by (10) and define $\mathscr{H}^k(M)$ by (11). Then $\mathscr{H}^k(M)$ is a finite-dimensional vector space, and $Z^k(M) = B^k(M)\oplus\mathscr{H}^k(M)$.*

By Proposition 1.1.1 it then follows that $\mathscr{H}^k(M)\cong H^k(M)$; thus one immediate consequence of Theorem 1.1.10 is that $H^k(M)$ is finite-dimensional for all k if M is a compact oriented smooth manifold.[6] One also obtains the Poincaré duality result that $\dim H^k(M) = \dim H^{n-k}(M)$ from the following:

PROPOSITION 1.1.11. *The Hodge star operator $\star$ restricts to $\mathscr{H}^k(M)$ as an isomorphism $\mathscr{H}^k(M)\to\mathscr{H}^{n-k}(M)$.*

[5]Throughout these notes all manifolds are manifolds *without boundary* unless otherwise stated; recall that Stokes' theorem in this context says that if M is an n-dimensional manifold without boundary and if $\alpha\in\Omega_c^{n-1}(M)$ then $\int_M d\omega = 0$. If we considered manifolds with boundary and allowed our differential forms to be nonzero at the boundary then additional terms would be required in the adjoint.

[6]Note that any smooth manifold admits a Riemannian metric, as follows from a partition-of-unity argument.

PROOF. Since $\star\star = (-1)^{k(n-k)}$ it is clear that $\star$ is a bijection between $\Omega^k(M)$ and $\Omega^{n-k}(M)$, so we just need to see that $\star$ maps $\mathcal{H}^k(M)$ to $\mathcal{H}^{n-k}(M)$ (and vice versa). But we have, for $\omega \in \Omega^k(M)$,

$$d(\star\omega) = \pm \star \star d \star \omega = \pm \star d^*\omega$$

so since $\star$ is an isomorphism we have

$$d^*\omega = 0 \Leftrightarrow d(\star\omega) = 0 \text{ and likewise } d^*(\star\omega) = 0 \Leftrightarrow d\omega = 0.$$

Thus $\omega \in \mathcal{H}^k(M) \Leftrightarrow \star\omega \in \mathcal{H}^{n-k}(M)$. □

The discussion so far should indicate that Theorem 1.1.10 would be easy to prove if $Z^k(M)$ were finite-dimensional, but of course $Z^k(M)$ is not finite-dimensional and so various analytical subtleties such as the one suggested by Proposition 1.1.2 become relevant. The proof of Theorem 1.1.10 will require a clearer understanding of certain properties of the Hodge Laplacian $d^*d + dd^*$; these properties are shared more generally by the class of elliptic differential operators which appear often in the geometry of manifolds. We will begin working toward this understanding in the next section. But first we will reformulate the Hodge theorem in a way that more closely reflects what the analytical argument will show, and draw out some consequences of this (including the original formulation Theorem 1.1.10).

THEOREM 1.1.12 (Hodge theorem, second version). *Let M be a compact oriented Riemannian manifold and endow $\Omega^k(M)$ with the inner product $\langle\cdot,\cdot\rangle$ from (9). Define $\Delta = d^*d + dd^*$, so that $\mathcal{H}^k(M) = \ker(\Delta)$. Then $\mathcal{H}^k(M)$ is finite-dimensional, and*

$$\operatorname{Im}(\Delta\colon \Omega^k(M) \to \Omega^k(M)) = \mathcal{H}^k(M)^\perp. \tag{12}$$

REMARK 1.1.13. Note that Δ is easily seen to be (formally) self-adjoint in the sense that $\langle \Delta\omega, \theta\rangle = \langle \omega, \Delta\theta\rangle$, as an immediate consequence of the fact that d and d^* are adjoint to each other. So if $\omega = \Delta\alpha \in \operatorname{Im}(\Delta)$ and $\theta \in \mathcal{H}^k(M)$ then certainly $\langle\omega,\theta\rangle = \langle\alpha,\Delta\theta\rangle = 0$; thus the inclusion "$\subset$" in (12) is straightforward. So the difficult parts of the Hodge theorem are the statements that $\mathcal{H}^k(M)$ is finite-dimensional and that any $\omega \in \Omega^k(M)$ satisfying the obviously-necessary condition that $\langle\omega,\theta\rangle = 0$ for all $\theta \in \mathcal{H}^k(M)$ can in fact be written as $\omega = \Delta\alpha$ for some $\alpha \in \Omega^k(M)$.

We now extract some consequences of this formulation of the Hodge theorem:

COROLLARY 1.1.14. *Assuming Theorem 1.1.12, we have an orthogonal direct sum decomposition*

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus \operatorname{Im}(\Delta\colon \Omega^k(M) \to \Omega^k(M)).$$

PROOF. We have already observed that the two summands are orthogonal and Theorem 1.1.12 says that $\operatorname{Im}(\Delta) = \mathcal{H}^k(M)^\perp$, so we just have to show that any $\omega \in \Omega^k(M)$ can be written as $\omega = x + y$ where $x \in \mathcal{H}^k(M)$ and $y \in \mathcal{H}^k(M)^\perp$. Although we noted earlier (cf. Proposition 1.1.2) that this sort of question is generally subtle, in this particular case it is easy because Theorem 1.1.12 also says that $\mathcal{H}^k(M)$ is finite-dimensional. So we can choose an orthonormal basis $\{e_1, \dots, e_N\}$ for $\mathcal{H}^k(M)$ and write

$$\omega = \left(\sum_{i=1}^N \langle\omega, e_i\rangle e_i\right) + \left(\omega - \sum_{i=1}^N \langle\omega, e_i\rangle e_i\right)$$

where the first expression in parentheses belongs to $\mathcal{H}^k(M)$ and the second belongs to $\mathcal{H}^k(M)^\perp$. □

COROLLARY 1.1.15. *Assuming Theorem 1.1.12, we have an orthogonal direct sum decomposition*

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus \operatorname{Im}(d\colon \Omega^{k-1}(M) \to \Omega^k(M)) \oplus \operatorname{Im}(d^*\colon \Omega^{k+1}(M) \to \Omega^k(M)).$$

PROOF. By Corollary 1.1.14 we just need to show that $\mathrm{Im}(\Delta) = \mathrm{Im}(d) + \mathrm{Im}(d^*)$ and that $\langle \omega, \theta \rangle = 0$ whenever $\omega \in \mathrm{Im}(d)$ and $\theta \in \mathrm{Im}(d^*)$. The latter statement is clear, since if $\omega = d\alpha$ and $\theta = d^*\beta$ then

$$\langle \omega, \theta \rangle = \langle d\alpha, d^*\beta \rangle = \langle dd\alpha, \beta \rangle = 0$$

since $d^2 = 0$. Also the statement that $\mathrm{Im}(\Delta) \subset \mathrm{Im}(d) + \mathrm{Im}(d^*)$ is clear from the equation $\Delta(\omega) = d(d^*\omega) + d^*(d\omega)$.

For the reverse inclusion, we note that if $\omega = d\alpha$, then for all $\phi \in \mathscr{H}^k(M)$ we have

$$\langle \omega, \phi \rangle = \langle \alpha, d^*\phi \rangle = 0$$

because Proposition 1.1.3 implies that $d^*\phi = 0$. Thus $\mathrm{Im}(d) \subset \mathscr{H}^k(M)^\perp$. The same argument (using instead that $d\phi = 0$ by Proposition 1.1.3) shows that $\mathrm{Im}(d^*) \subset \mathscr{H}^k(M)^\perp$. So we have

$$\mathrm{Im}(d) \oplus \mathrm{Im}(d^*) \subset \mathscr{H}^k(M)^\perp = \mathrm{Im}(\Delta)$$

where the last equality is given by Theorem 1.1.12, and the result follows. □

PROOF OF THEOREM 1.1.10, ASSUMING THEOREM 1.1.12. The statement that $\mathscr{H}^k(M)$ is finite-dimensional is part of Theorem 1.1.12, so we just need to show that $Z^k(M) = B^k(M) \oplus \mathscr{H}^k(M)$. If $\omega \in Z^k(M)$, then Corollary 1.1.15 allows us to write $\omega = \phi + d\alpha + d^*\beta$ for some $\phi \in \mathscr{H}^k(M)$, $\alpha \in \Omega^{k-1}(M)$, and $\beta \in \Omega^{k+1}(M)$. Moreover $\langle \phi, d^*\alpha \rangle = \langle d\alpha, d^*\beta \rangle = 0$, so we have

$$\langle d^*\beta, d^*\beta \rangle = \langle \omega, d^*\beta \rangle = \langle d\omega, \beta \rangle = 0,$$

so in fact $d^*\beta = 0$ and $\omega = \phi + d\alpha$.

This proves that $Z^k(M) \subset B^k(M) \oplus \mathscr{H}^k(M)$, while the reverse inclusion follows from the definitions (and Proposition 1.1.3). □

The "Hodge decomposition" in Corollary 1.1.15 leads to additional interesting consequences, including a PDE-free characterization of the space $\mathscr{H}^k(M)$ of harmonic forms. Recall that by definition a de Rham cohomology class $c \in H^k(M)$ is a coset of $B^k(M)$ in $Z^k(M)$, so that if we choose one element $\omega_0 \in c$ (necessarily having $d\omega_0 = 0$) then $c = \{\omega_0 + d\beta | \beta \in \Omega^{k-1}(M)\}$.

COROLLARY 1.1.16. *Assume Theorem 1.1.12. Then for any $c \in H^k(M)$ there is a unique element $\omega \in c$ such that, for all $\theta \in c$, we have*

$$\langle \omega, \omega \rangle \leq \langle \theta, \theta \rangle \text{ with equality only if } \theta = \omega$$

Moreover this element ω is the unique element of c that also belongs to $\mathscr{H}^k(M)$.

PROOF. Choose an aribitrary $\omega_0 \in c$. As was shown in the proof of Theorem 1.1.10, we can write $\omega_0 = \omega + d\alpha$ for some $\omega \in \mathscr{H}^k(M)$ and $\alpha \in \Omega^{k-1}(M)$ (the $\mathrm{Im}(d^*)$ term vanishes because $d\omega_0 = 0$). In particular $\omega \in c \cap \mathscr{H}^k(M)$. If there were another element $\omega' \in c \cap \mathscr{H}^k(M)$, then we would have $\omega - \omega' \in \mathscr{H}^k(M) \cap B^k(M) = \{0\}$, so ω is in fact the unique element of $c \cap \mathscr{H}^k(M)$.

We can then write any $\theta \in c$ as $\theta = \omega + d\beta$ for some $\beta \in \Omega^{k-1}(M)$. The orthogonality of the splitting in Corollary 1.1.15 then shows that

$$\langle \theta, \theta \rangle = \langle \omega, \omega \rangle + \langle d\beta, d\beta \rangle \geq \langle \omega, \omega \rangle,$$

with equality iff $d\beta = 0$, *i.e.* iff $\theta = \omega$. □

Thus harmonic forms can be characterized as precisely those closed forms ω which *minimize the "energy"* $\langle \omega, \omega \rangle = \int_M \omega \wedge \star\omega$ among all forms in their cohomology class $[\omega]$. Moreover in any cohomology class there is a unique such form.

The following exercise gives a notable consequence of the Hodge decomposition that is less directly related to cohomology:

EXERCISE 1.1.17. *Let M be a compact oriented Riemannian manifold, and assume Theorem 1.1.12 (and hence all of its corollaries). Prove that if ω is any exact form, then there is a unique $\alpha \in \mathrm{Im}(d^*)$ such that $\omega = d\alpha$. Moreover, prove that if $\beta \in \Omega^{k-1}(M)$ with $d\beta = \omega$, then $\langle \beta, \beta \rangle \geq \langle \alpha, \alpha \rangle$, with equality only if $\beta = \alpha$.*

(Hints: For the existence of α, use the facts that $\mathrm{Im}(d) \subset \mathrm{Im}(\Delta)$ (proven in the proof of Corollary 1.1.15) and that $\mathrm{Im}(d)$ and $\mathrm{Im}(d^*)$ are orthogonal. For uniqueness, you might draw inspiration from the proof of Proposition 1.1.3.)

Thus Hodge theory singles out a preferred solution to the equation $d\alpha = \omega$ whenever this equation has a solution; namely, if one takes α equal to the unique element of $\mathrm{Im}(d^*)$ (*i.e.* the unique "coexact" form) with $d\alpha = \omega$, then this choice of α will have the strictly smallest possible energy $\langle \alpha, \alpha \rangle$ among all solutions.

By Exercise 1.1.9, if $H^{n-k}(M) = \{0\}$ then a k-form is coexact if and only if it is coclosed (*i.e.*, in $\ker(d^*)$). So in this case if ω is an exact $(k+1)$-form then there is a unique $\beta \in \Omega^k(M)$ satisfying the system of differential equations

$$\begin{cases} d\beta &= \omega \\ d^*\beta &= 0 \end{cases}.$$

To relate this to physics, let $n = 3$ and $k = 1$. A 2-form ω written locally as $B_1 dy \wedge dz + B_2 dz \wedge dx + B_3 dx \wedge dy$ can be seen as corresponding to a magnetic field $\vec{B} = (B_1, B_2, B_3)$. One of Maxwell's equations (corresponding to the nonexistence of magnetic monopoles) says that $\nabla \cdot \vec{B} = 0$, *i.e.* that $d\omega = 0$. So if $H^2(M) = 0$ there is $\alpha \in \Omega^1(M)$ with $d\alpha = \omega$. We can likewise view $\alpha = A_1 dx + A_2 dy + A_3 dz$ as corresponding to a vector field $\vec{A} = (A_1, A_2, A_3)$, and the equation $d\alpha = \omega$ says that $\nabla \times \vec{A} = \vec{B}$, *i.e.* that $\vec{A}$ is what physicists call a "vector potential" for the magnetic field $\vec{B}$. There are many such vector potentials for any given $\vec{B}$, corresponding to all the solutions to $d\alpha = \omega$; one is said to "fix a gauge" when one makes a choice of the vector potential. One gauge-fixing criterion that is commonly used is the "Coulomb gauge," wherein one requires $\nabla \cdot \vec{A} = 0$. This equation is equivalent to $d^*\alpha = 0$. So it follows that any magnetic field on a compact oriented 3-manifold M with $H^2(M) = 0$ has a unique vector potential which is in Coulomb gauge.

1.2. Introductory Analysis of the Laplacian

1.2.1. The Hodge Laplacian in local coordinates. Our intention is to look for solutions $\omega \in \Omega^k(M)$ to the equation $\Delta\omega = \alpha$ for arbitrary $\alpha \in \mathscr{H}^k(M)^\perp$; doing this will require some information about what this equation says when ω is written in suitable local coordinates.

As before we assume that (M, g) is a compact oriented Riemannian manifold. The Hodge star operator $\star\colon \Omega^k(M) \to \Omega^{n-k}(M)$ is then induced pointwise by linear maps $\star\colon \Lambda^k T_pM^* \to \Lambda^{n-k} T_pM^*$ for each $p \in M$, which are characterized by the property that, if $\{e_1, \ldots, e_n\}$ is any oriented orthonormal basis for T_pM with dual basis $\{e^1, \ldots, e^n\}$, then $\star(e^1 \wedge \cdots \wedge e^k) = e^{k+1} \wedge \cdots \wedge e^n$. (The values of the various $\star(e^{i_1} \wedge \cdots \wedge e^{i_k})$ can then be inferred by reordering the oriented basis.) Then $d^*\colon \Omega^k(M) \to \Omega^{k-1}(M)$ is given by $d^* = (-1)^{n(k-1)+1} \star d \star$ and $\Delta = d^*d + dd^*$.

Choose a coordinate chart $\phi = (x_1, \ldots, x_n)\colon U \to \mathbb{R}^n$ for M, with $\phi(p) = 0$. Given an oriented orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM, by postcomposing ϕ with some linear map that sends the basis $\{\phi_* e_1, \ldots, \phi_* e_n\}$ to the standard basis for $\mathbb{R}^n$, we may assume that $\{\phi_* e_1, \ldots, \phi_* e_n\}$ is equal to the standard basis in $\mathbb{R}^n$. In other words, for each i we have $e_i = \left.\frac{\partial}{\partial x_i}\right|_p \in T_pM$ where we use the standard notation for the ith coordinate tangent vector at p induced by the coordinate chart.

Now the coordinate chart indeed gives us vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ throughout the open set U, and so the Riemannian metric g gives us functions

$$g_{ij}\colon U \to \mathbb{R} \quad \text{defined by} \quad g_{ij}(q) = g\left(\left.\frac{\partial}{\partial x_i}\right|_q, \left.\frac{\partial}{\partial x_j}\right|_q \right).$$

For each $q \in U$, the matrix $(g_{ij}(q))$ is symmetric and positive definite. Since the coordinate basis coincides with the orthonormal basis $\{e_1, \dots, e_n\}$ at p, we have $g_{ij}(p) = \delta_{ij}$; however this equality generally holds only at p and not elsewhere. In some special cases it may be possible to choose the coordinate chart ϕ so that $g_{ij} = \delta_{ij}$ everywhere, but if the curvature of the metric g is nonzero then this will not be the case.[7]

In spite of this warning, let us compute Δ in terms of the local coordinates $x_1, \dots, x_n$ on $U \subset M$ under the assumption that $g_{ij} = \delta_{ij}$ throughout U; after doing this we will explain qualitatively what happens in the more general situation. Thus everywhere in U, $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ gives an oriented orthonormal basis for the tangent space, and so (recalling that, essentially by definition, $\{dx_1, \dots, dx_n\}$ is the dual basis to $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$), the action of the Hodge star operator is given by the formula from (2).

By linearity and reordering coordinates it suffices to work out the action of $\Delta = d^*d + dd^*$ on a differential k-form of the shape $f\,dx_1 \wedge \cdots \wedge dx_k$. Beginning with the action of dd^*, we obtain (where $\widehat{\ }$ is used to signify omission of a term from a wedge product)

$$\begin{aligned}
dd^*(f\,dx_1 \wedge \cdots \wedge dx_k) &= (-1)^{n(k-1)+1} d \star d \star (f\,dx_1 \wedge \cdots \wedge dx_k) \\
&= (-1)^{n(k-1)+1} d \star d(f\,dx_{k+1} \wedge \cdots \wedge dx_n) = (-1)^{n(k-1)+1} \sum_{i=1}^{k} d \star \left(\frac{\partial f}{\partial x_i} dx_i \wedge dx_{k+1} \cdots \wedge dx_n \right) \\
&= (-1)^{n(k-1)+1} \sum_{i=1}^{k} (-1)^{(k-1)(n-k)+i-1} d\left(\frac{\partial f}{\partial x_i} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \right) \\
&= \sum_{i=1}^{k} (-1)^i \left(\frac{\partial^2 f}{\partial x_i^2} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k + \sum_{j=k+1}^{n} (-1)^{k-1} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \wedge dx_j \right) \\
&= -\left(\sum_{i=1}^{k} \frac{\partial^2 f}{\partial x_i^2} dx_1 \wedge \cdots \wedge dx_k \right) + \sum_{i=1}^{k} \sum_{j=k+1}^{n} (-1)^{i+k-1} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \wedge dx_j.
\end{aligned} \tag{13}$$

(The simplification of the sign in the second to last equality uses that $n(k-1) + (k-1)(n-k) = (k-1)(2n-k)$ which is even since $k(k-1)$ is always even.)

[7] More specifically, by using "exponential coordinates" one can arrange for the partial derivatives of the g_{ij} to be zero at p, but the curvature at p gives a coordinate-independent obstruction to the second derivatives all vanishing at p. See [**dC**, Chapter 4] or any other text on Riemannian geometry for more details.

Meanwhile,

$$d^*d(f\,dx_1\wedge\cdots\wedge dx_k) = (-1)^{nk+1}\star d\star(df\wedge dx_1\wedge\cdots\wedge dx_k) = (-1)^{nk+k+1}\star d\star\left(\sum_{j=k+1}^{n}\frac{\partial f}{\partial x_j}dx_1\wedge\cdots\wedge dx_k\wedge dx_j\right)$$

$$= (-1)^{nk+k+1}\star d\left(\sum_{j=k+1}^{n}(-1)^{j-k-1}\frac{\partial f}{\partial x_j}dx_{k+1}\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n\right)$$

$$= \sum_{j=k+1}^{n}(-1)^{nk+j}\star\left((-1)^{j-k-1}\frac{\partial^2 f}{\partial x_j^2}dx_{k+1}\wedge\cdots\wedge dx_n + \sum_{i=1}^{k}\frac{\partial^2 f}{\partial x_i\partial x_j}dx_i\wedge dx_{k+1}\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n\right)$$

$$= (-1)^{nk+k-1+k(n-k)}\sum_{j=k+1}^{n}\frac{\partial^2 f}{\partial x_j^2}dx_1\wedge\cdots\wedge dx_k$$

$$+\sum_{i=1}^{k}\sum_{j=k+1}^{n}(-1)^{nk+j+(i-1)(n-k)+(k-i)(n-k-1)+n-j}\frac{\partial^2 f}{\partial x_i\partial x_j}dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_k\wedge dx_j.$$

Now $nk+k-1+k(n-k) = -1+k(2n+1-k)$ is always odd (since $k(1-k)$ is always even). Meanwhile $nk+j+(i-1)(n-k)+(k-i)(n-k-1)+n-j = n(k+1)+(n-k)(k-1)-(k-i) = 2kn-k(k-1)-k-i$ has the same parity as $i+k$. So we obtain:

(14)

$$d^*d(f\,dx_1\wedge\cdots\wedge dx_k) = -\left(\sum_{j=k+1}^{n}\frac{\partial^2 f}{\partial x_j^2}dx_1\wedge\cdots\wedge dx_k\right)+\sum_{i=1}^{k}\sum_{j=k+1}^{n}(-1)^{i+k}\frac{\partial^2 f}{\partial x_i\partial x_j}dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_k\wedge dx_j.$$

Combining (13) with (14) we see significant cancellation yielding that

$$\Delta(f\,dx_1\wedge\cdots\wedge dx_k) = \left(-\sum_{i=1}^{n}\frac{\partial^2 f}{\partial x_i^2}\right)dx_1\wedge\cdots\wedge dx_k \quad \text{if } g_{ij}\equiv\delta_{ij}.$$

Of course this is consistent with what we found in Example 1.1.4.

As mentioned earlier, in general we can only hope to choose coordinates in which g_{ij} coincides with δ_{ij} at a prescribed point p, rather than throughout the domain U of the coordinate chart. In the more general situation, applying the Gram-Schmidt process pointwise to the bases $\{\frac{\partial}{\partial x_1}|_q,\ldots,\frac{\partial}{\partial x_n}|_q\}$ yields vector fields $e_1,\ldots,e_n$ on U with each $\{(e_1)_q,\ldots,(e_n)_q\}$ an oriented orthonomal basis for T_qM. Denoting by $\{e^1,\ldots,e^n\}$ the corresponding dual basis of one-forms, the Hodge star operator is determined not by (2) but rather by $\star(e^1\wedge\cdots\wedge e^k) = e^{k+1}\wedge\cdots\wedge e^n$ (and similar identities for the other $e^{i_1}\wedge\cdots\wedge e^{i_k}$, with the usual signs $\epsilon(I)$).

We could repeat the above calculation of Δ, working consistently in terms of the pointwise basis of forms $\{e^{i_1}\wedge\cdots\wedge e^{i_m}\}$ in order to compute $\Delta(f e^1\wedge\cdots\wedge e^k)$. This will lead to more complexity, however, because the one-forms e^i (unlike the coordinate forms dx_i) have no reason to be closed. For our purposes it will suffice to give a qualitative description of what ensues. We will encounter various derivatives of the form

$$d(he^{i_1}\wedge\cdots\wedge e^{i_m}) = (dh)\wedge e^{i_1}\wedge\cdots\wedge e^{i_m} + hd(e^{i_1}\wedge\cdots\wedge e^{i_m}). \tag{15}$$

Note that, although the e_i do not come from a coordinate system, we can still expand $dh = \sum_i(\nabla_{e_i}h)e^i$ where we define $\nabla_{e_i}h = dh(e_i)$ as the derivative of h along the vector field e_i. Thus the first term on the right hand side of (15) (which is the only term that involves the derivative of h) is

$\sum_i (\nabla_{e_i} h)(e^i \wedge e^{i_1} \wedge \cdots \wedge e^{i_k})$; this is just like what one sees in the simpler case that $e^i = dx_i$. On the other hand the second term involving $d(e^{i_1} \wedge \cdots \wedge e^{i_m})$ is rather different.

Fortunately, however, the terms of $\Delta(f e^1 \wedge \cdots \wedge e^k)$ that involve the most derivatives (*i.e.*, two derivatives) of f are the ones that will be most relevant to us. If one were to pay attention only to terms in Leibniz rule expansions that involve differentiating f rather than the e^i, then the discussion in the previous paragraph shows that the calculation of $\Delta(f e^1 \wedge \cdots \wedge e^k)$ is identical to the derivations of (13 and 14), just with dx_i's replaced by e^i's and $\frac{\partial}{\partial x_i}$'s replaced by ∇_{e_i}'s. I do not intend to work out anything more about the other terms that appear in $\Delta(f e^1 \wedge \cdots \wedge e^k)$ beyond the facts that they depend linearly on the input and involve differentiating f either zero or one times (and also involve some number of differentiations of the e^i).[8]

To make this more precise, using the abbreviation e^I for $e^{i_1} \wedge \cdots \wedge e^{i_k}$ for k-tuples $I = (i_1, \ldots, i_k)$, the following should be evident from the above discussion.

PROPOSITION 1.2.1. *Let $(e_1, \ldots, e_n)$ be an oriented orthonormal frame of vector fields throughout some coordinate chart U for a Riemannian manifold (M, g), and let $f \in C^\infty(U)$. Then for each $I = (i_1, \ldots, i_k)$ we have*

$$\Delta(f e^I) = \left(-\sum_{i=1}^n \nabla_{e_i}(\nabla_{e_i} f) \right) e^I + \sum_{i=1}^n \sum_J (\nabla_{e_i} f) \beta^i_{IJ} e^J + \sum_J \gamma_{IJ} f_I e^J$$

for some $\beta^i_{IJ}, \gamma_{IJ} \in C^\infty(U)$ depending only on the orthonormal frame and not on f.

1.2.2. Harmonic functions and the mean value property. For the rest of Section 1.2 we will work with real-valued functions defined on open subsets $\Omega \subset \mathbb{R}^n$, and we will regard the Laplacian on such functions as defined by

$$\Delta u = -\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}.$$

"Laplace's equation" is then the statement that $\Delta u = 0$; more generally one can prescribe a function $f : \Omega \to \mathbb{R}^n$ and consider "Poisson's equation" $\Delta u = f$. A function on $\mathbb{R}^n$ is said to be harmonic if it satisfies Laplace's equation.

Notice that, with our sign convention, $\Delta u = -\nabla \cdot (\nabla u) = -\mathrm{div}(\mathrm{grad}\, u)$ in terms of the familiar divergence and gradient operators from multivariable calculus. Consequently, if $u \in C^2(\Omega)$ is harmonic and if $x_0 \in \Omega$ with the closed radius-r ball $B_r(x_0) \subset \Omega$, we obtain:

$$0 = \int_{B_r(x_0)} (\Delta u) dV = -\int_{B_r(x_0)} \mathrm{div}(\mathrm{grad}\, u) dV = -\int_{\partial B_r(x_0)} (\mathrm{grad}\, u) \cdot \nu dS$$

where we have used the divergence theorem.[9] Now $(\mathrm{grad}\, u) \cdot \nu$ is of course equal to the directional derivative of f along ν, so if σ denotes the standard volume form on S^{n-1} the above shows that,

[8] While we won't pursue this, the detailed structure of these terms does have some interesting features and consequences for Riemannian geometry, see for instance [**P**, Section 7.3.3].

[9] Here $dV = dx_1 \wedge \cdots \wedge dx_n$ is the standard volume form on $\mathbb{R}^n$ and dS is the standard volume form on $\partial B_r(x_0)$ and ν is the outward normal vector along the boundary. So we have $dS = \iota_\nu dV$ as differential forms (ι_ν means insertion of ν as the first argument of dV). While the divergence theorem as taught in multivariable classes is usually phrased for domains in $\mathbb{R}^3$, it is equally valid in $\mathbb{R}^n$: to give a Hodge-star-based proof, if $\mathbf{F} = (F_1, \ldots, F_n)$ is a vector field on $\mathbb{R}^n$ we can construct the one-form $\phi = \sum F_i dx_i$, and then the reader can easily verify that $(\mathrm{div}\, \mathbf{F}) dV = d(\star\phi)$. So Stokes' theorem gives $\int_\Omega (\mathrm{div}\, F) dV = \int_{\partial\Omega} (\star\phi)$. Given a point $x \in \partial\Omega$, if we work in a rotated coordinate system in which $T_x \partial\Omega = \{x_1 = 0\}$, we then have

$$(\star\phi)|_{T_x\partial\Omega} = F_1 dx_2 \wedge \cdots \wedge dx_n = (\mathbf{F} \cdot \nu) dS|_{T_x\partial\Omega}.$$

This holds at each $x \in \partial\Omega$, so $\star\phi$ restricts to $\partial\Omega$ as $(\mathbf{F} \cdot \nu) dS$, and we indeed have $\int_\Omega (\mathrm{div}\, \mathbf{F}) dV = \int_{\partial\Omega} (\mathbf{F} \cdot \nu) dS$.

provided that $B_r(x_0) \subset \Omega$,

$$\int_{\nu \in S^{n-1}} \frac{d}{dt} u(x_0 + t\nu)\bigg|_{t=r} \sigma = 0.$$

This applies equally well with r replaced by an arbitrary value s between 0 and r; integrating over s then yields

$$0 = \int_0^r \left(\int_{\nu \in S^{n-1}} \frac{d}{dt} u(x_0 + t\nu)\bigg|_{t=s} \sigma \right) ds = \int_0^r \frac{d}{ds} \left(\int_{S^{n-1}} u(x_0 + s\nu)\sigma \right) ds = \int_{\nu \in S^{n-1}} (u(x_0 + r\nu) - u(x_0))\sigma$$

where we have used Fubini's theorem and the fundamental theorem of calculus.

In other words, if u is harmonic on Ω then we have $\int_{\partial B_r(x_0)} (u - u(x_0)) dS = 0$ whenever $B_r(x_0) \subset \Omega$. This is one version of the *mean value property* for harmonic functions. We will instead using a version that integrates over balls rather than spheres and follows almost immediately: we equally well have $\int_{\partial B_s(x_0)} (u - u(x_0)) dS = 0$ for $0 < s < r$, and then integrating over s (with a weight s^{n-1} to reflect a change from polar to Cartesian coordinates) and applying Fubini's theorem produces $\int_{B_r(x_0)} (u - u(x_0)) dV = 0$. In other words, denoting by $⨍_U f\, dV$ the average value $\frac{1}{vol(U)} \int f\, dV$ of a function f over a bounded set U, if u is harmonic in Ω then we have, on any closed ball $B_r(x_0)$ contained in Ω,

$$u(x_0) = ⨍_{B_r(x_0)} u\, dV. \tag{16}$$

This is called the *mean value property* for harmonic functions.

Conversely, if u is not harmonic, choose a point x_0 with $(\Delta u)(x_0) \neq 0$; for convenience (replacing u by $-u$ if necessary) let us assume that $\Delta u(x_0) > 0$. We assumed that u was C^2, so this implies $\Delta u > 0$ throughout some ball $B_r(x_0)$.

Reasoning in exactly the same way as before, the divergence theorem now gives $\int_{\partial B_s(x_0)} (\operatorname{grad} u) \cdot \nu dS < 0$ for all $s < r$, from which we infer via reasoning just like that above that $u(x_0) > ⨍_{B_r(x_0)} u\, dV$. To summarize:

DEFINITION 1.2.2. If $u \colon \Omega \to \mathbb{R}$ is any locally integrable[10] function on an open set $\Omega \subset \mathbb{R}^n$, we say that *$u$ satisfies the mean value property* provided that for all $x_0 \in \Omega$ and all $r > 0$ such that $B_r(x_0) \subset \Omega$, we have

$$u(x_0) = ⨍_{B_r(x_0)} u\, dV.$$

PROPOSITION 1.2.3. *Let $u \in C^2(\Omega)$ where $\Omega \subset \mathbb{R}^2$ is open. Then $\Delta u = 0$ if and only if u satisfies the mean value property.*

Of course, a function does not need to be differentiable in order for one to ask whether it satisfies the mean value property. We will see below that functions that satisfy the mean value property are however automatically C^2 (in fact C^∞) and hence are indeed solutions to Laplace's equations. To do this we will first need to record some facts about convolutions which will also be useful to us elsewhere.

1.2.3. Convolution and Mollifiers. For two functions $f, g \colon \mathbb{R}^n \to \mathbb{R}$ one may (attempt to, depending on how well-behaved f and g are) define a new function $f * g \colon \mathbb{R}^n \to \mathbb{R}$ by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) dV_y$$

[10] *i.e.*, Lebesgue integrable over each compact set

called the *convolution* of f and g (we use the notation dV_y for the volume form $dy_1 \wedge \cdots \wedge dy_n$). A simple change of variables to $z = x - y$ shows

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dV_y = \int_{\mathbb{R}^n} g(x-z)f(z)dV_z = (g * f)(x)$$

provided that both sides are defined, so $f * g$ is symmetric in f and g. However we will typically use convolution in a rather asymmetric way, in that often the function f will be known and quite well-behaved (*e.g.* smooth and compactly supported) while g will be potentially less well-behaved (*e.g.* we might just assume that g is locally integrable). Note that if $f \in C_0(\mathbb{R}^n)$ (where the subscript denotes compact support) and g is locally integrable then $f * g$ is certainly well-defined since for each x the integrand in the integral defining $(f * g)(x)$ has compact support and is obtained from multiplying g by a bounded function.

In general, when f is well-behaved, we can often expect $f * g$ to be similarly well-behaved regardless of g, as we now start to illustrate.

PROPOSITION 1.2.4. *Let $f \in C_0(\mathbb{R}^n)$ and let $g\colon \mathbb{R}^n \to \mathbb{R}$ be locally integrable. Then $f * g$ is continuous.*

PROOF. We have

$$|(f*g)(x+h)-(f*g)(x)| \le \int_{\mathbb{R}^n} |f(x-y+h)-f(x-y)||g(y)|dV_y = \int_{B_R(x)} |f(x-y+h)-f(x-y)||g(y)|dV_y$$

if we take $|h| < 1$ and R so large that $B_{R-1}(0)$ contains the support of f. Since f is continuous and has compact support it is uniformly continuous, so for any $\epsilon > 0$ there is $r \in (0,1)$ such that for all $z, h \in \mathbb{R}^n$ with $|h| < r$ we have $|f(z+h) - f(z)| < \epsilon$. So we obtain, for $|h| < r$, $|(f*g)(x+h) - (f*g)(x)| \le \epsilon \int_{B_R(x)} |g| dV$ for all $x \in \mathbb{R}^n$. Since g is locally integrable, $\int_{B_R(x)} |g| dV$ is just some finite constant, so this suffices to prove continuity. □

PROPOSITION 1.2.5. *Let $f \in C^1_0(\mathbb{R}^n)$ and let $g\colon \mathbb{R}^n \to \mathbb{R}$ be locally integrable. Then for each i the partial derivative $\frac{\partial}{\partial x_i}(f * g)$ exists and is equal to $\left(\frac{\partial f}{\partial x_i}\right) * g$.*

PROOF. For $x \in \mathbb{R}^n$ and $h \in (-1,1)$ we have

$$\frac{(f*g)(x+he_i)-(f*g)(x)}{h} = \int_{\mathbb{R}^n} \frac{f(x-y+he_i)-f(x-y)}{h} g(y)dV_y \tag{17}$$

Our assumptions on f imply that $\frac{\partial f}{\partial x_i}$ is compactly supported and continuous, and hence bounded in absolute value by some number M, and then it will hold that, for all $z \in \mathbb{R}^n$, $|f(z+he_i) - f(z)| \le M|h|$. Consequently the integrand in (17) has absolute value bounded by $|Mg|$, and moreover is supported in a compact set K_x (which can be chosen independent of $h \in (-1,1)$, though it will depend on x; specifically we can take $K_x = \{y : dist(x-y, supp(f)) \le 1\}$) on which g is integrable. So we can apply the Dominated Convergence Theorem to show that

$$\lim_{h\to 0} \frac{(f*g)(x+he_i)-(f*g)(x)}{h} = \lim_{h\to 0} \int_{K_x} \frac{f(x-y+he_i)-f(x-y)}{h} g(y)dV_y = \int_{K_x} \frac{\partial f}{\partial x_i}(x-y)g(y)dV_y = \left(\frac{\partial f}{\partial x_i}\right)*g(x),$$

as desired. □

A straightforward induction argument then yields:

COROLLARY 1.2.6. *Let $f \in C^\infty_0(\mathbb{R}^n)$ and let $g\colon \mathbb{R}^n \to \mathbb{R}$ be locally integrable. Then $f * g \in C^\infty(\mathbb{R}^n)$, with*

$$\frac{\partial^{\alpha_1+\cdots+\alpha_n}(f*g)}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} = \left(\frac{\partial^{\alpha_1+\cdots+\alpha_n} f}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\right)*g$$

for all $\alpha_1, \dots, \alpha_n \in \mathbb{Z}_{\geq 0}$.

The following inequality will help us prove that L^p functions can be approximated using certain (smooth) convolutions, see Theorem 1.2.10 below.

PROPOSITION 1.2.7 (Young's inequality). *Let* $f \in C_0(\mathbb{R}^n)$ *and let* $g \in L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$. *Then* $f * g \in L^p(\mathbb{R}^n)$, *with*

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

REMARK 1.2.8. One could make the weaker assumption that $f \in L^1(\mathbb{R}^n)$ here; the only reasons that I assumed $f \in C_0(\mathbb{R}^n)$ were that I didn't want to bother making a separate argument as to why $f * g$ is well-defined, and that in our applications f will be in $C_0(\mathbb{R}^n)$.

PROOF. If $p = \infty$ the statement is obvious:

$$|f * g(x)| = \left| \int_{\mathbb{R}^n} f(x-y) g(y) dV_y \right| \leq \|g\|_\infty \int_{\mathbb{R}^n} |f(x-y)| dV_y = \|f\|_1 \|g\|_\infty.$$

If $p = 1$ the result essentially follows from Fubini's theorem:

$$\begin{aligned} \|f * g\|_1 &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y) g(y) dV_y \right| dV_x \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)||g(y)| dV_y dV_x \\ &= \int_{\mathbb{R}^n} |g(y)| \left(\int_{\mathbb{R}^n} |f(x-y)| dV_x \right) dV_y = \|f\|_1 \|g\|_1. \end{aligned}$$

So for the rest of the proof assume that $1 < p < \infty$, and let $\frac{1}{p} + \frac{1}{q} = 1$ (so also $1 < q < \infty$, and $\frac{p}{q} = p - 1$). Then using Hölder's inequality we obtain, for any x,

$$\begin{aligned} |f * g(x)| &= \int_{\mathbb{R}^n} |f(x-y)|^{1/q} |f(x-y)|^{1/p} |g(y)| dV_y \leq \left(\int_{\mathbb{R}^n} |f(x-y)| dV_y \right)^{1/q} \left(\int_{\mathbb{R}^n} |f(x-y)||g(y)|^p dV_y \right)^{1/p} \\ &= \|f\|_1^{1/q} \left(\int_{\mathbb{R}^n} |f(x-y)||g(y)|^p dV_y \right)^{1/p} \end{aligned}$$

and so, using Fubini's theorem similarly to the $p = 1$ case:

$$\begin{aligned} \int_{\mathbb{R}^n} |f * g(x)|^p dV_x &\leq \|f\|_1^{p/q} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)||g(y)|^p dV_y dV_x \\ &= \|f\|_1^{p-1} \int_{\mathbb{R}^n} |g|^p dV \int_{\mathbb{R}^n} |f| dV = \|f\|_1^p \|g\|_p^p. \end{aligned}$$

□

Our preferred functions with which to convolve will be the functions η_r $(r > 0)$ constructed as follows. First choose a function $\eta \colon \mathbb{R}^n \to \mathbb{R}$ (a "standard mollifier") satisfying the following properties:

(i) $\eta \in C_0^\infty(\mathbb{R}^n)$.
(ii) $\eta(x) \geq 0$ for all x, and $\eta(x) = 0$ whenever $|x| \geq 1$.
(iii) $\eta(x)$ only depends on $|x|$ (*i.e.* each restriction $\eta|_{B_s(0)}$ is constant).
(iv) $\int_{\mathbb{R}^n} \eta dV = 1$

Now for $r > 0$ define

$$\eta_r(x) = r^{-n} \eta\left(\frac{x}{r}\right).$$

Then η_r likewise satisfies properties (i), (iii), (iv), while in place of (ii) we have $\eta_r(x) \geq 0$ and $\eta_r(x) = 0$ for $|x| \geq r$.

A convolution of the form $\eta_r * g$ is then given by (changing variables to $z = x - y$)

$$\eta_r * g(x) = \int_{\mathbb{R}^n} \eta_r(x-y)g(y)dV_y = \int_{\mathbb{R}^n} \eta_r(z)g(x-z)dV_z = \int_{B_r(0)} \eta_r(z)g(x-z)dV_z.$$

Thus to evaluate $\eta_r * g(x)$ we essentially average the value of g over the ball of radius r around x, with weights determined by the (spherically symmetric) function η_r. This should suggest that $\eta_r * g$ will approximate g when r is small, a notion that we begin to confirm as follows:

PROPOSITION 1.2.9. *If $g \in C_0(\mathbb{R}^n)$ then $\eta_r * g \to g$ uniformly (and hence also in L^p for any p) as $r \to 0$.*

PROOF. Bearing in mind that $\int_{\mathbb{R}^n} \eta_r dV = 1$ and $\eta_r \geq 0$, we have

$$\begin{aligned} |\eta_r * g(x) - g(x)| &= \left| \int_{\mathbb{R}^n} \eta_r(x-y)g(y)dV_y - g(x) \right| = \left| \int_{\mathbb{R}^n} \eta_r(x-y)(g(y)-g(x))dV_y \right| \\ &\leq \int_{\mathbb{R}^n} \eta_r(z)|g(x-z)-g(x)|dV_z = \int_{B_r(0)} \eta_r(z)|g(x-z)-g(x)|dV_z. \end{aligned}$$

Now g is continuous and compactly supported, and hence is uniformly continuous. So if $\epsilon > 0$ there is $r_0 > 0$ such that, for any $x \in \mathbb{R}^n$ and any $z \in B_{r_0}(0)$, $|g(x-z) - g(x)| < \epsilon$. So if $r \leq r_0$ then $\int_{\mathbb{R}^n} \eta_r(z)|g(x-z)-g(x)|dV_z \leq \epsilon \int_{\mathbb{R}^n} \eta_r dV = \epsilon$ and so $|\eta_r * g(x) - g(x)| \leq \epsilon$ independently of x, proving the desired uniform convergence.

Since the support of $\eta_r * g$ is contained in an r-neighborhood of the (compact) support of g, the fact that $\|\eta_r * g - g\|_\infty \to 0$ as $r \to 0$ immediately implies that $\|\eta_r * g - g\|_p \to 0$ for all p. □

This fairly quickly yields the following result which will be important to us several times:

THEOREM 1.2.10. *For $p < \infty$, if $g \in L^p(\mathbb{R}^n)$ then $\eta_r * g \to g$ in L^p as $r \to 0$.*

PROOF. Let $\epsilon > 0$. Standard constructions (relying on the assumption that $p < \infty$) show that g can be arbitrarily well-approximated in L^p norm by compactly supported continuous functions, so we can choose $h \in C_0(\mathbb{R}^n)$ with $\|h - g\|_p < \epsilon/3$. We then have

$$\|\eta_r * g - g\|_p \leq \|\eta_r * (g-h)\|_p + \|\eta_r * h - h\|_p + \|h - g\|_p.$$

By assumption $\|h-g\|_p < \epsilon/3$, and moreover $\|\eta_r * (g-h)\|_p \leq \|\eta_r\|_1 \|g-h\|_p < \epsilon/3$ using Proposition 1.2.7. Meanwhile Proposition 1.2.9 shows that there is r_0 such that $\|\eta_r * h - h\|_p < \epsilon/3$ for $r < r_0$. So altogether we obtain $\|\eta_r * g - g\|_p < \epsilon$ for $r < r_0$. □

1.2.4. Weak solutions to Laplace's equation and Weyl's lemma. Proposition 1.2.3 states that a C^2 function u satisfies Laplace's equation $\Delta u = 0$ if and only if u satisfies the mean value property that $u(x) = ⨍_{B_r(x)} u dV$ for all sufficiently small r. One can then think of the latter condition as a "weak" version of Laplace's equation, which unlike Laplace's equation does not require the function in question to have partial derivatives. (Another, more easily generalizable, weak form of Laplace's equation will be given shortly.) For elliptic equations such as Laplace's equation, a common and important feature is that solutions to the weak versions of the equation end up being smooth and consequently are solutions to the original equation. We are now in position to prove this for the case of Laplace's equation.

PROPOSITION 1.2.11. *Let $u\colon \Omega \to \mathbb{R}$ be a locally integrable function where $\Omega \subset \mathbb{R}^n$ is open, and assume that u satisfies the mean value property. Then $u \in C^\infty(\Omega)$ and $\Delta u = 0$.*

(In particular this shows that if $u \in C^2(\Omega)$ and $\Delta u = 0$ then $u \in C^\infty(\Omega)$, so even in the case where Δu is already known to be well-defined and zero the proposition has some content.)

PROOF. Denote $\Omega_r = \{x \in \Omega | B_r(x) \subset \Omega\}$. Since $B_r(x)$ denotes the closed ball and Ω is open, Ω_r is also open, and it suffices to show that, for each $r > 0$, $u|_{\Omega_r}$ is of class C^∞. So from now on we fix a specific $r > 0$.

Let η_r be the function constructed from the standard mollifier and defined above Proposition 1.2.9. Note that for each $x \in \Omega_r$, the integral $(\eta_r * u)(x) = \int_{\mathbb{R}^n} \eta_r(x-y)u(y)dV_y$ makes sense (even though we only defined u on Ω, not $\mathbb{R}^n$) because $\eta(x-y) = 0$ unless $y \in B_r(x) \subset \Omega$. So we can define the convolution $\eta_r * u \colon \Omega_r \to \mathbb{R}$.

Now recall that the functions η_r are spherically symmetric, in that there is a smooth function $g \colon [0,\infty) \to \mathbb{R}$, supported in $[0,r]$, such that $\eta_r(z) = g(|z|)$ for all z. I claim that this spherical symmetry, together with the mean value property satisfied by u, implies that $\eta_r * u$ is simply equal to u on Ω_r. This should be intuitively plausible, though the argument below runs into technicalities because our local integrability assumption on u is so modest.

Indeed the mean value property can be phrased as the statement that $\int_{B_s(0)}(u(x-z)-u(x))dV_z = 0$ for all $s \leq r$. Fubini's theorem shows that, for almost every $t \in [0,r]$, the function $u(x-\cdot)-u(x)$ is integrable on the sphere $\partial B_t(0)$, and that, if we denote $f(t) = \int_{S^{n-1}}(u(x-tz)-u(x))\sigma$, the statement that $\int_{B_s(0)}(u(x-z)-u(x))dV_z = 0$ is equivalent to the statement that $\int_0^s t^{n-1}f(t)dt = 0$. So for any $0 < s_1 < s_2 < r$ we have $\int_{s_1}^{s_2} t^{n-1}f(t) = 0$.

Meanwhile we have

$$\eta_r * u(x) - u(x) = \int_{B_r(x)} g(|x-y|)(u(y)-u(x))dV_y = \int_0^r t^{n-1}g(t)\left(\int_{S^{n-1}}(u(x-tz)-u(x))\sigma\right)dt$$
$$= \int_0^r g(t)t^{n-1}f(t)dt.$$

But we showed above that the integral of $t^{n-1}f(t)$ vanishes over every interval, so it's easy to see that the integral of $g(t)t^{n-1}f(t)$ vanishes (for instance, C^0-approximate g by step functions and then take a limit of the corresponding integrals). So we have in fact shown that $\eta_r * u(x) = u(x)$ for all $x \in \Omega_r$.

But Proposition 1.2.6 shows that $\eta_r * u$ is a C^∞ function, so the fact that $u|_{\Omega_r} = \eta_r * u$ means that u is C^∞ on Ω_r. The parameter $r > 0$ was arbitrary, and the sets $\{\Omega_r\}_{r>0}$ cover Ω, so we indeed have $u \in C^\infty(\Omega)$. Since u is a (better than) C^2 function that satisfies the mean value property, Proposition 1.2.3 shows that $\Delta u = 0$. □

We now, as promised, describe another notion of weak solution to Laplace's equation, which adapts more flexibly to other PDE's. We begin with the observation that if $u \in C^2(\Omega)$ is a genuine solution to the equation, then for any $\phi \in C_0^\infty(\Omega)$, we would have:

$$0 = \int_\Omega (\Delta u)\phi dV = -\sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)\phi dV = \sum_{i=1}^n \int_\Omega \left(\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial \phi}{\partial x_i}\right)dV = -\sum_{i=1}^n \int_\Omega u\frac{\partial^2\phi}{\partial x_i^2}dV$$
$$= \int_\Omega u(\Delta\phi)dV$$

(we have repeatedly integrated by parts, using that ϕ and each of the $\frac{\partial\phi}{\partial x_i}$ are compactly supported inside Ω and consequently integrals $\int_\Omega \frac{\partial}{\partial x_i}(f\phi)dV$ or $\int_\Omega \frac{\partial}{\partial x_i}\left(f\frac{\partial\phi}{\partial x_i}\right)dV$ always vanish.

Now for any locally integrable function u on Ω we can ask whether it is or is not the case that, for every $\phi \in C_0^\infty(\Omega)$, we have $\int_\Omega u\Delta\phi dV = 0$; if the answer is yes we call u a weak solution to Laplace's equation. If $u \in C^2(\Omega)$ the above calculation shows $\int_\Omega(\Delta u)\phi dV = \int_\Omega u\Delta\phi dV$, and u is a weak solution if $\Delta u = 0$, while if $\Delta u \neq 0$ then u is not a weak solution since then we can easily find $\phi \in C_0^\infty(\Omega)$ with $\int_\Omega(\Delta u)\phi dV \neq 0$. The following shows that any (locally integrable) weak solution is in fact a genuine solution in that its Laplacian is well-defined and equal to zero, at least modulo redefinition on a set of measure zero. (Note that it is clear from the definition of a weak solution that the criterion is preserved when one changes u on a set of measure zero.)

THEOREM 1.2.12 (Weyl's Lemma). *Assume that $\Omega \subset \mathbb{R}^n$ is open and that $u\colon \Omega \to \mathbb{R}$ is locally integrable and has the property that, for all $\phi \in C_0^\infty(\Omega)$, $\int_{\mathbb{R}^n} u\Delta\phi\, dV = 0$. Then, after possibly redefining u on a set of measure zero, $u \in C^\infty(\Omega)$, and $\Delta u = 0$.*

PROOF. Similarly to the proof of Proposition 1.2.11 let $\Omega_r = \{x \in \Omega | B_r(x) \subset \Omega\}$ and note that the mollified functions $\eta_r * u$ are well-defined and smooth on Ω_r. Fix $r_0 > 0$ and consider below only those r having $r \le r_0$ (so always $\Omega_{r_0} \subset \Omega_r$ and $\eta_r * u$ is defined on Ω_{r_0}).

CLAIM 1.2.13. For each $\phi \in C_0^\infty(\Omega_{r_0})$ we have $\int_{\Omega_{r_0}} (\eta_r * u)(x)\Delta\phi(x) dV_x = 0$

PROOF OF CLAIM 1.2.13. Note that the function $\eta_r * \phi$ has support in an r-neighborhood of the support of ϕ, and hence in Ω since ϕ is supported in Ω_{r_0} and $r < r_0$. Also Corollary 1.2.6 shows that $\eta_r * (\Delta\phi) = \Delta(\eta_r * \phi)$. Then, using the fact that $\eta_r(x-y) = \eta_r(y-x)$, we find:

$$\int_{\Omega_{r_0}} (\eta_r * u)(x)\Delta\phi(x)dV_x = \int_{\Omega_{r_0}}\int_\Omega \eta_r(x-y)u(y)\Delta\phi(x)dV_y dV_x = \int_\Omega\int_{\Omega_{r_0}} u(y)\eta_r(y-x)\Delta\phi(x)dV_x dV_y$$
$$= \int_\Omega u(y)(\eta_r * \Delta\phi)(y)dV_y = \int_\Omega u(y)\Delta(\eta_r * \phi)(y)dV_y.$$

But by the assumption on u in the theorem and the fact that $\eta_r * \phi \in C_0^\infty(\Omega)$ we have $\int_\Omega u(y)\Delta(\eta_r * \phi)(y)dV_y = 0$. ☐

Given Claim 1.2.13, the fact that $\eta_r * u \in C^\infty(\Omega_{r_0})$ implies that $\int_{\Omega_{r_0}} \Delta(\eta_r * u)(x)\phi(x)dV_x = 0$ for all $\phi \in C_0^\infty(\Omega_{r_0})$ and hence that $\Delta(\eta_r * u) = 0$. In particular $\eta_r * u$ satisfies the mean value property:

$$\eta_r * u(x) = \fint_{B_s(x)} \eta_r * u\, dV \quad \text{whenever } B_s(x) \subset \Omega_{r_0} \text{ and } r \le r_0.$$

Now Theorem 1.2.10 shows that, whenever $B_s(x) \subset \Omega_{r_0}$, we have $\eta_r * u \to u$ as $r \to 0$ in $L^1(B_s(x))$, and hence $\fint_{B_s(x)} \eta_r * u\, dV \to \fint_{B_s(x)} u\, dV$.

Meanwhile, the fact that $\eta_r * u \to u$ in L^1 on all compact sets implies (see, *e.g.*, [**F**, Proposition 2.29 and Theorem 2.30]) that there is a sequence $r_k \searrow 0$ such that $\eta_{r_k} * u \to u$ almost everywhere on Ω_{r_0}. Let $E = \{x \in \Omega_{r_0} | \eta_{r_k} * u(x) \to u(x)\}$. We then have

$$\text{(18) For each } x \in E: \quad u(x) = \lim_{r_k \to 0} \eta_{r_k} * u(x) = \lim_{k\to\infty} \fint_{B_s(x)} \eta_{r_k} * u\, dV = \fint_{B_s(x)} u\, dV \quad \text{if } B_s(x) \subset \Omega_{r_0}.$$

So u satisfies the mean value property at almost every point of Ω_{r_0}; we will modify u on a set of measure zero so that it satisfies the mean value property on all of Ω_{r_0}.

Let us choose a smooth function $s\colon \Omega_{r_0} \to (0,\infty)$ such that $B_{s(x)}(x) \subset \Omega_{r_0}$ (as is easily seen to be possible since Ω_{r_0} is open) and define $\tilde{u}(x) = (\eta_{s(x)} \star u)(x)$. Due to Corollary 1.2.6 and the fact that η_r depends smoothly on positive parameters r, $\tilde{u}$ is a smooth function. Also if $x \in E$, (18) shows that u satisfies the mean value property at x for balls of radii less than or equal to $s(x)$, and so just as in the proof of Proposition 1.2.11 we will have $\eta_{s(x)} * u(x) = u(x)$. So $\tilde{u}(x) = u(x)$ almost everywhere on Ω_{r_0}. But a *continuous* function such as $\tilde{u}$ that satisfies the mean value property at almost every point must in fact satisfy it at every point by a simple limiting argument. So $\tilde{u}$ is a smooth function on Ω_{r_0} satisfying the mean value property; thus $\Delta\tilde{u} = 0$.

We have thus shown, for any $r_0 > 0$, how to redefine u on a set of measure zero so as to make it smooth and harmonic on Ω_{r_0}. If $r_1 < r_0$ (so $\Omega_{r_0} \subset \Omega_{r_1}$), the functions $\tilde{u}^{(r_0)}$ and $\tilde{u}^{(r_1)}$ obtained by this procedure will coincide on Ω_{r_0} for the simple reason that they are smooth functions which coincide almost everywhere, and hence everywhere by continuity. So repeating the process for a sequence $r_i \searrow 0$ yields a function $\tilde{u}\colon \Omega \to \mathbb{R}$, coinciding with u almost everywhere, such that $\Delta\tilde{u} = 0$. □

1.3. Weak derivatives and Sobolev spaces

We motivated Weyl's Lemma by observing via integration by parts that a smooth function $u\colon \mathbb{R}^n \to \mathbb{R}$ necessarily satisfies, for every $\phi \in C_0^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}(\Delta u)\phi dV = \int_{\mathbb{R}^n} u\Delta\phi dV$ (for simplicity we just consider the case that $\Omega = \mathbb{R}^n$ here), based on which we said that a weak solution to the equation $\Delta u = 0$ should be a (*a priori* not necessarily differentiable) function u such that $\int_{\mathbb{R}^n} u\Delta\phi dV = 0$ for all $\phi \in C_0^\infty(\mathbb{R}^n)$. The Laplacian Δ could be replaced in such considerations by any number of other differential operators L; in general one can use integration by parts to get an identity $\int_{\mathbb{R}^n}(Lu)\phi dV = \int_{\mathbb{R}^n} u(L^*\phi)dV$ for some other operator L^*, which leads to a definition of a weak solution to $Lu = 0$ or more generally to $Lu = f$. (The Laplacian is somewhat unusual in that it is formally self-adjoint and so $\Delta^* = \Delta$ in this notation.)

The simplest operator L to which one might apply this—simpler indeed than the Laplacian—is the single partial derivative operator $\frac{\partial}{\partial x_i}$. Here one observes that, for $\phi \in C_0^\infty(\mathbb{R}^n)$ and $u \in C^1(\mathbb{R}^n)$, one has by integration by parts (since ϕ is compactly supported) $\int_{\mathbb{R}^n}\left(\frac{\partial u}{\partial x_i}\right)\phi dV = -\int_{\mathbb{R}^n} u\frac{\partial\phi}{\partial x_i}dV$. So if $u \in C^1(\mathbb{R}^n)$ the partial derivative $\frac{\partial u}{\partial x_i}$ is the unique continuous function f having the property that, for every $\phi \in C_0^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} u\frac{\partial\phi}{\partial x_i}dV = -\int_{\mathbb{R}^n} f\phi dV$. Since the latter condition makes sense if u is just locally integrable, we make the following definition:

DEFINITION 1.3.1. Let $u, f : \mathbb{R}^n \to \mathbb{R}$ be locally integrable functions. We say that the *weak ith partial derivative* of u is defined and equal to f, and write $\frac{\partial u}{\partial x_i} \overset{\mathrm{W}}{=} f$, provided that, for all $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} u\frac{\partial\phi}{\partial x_i}dV = -\int_{\mathbb{R}^n} f\phi dV.$$

Note that the truth or falsehood of the statement that $\frac{\partial u}{\partial x_i} \overset{\mathrm{W}}{=} f$ is unaffected if either u or f is modified on a set of measure zero.

EXERCISE 1.3.2. *Prove that weak derivatives satisfy the product rule,* i.e. *that if* $\frac{\partial u}{\partial x_i} \overset{\mathrm{W}}{=} f$ *and* $\frac{\partial v}{\partial x_i} \overset{\mathrm{W}}{=} g$ *then* $\frac{\partial(uv)}{\partial x_i} \overset{\mathrm{W}}{=} ug + fv$*. You may assume that, for some* p, q *with* $\frac{1}{p} + \frac{1}{q} = 1$*, we have* $u, f \in L^p(\mathbb{R}^n)$ *and* $v, g \in L^q(\mathbb{R}^n)$ *(which ensures that* $uv, ug, fv \in L^1(\mathbb{R}^n)$ *by Hölder's inequality). Suggestion: First do the problem in the case that* $v \in C^\infty(\mathbb{R}^n)$*, then approximate.*

EXAMPLE 1.3.3. *If* $n = 1$ *and* $u(x) = |x|$*, it probably won't be too surprising to learn that* $\frac{\partial u}{\partial x} \overset{\mathrm{W}}{=} \sigma$*, where* $\sigma(x) = 1$ *if* $x \geq 0$ *and* $\sigma(x) = -1$ *if* $x < 0$*. To check this, note that if* $\phi \in C_0^\infty(\mathbb{R})$ *then*

$$\int_{-\infty}^{\infty}|x|\phi'(x)dx = -\int_{-\infty}^{0} x\phi'(x)dx + \int_{0}^{\infty} x\phi'(x)dx = \int_{-\infty}^{0}\phi(x)dx - \int_{-\infty}^{0}\phi(x)dx = -\int_{-\infty}^{\infty}\sigma(x)\phi(x)dx$$

where we have integrated by parts and used the fact that the function $x \mapsto x\phi(x)$ *vanishes at* $\pm\infty$ *and at* 0*. Thus* u *has a weak derivative even though it cannot be redefined on a set of measure zero to be differentiable everywhere.*

EXAMPLE 1.3.4. *Consider* $u(x) = |x|^{-\alpha}$ *where* $0 < \alpha < n$*, as a function on* $\mathbb{R}^n$ *(extended arbitrarily to* $x = 0$*; the condition* $\alpha < n$ *is needed to keep* u *locally integrable). Away from* 0 *we have a genuine partial derivative* $\frac{\partial u}{\partial x_i} = -\frac{\alpha x_i}{|x|^{\alpha+2}}$*, which is locally integrable provided that* $\alpha < n-1$*. Now if* $\epsilon > 0$ *and* $\phi \in C_0^\infty(\mathbb{R}^n)$ *we can write* $\phi = \phi_1^\epsilon + \phi_2^\epsilon$ *where* ϕ_1^ϵ *is supported in* $\{\|x\| \geq \frac{\epsilon}{2}\}$ *and* ϕ_2^ϵ *is supported in*

$\{|x| \leq \epsilon\}$ *(namely take a smooth function* $\beta\colon \mathbb{R} \to [0,1]$ *such that* $\beta(s) = 0$ *for* $s \leq \epsilon/2$ *and* $\beta(s) = 1$ *for* $s \geq \epsilon$ *and put* $\phi_1^\epsilon = \beta\phi$ *and* $\phi_2^\epsilon = (1-\beta)\phi$*). Moreover we can do this in such a way that, for some constant* M *depending on* ϕ *but independent of* ϵ*, we have* $\|\phi_i^\epsilon\|_\infty < M$ *and* $\|\nabla\phi_i^\epsilon\|_\infty < M/\epsilon$*. Then if* $\alpha < n-1$*, both* $\int_{\mathbb{R}^n} |x|^{-\alpha}\frac{\partial\phi_2^\epsilon}{\partial x_i} dV$ *and* $\int_{\mathbb{R}^n} \frac{\alpha x_i}{|x|^{\alpha+2}}\phi_2^\epsilon dV$ *converge to zero as* $\epsilon \to 0$*, as both of these are bounded above by a constant multiple of the integral of* $|x|^{-\alpha-1}$ *over an* n*-ball of radius* ϵ*, which is equal to a constant multiple of* $\epsilon^{n-1-\alpha}$*. On the other hand we have* $\int_{\mathbb{R}^n} |x|^{-\alpha}\frac{\partial\phi_1^\epsilon}{x_i} dV = \int_{\mathbb{R}^n} \frac{\alpha x_i}{|x|^{\alpha+2}}\phi_1^\epsilon dV$ *simply because* $\frac{\partial}{\partial x_i}|x|^{-\alpha} = -\frac{\alpha x_i}{|x|^{\alpha+2}}$ *(in the usual sense) throughout the support of* ϕ_1^ϵ*. Sending* $\epsilon \to 0$ *and using that* $\phi = \phi_1^\epsilon + \phi_2^\epsilon$*, we deduce that*

$$\frac{\partial}{\partial x_i}|x|^{-\alpha} \overset{\mathrm{W}}{=} -\frac{\alpha x_i}{|x|^{\alpha+2}} \quad \textit{provided that } \alpha < n-1.$$

So this gives an example of a weakly differentiable function that fails to be continuous at the origin; this example can be made quite a lot worse in the following way (cf. [**E**, Example 5.2.2.4]*). Let* $\{r_k\}_{k=1}^\infty$ *be a sequence which is dense in* $\mathbb{R}^n$*, and define (for* $x \notin \{r_k\}_{k=1}^\infty$*; the values* $u(r_k)$ *can be prescribed arbitrarily):*

$$u(x) = \sum_{k=1}^\infty 2^{-k}|x-r_k|^{-\alpha}$$

where $0 < \alpha < n-1$*. Also let*

$$f(x) = -\sum_{k=1}^\infty 2^{-k}\frac{\alpha(x-r_k)_i}{|x-r_k|^{\alpha+2}}.$$

The previous paragraph obviously extends, for any $K \in \mathbb{N}$ *and any* $\phi \in C_0^\infty(\mathbb{R}^n)$*, to give*

$$\int_{\mathbb{R}^n}\left(\sum_{k=1}^K 2^{-k}|x-r_k|^{-\alpha}\right)\frac{\partial\phi}{\partial x_i} dV = \int_{\mathbb{R}^n}\left(\sum_{k=1}^K 2^{-k}\frac{\alpha(x-r_k)_i}{|x-r_k|^{\alpha+2}}\right)\phi\, dV,$$

and it is not hard to see from the Dominated Convergence Theorem that taking a limit as $K \to \infty$ *shows that* $\int_{\mathbb{R}^n} u\frac{\partial\phi}{\partial x_i} dV = -\int_{\mathbb{R}^n} f\phi\, dV$*. Thus* $\frac{\partial u}{\partial x_i} \overset{\mathrm{W}}{=} f$*. Thus* u *has weak partial derivatives even though it is an extremely poorly behaved function—indeed there is no open set on which* u *is bounded, so* u *is not continuous anywhere.*

Analogously to Proposition 1.2.5 (and to the start of the proof of Weyl's Lemma), we have:

PROPOSITION 1.3.5. *Suppose that* $\frac{\partial u}{\partial x_i} \overset{\mathrm{W}}{=} f$*, and let* $g \in C_0(\mathbb{R}^n)$*. Then* $\frac{\partial}{\partial x_i}(g * u) \overset{\mathrm{W}}{=} g * f$*.*

PROOF. Define $\bar{g}(z) = g(-z)$. For $\phi \in C_0^\infty(\mathbb{R}^n)$ we have:

$$\begin{aligned}\int_{\mathbb{R}^n} g*u(x)\frac{\partial\phi}{\partial x_i}(x)dV_x &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y)u(y)\frac{\partial\phi}{\partial x_i}(x)dV_y dV_x \\ &= \int_{\mathbb{R}^n} u(y)\left(\int_{\mathbb{R}^n}\bar{g}(y-x)\frac{\partial\phi}{\partial x_i}(x)dV_x\right)dV_y = \int_{\mathbb{R}^n} u(y)\left(\bar{g}*\frac{\partial\phi}{\partial x_i}\right)(y)dV_y \\ &= \int_{\mathbb{R}^n} u(y)\frac{\partial}{\partial x_i}(\bar{g}*\phi)(y)dV_y\end{aligned}$$

where we have used Proposition 1.2.5 to obtain $\bar{g}*\frac{\partial\phi}{\partial x_i} = \frac{\partial}{\partial x_i}(\bar{g}*\phi)$. Now since g was assumed compactly supported and $\phi \in C_0^\infty(\mathbb{R}^n)$, Corollary 1.2.6 shows that $\bar{g}*\phi \in C_0^\infty(\mathbb{R}^n)$. Hence the fact that $\frac{\partial u}{\partial x_i} \overset{\mathrm{W}}{=} f$ implies that the last line displayed above is equal to $-\int_{\mathbb{R}^n} f(y)(\bar{g}*\phi)(y)dV_y$. So we obtain

$$\int_{\mathbb{R}^n} g * u(x)\frac{\partial \phi}{\partial x_i}(x)dV_x = -\int_{\mathbb{R}^n} f(y)\left(\int_{\mathbb{R}^n} \bar{g}(y-x)\phi(x)dV_x\right)dV_y$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y)f(y)\phi(x)dV_y dV_x = \int_{\mathbb{R}^n}(g*f)(x)\phi(x)dV_x.$$

This holds for all $\phi \in C_0^\infty(\mathbb{R}^n)$, so we have shown that $\frac{\partial}{\partial x_i}(g*u) \overset{\mathrm{W}}{=} g*f$. ☐

COROLLARY 1.3.6. *Let $g \in C_0^1(\mathbb{R}^n)$ and suppose that $u\colon \mathbb{R}^n \to \mathbb{R}$ is locally integrable with $\frac{\partial u}{\partial x_i} \overset{\mathrm{W}}{=} f$. Then the partial derivative $\frac{\partial}{\partial x_i}(g*u)$ exists (in the ordinary sense) and is equal to $g*f$.*

PROOF. Propositions 1.2.4 and 1.2.5 show that that $g*u$ is a C^1 function, with partial derivatives $\left(\frac{\partial g}{\partial x_i}\right)*u$. Integration by parts then shows that also $\frac{\partial}{\partial x_i}(g*u) \overset{\mathrm{W}}{=} \left(\frac{\partial g}{\partial x_i}\right)*u$. But the preceding proposition shows that, at the same time, $\frac{\partial}{\partial x_i}(g*u) \overset{\mathrm{W}}{=} g*f$. From the definition of a weak derivative, this implies that the functions $h_1 = \left(\frac{\partial g}{\partial x_i}\right)*u$ and $h_2 = g*f$ have the property that, for all $\phi \in C_0^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} h_1\phi dV = \int_{\mathbb{R}^n} h_2\phi dV$. Moreover h_1 and h_2 are both continuous by Proposition 1.2.4, so it is easy to see from this that they are equal. So $g*f$ is indeed equal to the genuine ith partial derivative of $g*u$. ☐

We will also be dealing with higher order (both weak and genuine) partial derivatives; let us introduce a standard notation for doing this. Consider multi-indices $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n_{\geq 0}$. For any such α define the "order" of α to be $|\alpha| = \sum_{i=1}^n \alpha_i$. We then define

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}.$$

So for instance Corollary 1.2.6 says that if $f \in C_0^\infty(\mathbb{R}^n)$ and g is locally integrable then $f*g$ is smooth with $D^\alpha(f*g) = (D^\alpha f)*g$ for all α. As for weak derivatives, iterating Definition 1.3.1, we say that $D^\alpha u \overset{\mathrm{W}}{=} f$ (for u, f locally integrable) provided that $\int_{\mathbb{R}^n} uD^\alpha\phi dV = (-1)^{|\alpha|}\int_{\mathbb{R}^n} f\phi dV$ for all $\phi \in C_0^\infty(\mathbb{R}^n)$.

Weak derivatives are used to define the following **Sobolev spaces**, which provide an effective way of interpolating between L^p functions and smooth functions.

DEFINITION 1.3.7. Let $k \in \mathbb{Z}_{\geq 0}$ and $1 \leq p < \infty$. The (k,p)-Sobolev space on $\mathbb{R}^n$ is

$$W^{k,p}(\mathbb{R}^n) = \left\{u \in L^p(\mathbb{R}^n) | \text{For each } \alpha \text{ with } |\alpha| \leq k, \text{there is } f_\alpha \in L^p(\mathbb{R}^n) \text{ such that } D^\alpha u \overset{\mathrm{W}}{=} f_\alpha\right\}.$$

If $u \in W^{k,p}(\mathbb{R}^n)$ with $D^\alpha u \overset{\mathrm{W}}{=} f_\alpha$, the (k,p)-Sobolev norm of u is

$$\|u\|_{k,p} = \left(\|u\|_p^p + \sum_{1\leq|\alpha|\leq k}\|f_\alpha\|_p^p\right)^{1/p}$$

where $\|\cdot\|_p$ denotes the usual L^p norm.

So $W^{k,p}(\mathbb{R}^n)$ is the space of functions having all weak derivatives of order up to k belonging to the class L^p, and the $W^{k,p}$ norm is a natural combination of the L^p norms of these weak derivatives (including the zeroth).

Recall the mollifying functions $\eta_r \in C_0^\infty(\mathbb{R}^n)$ defined above Proposition 1.2.9.

PROPOSITION 1.3.8. *If $u \in W^{k,p}(\mathbb{R}^n)$ then for all $r > 0$ we have $\eta_r * u \in W^{k,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, and $\|\eta_r * u - u\|_{k,p} \to 0$ as $r \to 0$.*

PROOF. If $D^\alpha u = f_\alpha \in L^p(\mathbb{R}^n)$ where $|\alpha| \le k$, then iterating Proposition 1.3.5 shows that $D^\alpha(\eta_r * u) \overset{W}{=} \eta_r * f_\alpha$, which lies in $L^p(\mathbb{R}^n)$ by Proposition 1.2.7. Thus $\eta_r * u \in W^{k,p}(\mathbb{R}^n)$, while of course $\eta_r * u \in C^\infty(\mathbb{R}^n)$ by Corollary 1.2.6. Moreover we have $\|\eta_r * f_\alpha - f_\alpha\|_p \to 0$ as $r \to 0$ by Theorem 1.2.10. So since $D^\alpha(\eta_r * u - u) \overset{W}{=} \eta_r * f_\alpha - f_\alpha$ for each α with $|\alpha| \le k$ it immediately follows from the definition of $\|\cdot\|_{k,p}$ that $\|\eta_r * u - u\|_{k,p} \to 0$. □

An important feature of the L^p spaces is that they are complete with respect to the norm $\|\cdot\|_p$; analogously we have:

PROPOSITION 1.3.9. *Each $(W^{k,p}(\mathbb{R}^n), \|\cdot\|_{k,p})$ is a complete normed space.*

PROOF. Let $\{u_m\}_{m=1}^\infty$ be a sequence which is Cauchy in the norm $\|\cdot\|_{k,p}$. So whenever $|\alpha| \le k$ we have $f_m^\alpha \in L^p(\mathbb{R}^n)$ with $D^\alpha u_m \overset{W}{=} f_m^\alpha$, and the Cauchy condition on u_m amounts to the statement that each $\{f_m^\alpha\}_{m=1}^\infty$ is a Cauchy sequence in L^p. (In particular for the zero multi-index we see that $\{u_m\}_{m=1}^\infty$ is Cauchy in L^p.) So by the completeness of $L^p(\mathbb{R}^n)$ we have $u \in L^p(\mathbb{R}^n)$ and (for $1 \le |\alpha| \le k$) $f^\alpha \in L^p(\mathbb{R}^n)$ such that $u_m \to u$ in L^p and each $f_m^\alpha \to f^\alpha$ in L^p. Now the Hölder inequality readily implies that, for any $\phi \in C_0^\infty(\mathbb{R}^n)$ and any α with $|\alpha| \le k$,

$$\int_{\mathbb{R}^n} u D^\alpha \phi \, dV = \lim_{m\to\infty} \int_{\mathbb{R}^n} u_m D^\alpha \phi \, dV = (-1)^{|\alpha|} \lim_{m\to\infty} \int_{\mathbb{R}^n} f_m^\alpha \phi \, dV = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f^\alpha \phi \, dV.$$

Thus we have $D^\alpha u \overset{W}{=} f^\alpha$ whenever $|\alpha| \le k$. Since $u, f^\alpha \in L^p(\mathbb{R}^n)$ this shows that $u \in W^{k,p}(\mathbb{R}^n)$, and we have

$$\|u - u_m\|_{k,p} = \|u - u_m\|_p + \sum_{1\le|\alpha|\le k} \|f^\alpha - f_m^\alpha\|_p \to 0 \quad \text{as } m \to \infty.$$

Thus our arbitrary Cauchy sequence $\{u_m\}_{m=1}^\infty$ in $W^{k,p}(\mathbb{R}^n)$ converges to a limit $u \in W^{k,p}(\mathbb{R}^n)$ with respect to $\|\cdot\|_{k,p}$. □

COROLLARY 1.3.10. *$W^{k,p}(\mathbb{R}^n)$ is (Banach-space-isomorphic to) the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{k,p}$.*

PROOF. Clearly we have $C_0^\infty(\mathbb{R}^n) \subset W^{k,p}(\mathbb{R}^n)$, and we have just shown that $W^{k,p}(\mathbb{R}^n)$ is complete. So it suffices to show that $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.

Let $u \in W^{k,p}(\mathbb{R}^n)$ and $\epsilon > 0$. Proposition 1.3.8 shows that, by taking $v = \eta_r * u$ for small r, we can find $v \in W^{k,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ with $\|v - u\|_{k,p} < \frac{\epsilon}{2}$. So given this $v \in W^{k,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ it suffices to find $w \in C_0^\infty(\mathbb{R}^n)$ with $\|v - w\|_{k,p} < \frac{\epsilon}{2}$.

Now since v is smooth the weak derivatives of v are equal to its genuine partial derivatives $D^\alpha v$. Since $v \in W^{k,p}(\mathbb{R}^n)$ we have $D^\alpha v \in L^p(\mathbb{R}^n)$ whenever $|\alpha| \le k$, so given $\delta > 0$ we can find R such that, whenever $|\alpha| \le k$, $\int_{\mathbb{R}^n \setminus B_R(0)} |D^\alpha v|^p dV \le \delta$. Let $\chi \colon \mathbb{R}^n \to [0,1]$ be a compactly supported smooth function which is identically equal to 1 on $B_R(0)$ and which obeys $|D^\alpha \chi(x)| \le 1$ for all $x \in \mathbb{R}^n$ whenever $|\alpha| \le k$. Now define $w = \chi v$, so certainly $w \in C_0^\infty(\mathbb{R}^n)$.

The function $v - w = (1-\chi)v$ is supported outside of $B_R(0)$, and obeys, whenever $|\alpha| \le k$ and $x \in \mathbb{R}^n$,

$$|D^\alpha(v-w)(x)| = \left| \sum_{\beta+\gamma=\alpha} (D^\beta(1-\chi))(D^\gamma v)(x) \right| \le \sum_{\beta+\gamma=\alpha} |D^\gamma v(x)|.$$

Since the integral of each $|D^\gamma v|^p$ over $\mathbb{R}^n \setminus B_R(0)$ is at most δ, it follows that each $\int_{\mathbb{R}^n} |D^\alpha(v-w)|^p dV$ is bounded above by a quantity that tends to zero as $\delta \to 0$. So by taking δ sufficiently small in this construction we can arrange that $\|v - w\|_{k,p} < \frac{\epsilon}{2}$. □

1.3.1. Embedding theorems. As Example 1.3.4 shows, the existence of the weak derivatives such as those that appear in the definition of $W^{k,p}(\mathbb{R}^n)$ is not necessarily enough to directly guarantee that a function is differentiable, even after redefinition on a set of measure zero. We cannot generally expect a function that lies in $W^{k,p}(\mathbb{R}^n)$ to be k-times differentiable. However our use of Sobolev spaces in proving major results such as Theorem 1.1.10 is premised in part on the fact that, given $p \geq 1$ a function which belongs to the Sobolev space $W^{k,p}(\mathbb{R}^n)$ for *every* k is necessarily of class C^∞ after redefinition on a set of measure zero (see Corollary 1.3.22 below for a more precise statement). This is guaranteed by two separate "Sobolev embedding theorems" (one for large p and one for small p), to which we turn now.

THEOREM 1.3.11 (Morrey's inequality). *Fix a number $p > n$. Then there is a constant C such that if $u \in W^{1,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ then, for all $x, y \in \mathbb{R}^n$,*

$$|u(x)| \leq C\|u\|_{1,p} \qquad \text{and} \qquad |u(x) - u(y)| \leq C\|\nabla u\|_p |x-y|^{1-n/p}. \tag{19}$$

REMARK 1.3.12. Here the notation $\|\nabla u\|_p$ just means $\left(\sum_{i=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial x_i}\right|^p dV\right)^{1/p}$. So $\|u\|_{1,p}^p = \|u\|_p^p + \|\nabla u\|_p^p$.

PROOF. The key is the following lemma:

LEMMA 1.3.13. *If $p > n$, there is a constant $C_0 > 0$ such that for all $x \in \mathbb{R}^n$, $r > 0$, and $u \in C^\infty(B_r(x))$ we have*

$$⨍_{B_r(x)} |u - u(x)| dV \leq C_0 \|\nabla u\|_p r^{1-n/p}.$$

PROOF OF LEMMA 1.3.13. As usual let σ denote the standard volume form on S^{n-1}. For $0 \leq s \leq r$ we have:

$$\begin{aligned}\int_{w\in S^{n-1}} |u(x+sw) - u(x)|\sigma &= \int_{w\in S^{n-1}} \left|\int_0^s \nabla u(x+tw)\cdot w\,dt\right|\sigma \\ &\leq \int_0^s \int_{w\in S^{n-1}} |\nabla u(x+tw)|\sigma\,dt = \int_{B_s(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} dV_y\end{aligned}$$

where in the last equality we have converted from (n-dimensional) polar to Cartesian coordinates using the change of variables $y = x + tw$.

So we obtain

$$\begin{aligned}\int_{B_r(x)} |u - u(x)| dV &= \int_0^r \left(s^{n-1}\int_{w\in S^{n-1}} |u(x+sw) - u(x)|\sigma\right) ds \\ &\leq \int_0^r s^{n-1}\int_{B_s(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} dV_y ds \leq \frac{r^n}{n}\int_{B_r(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} dV_y.\end{aligned} \tag{20}$$

Now the assumption that $p > n$ implies that, where q is given by $\frac{1}{p} + \frac{1}{q} = 1$, we have $\frac{n}{q} = n(1-1/p) > n-1$, and hence that $(n-1)q < n$, in view of which the function $y \mapsto \frac{1}{|y-x|^{n-1}}$ belongs to $L^q(B_r(x))$, with

$$\left\|\frac{1}{|y-x|^{n-1}}\right\|_q = \left(\int_0^r t^{n-1}\int_{S^{n-1}} \frac{1}{t^{(n-1)q}}\sigma\,dt\right)^{1/q} = \left(\frac{\alpha_n}{n-(n-1)q} r^{n-(n-1)q}\right)^{1/q} = c_n r^{n/q-n-1}$$

where α_n is the ($(n-1)$-dimensional) volume of S^{n-1} and c_n is some dimensional constant. Now $n/q - (n-1) = n(1-1/p) - (n-1) = 1 - n/p$, so applying Hölder's inequality to (20) shows (perhaps after redefining c_n)

$$\int_{B_r(x)} |u-u(x)|dV \le r^n c_n \|\nabla u\|_p r^{1-n/p}.$$

Dividing both sides by the volume of $B_r(x_0)$ then proves the lemma. □

The first inequality in (19) follows easily from Lemma 1.3.13 once we observe that

$$|u(x)| \le \fint_{B_1(x)} |u-u(x)|dV + \fint_{B_1(x)} |u|dV.$$

The first term on the right is bounded above by a constant times $\|\nabla u\|_p$ by Lemma 1.3.13, and the second is bounded above by a constant times $\|u\|_p$ by Hölder's inequality.

As for the second inequality in (19), let $x, y \in \mathbb{R}^n$ and put $r = |x-y|$. Let $U = B_r(x) \cap B_r(y)$, and notice that the volume of U is bounded below by a positive constant a_n times the volume of $B_r(x)$ or of $B_r(y)$ (for instance we could take $a_n = 2^{-n}$ since U contains a ball of radius $r/2$ around $\frac{x+y}{2}$). Now observe that

$$|u(x)–u(y)| \le \fint_U |u–u(x)|dV + \fint_U |u–u(y)|dV \le a_n^{-1}\left(\int_{B_r(x)} |u-u(x)|dV + \int_{B_r(y)} |u-u(y)|dV\right) \le 2a_n^{-1} C_0 \|\nabla u\|_p r^{1-n/p}$$

where the last inequality uses Lemma 1.3.13. Since $r = |x-y|$ this proves the result. □

THEOREM 1.3.14 (Sobolev embedding $W^{k,p} \hookrightarrow C^{k-1,\gamma}$ for $p > n$). *Let $k \ge 1$ and $p > n$, and define $\gamma = 1-n/p$. Then any function $u \in W^{k,p}(\mathbb{R}^n)$ coincides almost everywhere with a unique function (still denoted u) that lies in the space $C^{k-1,\gamma}(\mathbb{R}^n)$ of continuous functions whose derivatives $D^\alpha u$ of all orders α with $0 \le |\alpha| \le k-1$ exist and are Hölder continuous with exponent γ. Moreover there is a constant C independent of u such that, for $0 \le |\alpha| \le k-1$,*

$$\max_x |D^\alpha u(x)| + \sup_{x,y,x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} \le C\|u\|_{k,p}. \tag{21}$$

PROOF. For each $u \in W^{k,p}(\mathbb{R}^n)$, Proposition 1.3.8 produces a sequence of functions $u_m \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ such that $\|u_m–u\|_{k,p} \to 0$. In particular this implies that, if $0 \le |\alpha| \le k–1$, then $\{D^\alpha u_m\}_{m=1}^\infty$ is a Cauchy sequence in the norm $W^{1,p}(\mathbb{R}^n)$ (with limit equal to the weak α-derivative of u). But then Theorem 1.3.11 gives an inequality $\max_{x\in\mathbb{R}^n} |D^\alpha u_l(x)–D^\alpha u_m(x)| \le C\|D^\alpha u_l – D^\alpha u_m\|_{1,p}$, so that, for $0 \le |\alpha| \le k-1$, $\{D^\alpha u_m\}_{m=1}^\infty$ is uniformly Cauchy. So each sequence $D^\alpha u_m$ converges uniformly to some continuous function which we temporarily denote by v^α. In particular (in case $\alpha = (0,\dots,0)$), u_m converges uniformly to a continuous function, which since $u_m \to u$ in L^p must coincide almost everywhere with u. So if necessary we redefine u on a set of measure zero to coincide with the uniform limit of the u_m.

Now in general if we have a sequence of smooth functions f_m with uniform limits $f_m \to f$ and $\frac{\partial f}{\partial x_i} \to g$ (so f and g are both continuous), then

$$f(x+he_i) - f(x) = \lim_{m\to\infty} \int_0^h \frac{\partial f_m}{\partial x_i}(x+te_i)dt = \int_0^h g(x+te_i)dt,$$

and so dividing by h and sending $h \to 0$ shows that $\frac{\partial f}{\partial x_i}$ exists and is equal to g. Applying this to our functions v^α from the previous paragraph which were uniform limits of the sequences $D^\alpha u_m$ (with $u_m \to u$ uniformly) shows that, for each α, $D^\alpha u$ exists and is equal to v^α. Thus the function u is indeed $(k-1)$-times continuously differentiable, and for $0 \le |\alpha| \le k-1$ we have $D^\alpha u_m \to D^\alpha u$ both in $W^{1,p}$ and unariformly, with

$$\max |D^\alpha u(x)| = \lim_{m\to\infty} \max |D^\alpha u_m(x)| \le \lim_{m\to\infty} C\|D^\alpha u_m\|_{1,p} = C\|D^\alpha u\|_{1,p} \le C\|u\|_{k,p}$$

where C is the constant from Theorem 1.3.11.

Likewise, for $0 \le |\alpha| \le k-1$ and $\gamma = 1 - n/p$, and for distinct points $x, y \in \mathbb{R}^n$,

$$\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} = \lim_{m\to\infty} \frac{|D^\alpha u_m(x) - D^\alpha u_m(y)|}{|x-y|^\gamma} \le C \lim_{m\to\infty} \|\nabla(D^\alpha u_m)\|_p \le C\|u\|_{k,p}$$

where we have used Theorem 1.3.11. In particular each $D^\alpha u$ with $0 \le |\alpha| \le k-1$ is Hölder continuous with exponent γ, and the estimate (21) follows immediately. □

This suffices to show that, if $p > n$, then a function lying in $W^{k,p}(\mathbb{R}^n)$ for all p will be smooth. Note that most of the hard work in the proof was a statement, namely Morrey's inequality, about functions that were *already smooth*. The point was that since smooth functions are dense in $W^{k,p}$ we could combine this with an approximation argument, and Morrey's inequality converts the L^p convergence used in Sobolev spaces (which does not behave well with respect to classical differentiation, hence the introduction of weak derivatives) to uniform convergence (which behaves better, see the start of the second paragraph of the proof of Theorem 1.3.14).

While it might be nice to imagine that Theorem 1.3.14 would extend to the case that $p = n$ to at least give that $W^{1,n}$ functions are continuous, the following shows that this is not the case:

EXERCISE 1.3.15. *Show that the function $u\colon \mathbb{R}^n \to \mathbb{R}$ defined by $u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$ for $x \neq 0$ and $u(0) = 0$ has $\int_{B_1(0)} |\nabla u|^n dV < \infty$ for all $n \ge 2$, and hence that $\chi u \in W^{1,n}(\mathbb{R}^n)$ for every $\chi \in C_0^\infty(\mathbb{R}^n)$.*

The following consequence of Theorem 1.3.14 (and the Arzelà-Ascoli theorem) will be helpful later:

THEOREM 1.3.16 (Rellich-Kondrachov compactness for $p > n$). *Let $p > n$ and $k \ge 1$. Suppose that $u_m \in W^{k,p}(\mathbb{R}^n)$ ($m \in \mathbb{N}$) and that there is a bounded set $\Omega \subset \mathbb{R}^n$ such that each $supp(u_m) \subset \Omega$ and a constant C such that $\|u_m\|_{k,p} \le C$. Then there is $u \in C^{k-1}(\mathbb{R}^n)$ and a subsequence $\{u_{m_j}\}_{j=1}^\infty$ of u_m such that $u_{m_j} \to u$ in C^{k-1} norm (and hence in $W^{k-1,q}$ norm for all q).*

PROOF. By Theorem 1.3.14, after we redefine the u_m on a set of measure zero, each u_m is a class C^{k-1} function, supported within Ω, such that the quantities $|D^\alpha u_m(x)|$ and $\frac{|D^\alpha u_m(x) - D^\alpha u_m(y)|}{|x-y|^{1-n/p}}$ are bounded independently of m, x, y when $0 \le |\alpha| \le k-1$. Thus each $\{D^\alpha u_m\}_{m=1}^\infty$ is a uniformly bounded and equicontinuous sequence of functions supported in the bounded set Ω, so the Arzelà-Ascoli theorem (*e.g.* [**F**, Theorem 4.44]) shows that there is a subsequence $\{u_{m_j}\}_{j=1}^\infty$ such that each $\{D^\alpha u_{m_j}\}_{j=1}^\infty$ converges uniformly. If $u = \lim_{j\to\infty} u_{m_j}$, then the same argument as at the start of the second paragraph of the proof of Theorem 1.3.14 shows that, given that the $D^\alpha u_{m_j}$ uniformly converge for $1 \le |\alpha| \le k-1$, their limits must respectively be $D^\alpha u$. Thus $u_{m_j} \to u$ in C^{k-1} norm. Since the u_{m_j} and hence u are supported in the bounded set Ω this immediately implies that $u_{m_j} \to u$ in each $W^{k-1,q}$ norm (since $\int_\Omega |D^\alpha u_{m_j} - D^\alpha u|^q dV \le vol(\Omega)\max|D^\alpha u_{m_j} - D^\alpha u|^q \to 0$). □

Now we turn to a Sobolev embedding theorem that applies to spaces $W^{k,p}(\mathbb{R}^n)$ with $p < n$. Again this follows from a certain inequality for smooth functions. First we consider the rather specific case of $W^{1,1}(\mathbb{R}^n)$; the general case will be reduced to this one.

LEMMA 1.3.17. *For every $u \in C_0^1(\mathbb{R}^n)$ we have*

$$\|u\|_{\frac{n}{n-1}} \le \|\nabla u\|_1.$$

REMARK 1.3.18. While our focus is on the case that $n > 1$, the statement is still true if $n = 1$ provided that we interpret $\frac{n}{n-1} = \infty$, since the assumption that $u \in C_0^1(\mathbb{R})$ implies that $|u(x)| = \left|\int_{-\infty}^x u'(t)dt\right| \le \int_{-\infty}^\infty |u'(t)|dt$.

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PROOF. It is perhaps instructive to warm up with the case that $n = 2$, so $\frac{n}{n-1} = 1$. In this case (much like in Remark 1.3.18) for any $x = (x_1, x_2) \in \mathbb{R}^2$ we have

$$|u(x_1, x_2)| = \left|\int_{-\infty}^{x_1} \frac{\partial u}{\partial x_1}(y_1, x_2) dy_1\right| \le \int_{-\infty}^{x_1} \left|\frac{\partial u}{\partial x_1}(y_1, x_2)\right| dy_1 \le \int_{-\infty}^{\infty} |\nabla u(y_1, x_2)| dy_1,$$

and similarly

$$|u(x_1, x_2)| \le \int_{-\infty}^{\infty} |\nabla u(x_1, y_2)| dy_2.$$

Let us write

$$\nu_{0,1}(x_2) = \int_{-\infty}^{\infty} |\nabla u(y_1, x_2)| dy_1 \qquad \nu_{0,2}(x_1) = \int_{-\infty}^{\infty} |\nabla u(x_1, y_2)| dy_1,$$

so the above shows that $|u(x_1, x_2)| \le \nu_{0,1}(x_2)$ and $|u(x_1, x_2)| \le \nu_{0,2}(x_1)$ for all $(x_1, x_2) \in \mathbb{R}^2$. Hence $|u(x_1, x_2)|^2 \le \nu_{0,1}(x_2)\nu_{0,2}(x_1)$. Integrating then yields

$$\|u\|_2^2 = \int_{\mathbb{R}^2} |u(x_1, x_2)| dx_1 dx_2 \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \nu_{0,1}(x_2)\nu_{0,2}(x_1) dx_1 dx_2 = \left(\int_{-\infty}^{\infty} \nu_{0,1}(x_2) dx_2\right)\left(\int_{\infty}^{\infty} \nu_{0,2}(x_1) dx_1\right).$$

But inspection of the definitions of $\nu_{0,1}$, $\nu_{0,2}$ shows that the integral of either one of these functions is equal to $\|\nabla u\|_1$. So we have shown (when $n = 2$) that $\|u\|_2^2 \le \|\nabla u\|_1^2$, which is precisely the content of the lemma in this case.

Now we consider an arbitrary $n \ge 2$. Let us define some functions as follows, where $\widehat{\ }$ denotes omission:

$$\text{For } 0 \le i \le n-2 < j \le n: \quad \nu_{i,j}(x_{i+1}, \dots, \hat{x}_j, \dots, x_n) = \int_{\mathbb{R}^{i+1}} |\nabla u(y_1, \dots, y_i, x_{i+1}, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_n)| dy_1 \cdots dy_i dy_j$$

$$\text{For } 0 \le i \le n-1: \quad \mu_i(x_{i+1}, \dots, x_n) = \int_{\mathbb{R}^i} |\nabla u(y_1, \dots, y_i, x_{i+1}, \dots, x_n)| dy_1 \cdots dy_i.$$

Note that $\nu_{i,i+1} = \mu_{i+1}$. For notational consistency we also let $\nu_{n-1,n}$ be the number (function with no arguments) $\nu_{n-1,n} = \int_{\mathbb{R}^n} |\nabla u(y_1, \dots, y_n)| dy_1 \dots dy_n = \|\nabla u\|_1$.

Now as in the $n = 2$ case we have, for each j and each $x = (x_1, \dots, x_j, \dots, x_n) \in \mathbb{R}^n$,

$$\begin{aligned} |u(x_1, \dots, x_j, \dots, x_n)| &= \left|\int_{-\infty}^{x_j} \frac{\partial u}{\partial x_j}(x_1, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_n) dy_j\right| \\ &\le \int_{-\infty}^{\infty} |\nabla u(x_1, \dots, y_j, \dots, x_n)| dy_j = \nu_{0,j}(x_1, \dots, \hat{x}_j, \dots, x_n). \end{aligned}$$

So

$$|u(x_1, \dots, x_n)|^{\frac{n}{n-1}} \le \prod_{j=1}^{n} \nu_{0,j}(x_1, \dots, \hat{x}_j, \dots, x_n)^{\frac{1}{n-1}}.$$

If we integrate the above inequality with respect to x_1, then the factor $\nu_{0,1}(x_2,\dots,x_n)^{\frac{1}{n-1}}$ comes out as a constant, so we get

$$\begin{aligned}\int_{\mathbb{R}} |u(x_1,\dots,x_n)|^{\frac{n}{n-1}}dx_1 &\le \nu_{0,1}(x_2,\dots,x_n)^{\frac{1}{n-1}}\int_{\mathbb{R}}\prod_{j=2}^{n}\nu_{0,j}(x_1,\dots,\hat{x}_j,\dots,x_n)^{\frac{1}{n-1}}dx_1\\ &\le \nu_{0,1}(x_2,\dots,x_n)^{\frac{1}{n-1}}\prod_{j=2}^{n}\left(\int_{\mathbb{R}}\nu_{0,j}(y_1,x_2,\dots,\hat{x}_j,\dots,x_n)dy_1\right)^{\frac{1}{n-1}}\\ &= \mu_1(x_2,\dots,x_n)^{\frac{1}{n-1}}\prod_{j=2}^{n}\nu_{1,j}(x_2,\dots,\hat{x}_j,\dots,x_n)^{\frac{1}{n-1}}.\end{aligned}$$

Here in the second line we use the generalized Hölder inequality[11] and in the last line we use the definitions of the $\nu_{i,j}$ and μ_i.

We then continue integrating with respect to the variables x_i. We may inductively assume that $1 \le i \le n-2$ and that we have shown that

$$\int_{\mathbb{R}^i}|u(x_1,\dots,x_n)|^{\frac{n}{n-1}}dx_1\cdots dx_i \le \mu_i(x_{i+1},\dots,x_n)^{\frac{i}{n-1}}\prod_{j=i+1}^{n}\nu_{i,j}(x_{i+1},\dots,\hat{x}_j,\dots,x_n)^{\frac{1}{n-1}}.$$

Now integrate with respect to x_{i+1}. The $\nu_{i,i+1}$ term does not depend on this variable and so comes out as a constant (and is equal to $\mu_{i+1}(x_{i+2},\dots,x_n)^{\frac{1}{n-1}}$); the integral of the remaining product is bounded above via the generalized Hölder inequality just as before: suppressing argument names we have

$$\int_{\mathbb{R}}\mu_i^{\frac{i}{n-1}}\prod_{j=i+2}^{n}\nu_{i,j}^{\frac{1}{n-1}}dx_{i+1} \le \left(\int_{\mathbb{R}}\mu_i dx_{i+1}\right)^{\frac{i}{n-1}}\prod_{j=i+2}^{n}\left(\int_{\mathbb{R}}\nu_{i,j}dx_{i+1}\right)^{\frac{1}{n-1}} = \mu_{i+1}^{\frac{i}{n+1}}\prod_{j=i+2}^{n}\nu_{i+1,j}^{\frac{1}{n-1}}.$$

Combining this from the factor of $\mu_{i+1}^{\frac{1}{n-1}}$ that came from the $\nu_{i,i+1}$ term, this shows that

$$\int_{R^{i+1}}|u(x_1,\dots,x_n)|^{\frac{n}{n-1}}dx_1\cdots dx_{i+1} \le \mu_{i+1}(x_{i+2},\dots,x_n)^{\frac{i+1}{n-1}}\prod_{j=i+2}^{n}\nu_{i+1,j}(x_{i+2},\dots,\hat{x}_j,\dots,x_n)^{\frac{1}{n-1}},$$

thus continuing the induction until we arrive at $i = n-1$. At this stage the inequality becomes

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}|u(x_1,\dots,x_{n-1},x_n)|^{\frac{n}{n-1}}dx_1\cdots dx_{n-1} &\le \mu_{n-1}(x_n)\nu_{n-1,n}^{\frac{1}{n-1}}\\ &= \|\nabla u\|_1^{\frac{1}{n-1}}\int_{\mathbb{R}^{n-1}}|\nabla u(x_1,\dots,x_{n-1},x_n)|dx_1\cdots dx_{n-1}.\end{aligned}$$

Integrating a final time with respect to x_n then indeed gives

$$\|u\|_{\frac{n}{n-1}} \le \|\nabla u\|_1^{\frac{n}{n-1}}.$$

□

The following is intended to motivate the choice of p^* in Theorem 1.3.20 and Corollary 1.3.21.

[11] This inequality says that if finitely many functions $f_i \in L^{p_i}$ where $\sum_i \frac{1}{p_i} = 1$, then $\int\left|\prod_i f_i\right| \le \prod_i \|f_i\|_{p_i}$. To prove it, rescale the f_i to each have L^{p_i} norm equal to one (unless one of the f_i is zero a.e. in which case there is nothing to show) and use the inequality $\prod_i a_i \le \sum_i \frac{a_i^{p_i}}{p_i}$ for nonnegative real numbers a_i. The latter inequality is trivial if some $a_i = 0$, and otherwise can be proven by putting $x_i = a_i^{p_i}$ and observing that the concavity of the logarithm function implies that $\sum_i \frac{1}{p_i}\log(x_i) \le \log\left(\sum_i \frac{x_i}{p_i}\right)$.

EXERCISE 1.3.19. *For $\lambda \in \mathbb{R}$ and a nonzero $u \in C_0(\mathbb{R}^n)$ define $u_\lambda(x) = u(\lambda x)$. For any $1 \le p, q < \infty$ compute*

$$\frac{\|u_\lambda\|_q}{\|u\|_q} \quad \text{and} \quad \frac{\|\nabla u_\lambda\|_p}{\|\nabla u\|_p}.$$

Deduce from this that if there is a constant $C > 0$ such that $\|v\|_q \le C\|\nabla v\|_p$ for all $v \in C_0^\infty(\mathbb{R}^n)$ we must have $q = \frac{np}{n-p}$.

That such an inequality indeed holds follows without too much work from Lemma 1.3.17:

THEOREM 1.3.20 (Gagliardo-Nirenberg-Sobolev Inequality). *Let $1 \le p < n$ and define p^* by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ (so $p^* = \frac{np}{n-p}$). Then there is a constant $C > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$ we have*

$$\|u\|_{p^*} \le C\|\nabla u\|_p.$$

PROOF. If $p = 1$ this is immediate from Lemma 1.3.17 (and we can even take $C = 1$), so assume $p > 1$. Define $\gamma = \frac{p^*(n-1)}{n} = \frac{p(n-1)}{n-p}$. Note that $\gamma > 1$ since the assumption that $p > 1$ implies that $np - p > n - p$. So if $u \in C_0^\infty(\mathbb{R}^n)$ then $|u|^\gamma \in C_0^1(\mathbb{R}^n)$ with $|\nabla |u|^\gamma| = \gamma |u|^{\gamma-1}|\nabla u|$.

Also, if q is given by $\frac{1}{p} + \frac{1}{q} = 1$ (so $q = p/(p-1)$), then

$$(\gamma - 1)q = \frac{p}{p-1}\frac{(np-p)-(n-p)}{n-p} = \frac{p}{p-1}\frac{n(p-1)}{n-p} = \frac{np}{n-p} = p^*.$$

Hence, using Lemma 1.3.17 (applied to $|u|^\gamma$) and Hölder's inequality,

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{p^*} dV\right)^{\frac{n-1}{n}} &= \left(\int_{\mathbb{R}^n} (|u|^\gamma)^{\frac{n}{n-1}} dV\right)^{\frac{n-1}{n}} = \||u|^\gamma\|_{\frac{n-1}{n}} \le \int_{\mathbb{R}^n} |\nabla |u|^\gamma|\, dV \\ &= \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1}|\nabla u| dV \le \gamma \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)q} dV\right)^{1/q} \left(\int_{\mathbb{R}^n} |\nabla u|^p\right)^{1/p} \le C\left(\int_{\mathbb{R}^n} |u|^{p^*} dV\right)^{\frac{p-1}{p}} \|\nabla u\|_p\end{aligned}$$

for some constant C. Now

$$\frac{n-1}{n} - \frac{p-1}{p} = \frac{1}{p} - \frac{1}{n} = \frac{1}{p^*},$$

so dividing both sides of the above by $\left(\int_{\mathbb{R}^n} |u|^{p^*} dV\right)^{\frac{p-1}{p}}$ gives

$$\|u\|_{p^*} \le C\|\nabla u\|_p,$$

as desired. □

COROLLARY 1.3.21 (Sobolev embedding $W^{k,p} \hookrightarrow W^{k-1,p^*}$ for $p < n$). *Let $u \in W^{k,p}(\mathbb{R}^n)$ with $k \ge 1$ and $p < n$, and define p^* by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Then $u \in W^{k-1,p^*}(\mathbb{R}^n)$, and there is an inequality $\|u\|_{k-1,p^*} \le C\|u\|_{k,p}$ where C depends only on k, p, n and not on u.*

PROOF. By Corollary 1.3.10 we can find $u_m \in C_0^\infty(\mathbb{R}^n)$ with $\|u_m - u\|_{k,p} \to 0$. Then for all α with $0 \le |\alpha| \le k-1$, $\{D^\alpha u_m\}_{m=1}^\infty$ is Cauchy in $W^{1,p}(\mathbb{R}^n)$, hence also in $L^{p^*}(\mathbb{R}^n)$ by Theorem 1.3.20. If $D^\alpha u \overset{\mathrm{W}}{=} f_\alpha$ where $0 \le |\alpha| \le k-1$, we know both that $D^\alpha u_m \to f_\alpha$ in $W^{1,p}$ (in particular in L^p) and that $D^\alpha u_m$ has a limit in L^{p^*}; it follows from this that f_α must be equal (almost everywhere) to this L^{p^*}-limit and hence in particular that $f_\alpha \in L^{p^*}(\mathbb{R}^n)$. This holds for all α with $0 \le |\alpha| \le k-1$, so indeed $u \in W^{k-1,p^*}(\mathbb{R}^n)$. Moreover when $0 \le |\alpha| \le k-1$ we have

$$\|f_\alpha\|_{p^*} = \lim_{m\to\infty} \|D^\alpha u_m\|_{p^*} \le \lim_{m\to\infty} C\|u_m\|_{1,p} = \|f_\alpha\|_{1,p}$$

where C is the constant from Theorem 1.3.20, so the estimate $\|u\|_{k-1,p^*} \le C\|u\|_{k,p}$ follows directly. □

The above Sobolev embedding theorems can be combined as follows:

COROLLARY 1.3.22. *If l is a positive integer and if $u \in W^{k,p}(\mathbb{R}^n)$ where $k - n/p > l$ then, after possibly redefining u on a set of measure zero, $u \in C^l(\mathbb{R}^n)$.*

PROOF. Let us first assume (for reasons of convenience that will become clear later) that $\frac{n}{p}$ is not an integer. Let h be the greatest integer that is less than n/p, so since n/p is not an integer we have $h < \frac{n}{p} < h+1$. If $h = 0$ then $n < p$ and the result is immediate from Theorem 1.3.14 (since for k, l to be integers with $k - n/p > l$ we must have $l \leq k-1$), so assume that $h \geq 1$. We may apply Corollary 1.3.21 h times to get a sequence of embeddings (writing $p_0 = p$)

$$W^{k,p_0}(\mathbb{R}^n) \hookrightarrow W^{k-1,p_1}(\mathbb{R}^n) \hookrightarrow \cdots \hookrightarrow W^{k-h,p_h}(\mathbb{R}^n)$$

where $\frac{1}{p_{i+1}} = \frac{1}{p_i} - \frac{1}{n}$ and hence $\frac{1}{p_h} = \frac{1}{p} - \frac{h}{n}$. By the definition of h we have $\frac{h+1}{n} > \frac{1}{p} > \frac{h}{n}$ and so $\frac{1}{n} > \frac{1}{p_h}$, *i.e.* $p_h > n$. So Theorem 1.3.14 gives an embedding $W^{k-h,p_h}(\mathbb{R}^n) \hookrightarrow C^{k-h-1,1-n/p_h}(\mathbb{R}^n)$, which combines with what we have previously done to give an embedding $W^{k,p}(\mathbb{R}^n) \hookrightarrow C^{k-h-1,1-n/p_h}(\mathbb{R}^n)$. So our arbitrary function $u \in W^{k,p}(\mathbb{R}^n)$ is $(k-h-1)$-times differentiable (after redefinition on a set of measure zero in order to apply Theorem 1.3.14). The definition of h was such that $k-h-1$ is the largest integer that is smaller than $k - \frac{n}{p}$, so u is of class C^l for any integer $l < k - \frac{n}{p}$ whenever $\frac{n}{p}$ is not an integer.

If $\frac{n}{p}$ is an integer, the argument in the previous paragraph runs into the inconvenient fact that we do not have a Sobolev embedding theorem for the "borderline" case of spaces $W^{k,n}(\mathbb{R}^n)$. We can argue around this as follows. If $u \in W^{k,p}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ then u is of class C^l on a neighborhood of x provided that $\zeta u \in C^l(\mathbb{R}^n)$ where $\zeta \in C_0^\infty(\mathbb{R}^n)$ is smooth and equal to one on a neighborhood of x. The fact that $u \in W^{k,p}(\mathbb{R}^n)$ implies that $\zeta u \in W^{k,p}(\mathbb{R}^n)$, for instance by Exercise 1.3.2. But ζu has compact support, which together with the fact that $\zeta u \in W^{k,p}(\mathbb{R}^n)$ implies that $\zeta u \in W^{k,p'}(\mathbb{R}^n)$ for all $p' < p$. Choose $p' < p$ with $l < k - \frac{n}{p'}$ and $\frac{n}{p'}$ not an integer. Then the previous paragraph shows that $\zeta u \in C^l(\mathbb{R}^n)$, and hence that u is C^l on a neighborhood of x. $x \in \mathbb{R}^n$ was arbitrary, so this proves that $u \in C^l(\mathbb{R}^n)$. □

We'll also need the following compactness result, complementing Theorem 1.3.16:

THEOREM 1.3.23 (Rellich-Kondrachov compactness for $p < n$). *Let $p < n$, and set $p^* = \frac{np}{n-p}$. Fix a bounded open subset $\Omega \subset \mathbb{R}^n$, and let $\{u_m\}_{m=1}^\infty$ be a sequence of functions with support contained in Ω such that, for some $C > 0$ independent of m, we have $u_m \in W^{1,p}(\mathbb{R}^n)$ with $\|u_m\|_{1,p} \leq C$ for all m. Then there is a subsequence $\{u_{m_k}\}_{k=1}^\infty$ which is Cauchy in the L^q norm for all $q \in [1, p^*)$.*

PROOF. Fix $r > 0$ and consider the sequence of r-mollifications $\{\eta_r * u_m\}_{m=1}^\infty$. Note that we have, for each $x \in \mathbb{R}^n$,

$$|\eta_r * u_m(x)| \leq \int_{\mathbb{R}^n} \eta_r(x-y)|u_m(y)|dV_y \leq \|\eta_r\|_\infty \int_\Omega |u_m| dV \leq \|\eta_r\|_\infty \|u_m\|_p (vol(\Omega))^{1-1/p} \leq C\|\eta_r\|_\infty (vol(\Omega))^{1-1/p}$$

and similarly

$$\left|\frac{\partial(\eta_r * u_m)}{\partial x_i}\right| \leq \int_{\mathbb{R}^n} \left|\frac{\partial \eta_r}{\partial x_i}(x-y)\right| |u_m(y)| dV_y \leq C \left\|\frac{\partial \eta_r}{\partial x_i}\right\|_\infty (vol(\Omega))^{1-1/p}.$$

Here of course we use both the assumptions that the u_m are supported in the bounded subset Ω and that they obey $\|u\|_{1,p} \leq C$. Consequently, for any fixed r, the sequence of functions $\{\eta_r * u_m\}_{m=1}^\infty$ is uniformly bounded and equicontinuous, so since each $\eta_r * u_m$ is supported in an r-neighborhood of u it follows from the Arzelà-Ascoli theorem that we can find a subsequence $\{u_{m_k^r}\}_{k=1}^\infty$ (depending on r) such that $\{\eta_r * u_{m_k^r}\}_{k=1}^\infty$ is uniformly Cauchy. If we apply this sequentially to the values $r = \frac{1}{j}$ for

$j \in \mathbb{Z}_{>0}$, using $\{u_{m_k^{1/j}}\}$ as the input (in place of the whole sequence $\{u_m\}_{m=1}^\infty$) for the construction of $\{u_{m_k^{1/(j+1)}}\}_{k=1}^\infty$, we can arrange that each $\{u_{m_k^{1/(j+1)}}\}_{k=1}^\infty$ is a subsequence of $\{u_{m_k^{1/j}}\}_{k=1}^\infty$. Then the standard diagonal trick of setting $m_k = m_k^{1/k}$ yields a subsequence $\{u_{m_k}\}_{k=1}^\infty$ of the original sequence $\{u_m\}_{m=1}^\infty$ such that, for every $j \in \mathbb{Z}_{>0}$, $\{\eta_{1/j} * u_{m_k}\}_{k=1}^\infty$ is uniformly Cauchy.

Now let us consider how the $\eta_r * u_m$ converge as $r \to 0$ and m is fixed. Recall that $\eta_r(x) = r^{-n}\eta(x/r)$ for a smooth function η supported in $B_1(0)$ with $\int_{B_1(0)} \eta dV = 1$. Suppose that $v \in C^1(\mathbb{R}^n)$. Then we can write

$$\eta_r * v(x) = \int_{B_1(0)} r^{-n}\eta(z/r)v(x-z)dV_z = \int_{B_1(0)} \eta(y)v(x-ry)dV_y$$

and so

$$\begin{aligned}|\eta_r * v(x) - v(x)| &= \left|\int_{B_1(0)} \eta(y)(v(x-ry)-v(x))\,dV_y\right| \leq \int_{B_1(0)} \eta(y)\int_0^1 \left|\frac{d}{dt}(v(x-try))\right| dtdV_y \\ &\leq r\int_0^1 \int_{B_1(0)} \eta(y)|\nabla v(x-try)|dV_y dt\end{aligned}$$

Integrating over $\mathbb{R}^n$ gives

$$\|\eta_r * v - v\|_1 \leq r\int_{B_1(0)} \eta(y)\int_0^1 \int_{\mathbb{R}^n} |\nabla v(x-try)|dV_x dtdV_y = r\int_{B_1(0)} \eta(y)\int_0^1 \|\nabla v\|_1 dtdV_y = r\|\nabla v\|_1.$$

We would like to apply the above with $v = u_m$, which superficially may not be possible because u_m may not be C^1, but by approximating u_m by smooth functions supported on a small neighborhood of Ω and using Young's inequality, the fact that $\|\eta_r * v - v\|_1 \leq \|\nabla v\|_1$ for smooth v implies that $\|\eta_r * u_m - u_m\|_1 \leq \|\nabla u_m\|_1$. Now since the u_m are supported in the bounded set Ω we have $\|\nabla u_m\|_1 \leq \|\nabla u_m\|_p vol(\Omega)^{1-1/p}$. Moreover our hypothesis gives that $\|\nabla u_m\|_p \leq C$ independently of m, so where $A = C vol(\Omega)^{1-1/p}$ we have shown that

$$\|\eta_r * u_m - u_m\|_1 \leq Ar, \tag{22}$$

so the L^1-convergence of $\eta_r * u_m$ to u_m as $r \to 0$ is uniform in m.

We can improve this to uniform-in-m convergence in the L^q norm for any $q \in [1, p^*)$ by using a standard L^p interpolation technique: Note in general that if $f \in L^1(\mathbb{R}^n) \cap L^{p^*}(\mathbb{R}^n)$ then for $r \in [1, p^*]$ we have, choosing $\theta \in [0,1]$ such that $\theta + \frac{1-\theta}{p^*} = \frac{1}{r}$ and using Hölder's inequality,

$$\int_{\mathbb{R}^n} |f|^r dV = \int_{\mathbb{R}^n} |f|^{\theta r}|f|^{(1-\theta)r} dV \leq \left(\int_{\mathbb{R}^n} |f| dV\right)^{\theta r}\left(\int_{\mathbb{R}^n} |f|^{p^*} dV\right)^{\frac{(1-\theta)r}{p^*}}$$

and so $f \in L^r(\mathbb{R}^n)$ with $\|f\|_r \leq \|f\|_1^\theta \|f\|_{p^*}^{1-\theta}$. Applying this to (22) and using Corollary 1.3.21 to convert our assumed bound on $\|u_m\|_{1,p}$ to a bound on $\|u_m\|_{p^*}$, and then using Young's inequality to obtain $\|\eta_r * u_m\|_{p^*} \leq \|u_m\|_{p^*}$, for any $q \in [1, p^*)$ we obtain a constant B depending on Ω but not on r and m, and a constant $\theta > 0$ (given by $\frac{1}{q} = \theta + \frac{1-\theta}{p^*}$) such that

$$\|\eta_r * u_m - u_m\|_q \leq Br^\theta.$$

In other words, for any $q < p^*$ the convergence of $\eta_r * u_m$ to u_m in L^q norm is uniform in m.

Recall from earlier that we have a subsequence $\{u_{m_k}\}_{k=1}^\infty$ such that, for all j, $\{\eta_{1/j} * u_{m_k}\}_{k=1}^\infty$ is uniformly Cauchy, and hence Cauchy in L^q by the boundedness of the support Ω of the u_m. Given $\epsilon > 0$ and $q < p^*$ (yielding $\theta > 0$ and B as in the previous paragraph), let us choose j so large that

$Bj^{-\theta} < \frac{\epsilon}{3}$, and choose K so large that, whenever $k, l \geq K$, we have $\|\eta_{1/j} * u_{m_k} - \eta_{1/j} * u_{m_l}\|_q < \frac{\epsilon}{3}$. Then for $k, l \geq K$ we have

$$\|u_{m_k} - u_{m_l}\|_q \leq \|u_{m_k} - \eta_{1/j} * u_{m_k}\|_q + \|\eta_{1/j} * u_{m_k} - \eta_{1/j} * u_{m_l}\|_q + \|\eta_{1/j} * u_{m_l} - u_{m_l}\|_q < \epsilon,$$

so the subsequence $\{u_{m_k}\}_{k=1}^{\infty}$ constructed at the start of the proof is Cauchy in L^q. □

1.4. Hilbert spaces, weak convergence, and $W^{1,2}$

We now recall some standard facts about Hilbert spaces, which will lead to a criterion for membership in $W^{1,2}(\mathbb{R}^n)$ in terms of difference quotients; this will be useful to us in the proof of the main regularity theorem in the following section.

Recall that a (real) Hilbert space is by definition an inner product space $(H, \langle\cdot,\cdot\rangle)$ which is complete with respect to the metric induced by the inner product. Examples include $W^{k,2}(\mathbb{R}^n)$ for any $k \geq 0$, with inner product given by

$$\langle u, v\rangle = \sum_{0\leq|\alpha|\leq k} \int_{\mathbb{R}^n} (D^\alpha u)(D^\alpha v) dV$$

where we slightly abuse notation and write D^α for weak derivatives rather than genuine ones. As usual we define denote the norm induced by the inner product on a Hilbert space by $\|\cdot\|$ (so $\|x\| = \sqrt{\langle x, x\rangle}$).

PROPOSITION 1.4.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $V \leq H$ be a proper closed subspace. Then there is $z_0 \in H$ such that $\|z_0\| = 1$ and $\langle z_0, v\rangle = 0$ for all $v \in V$.*

PROOF. Since V is closed and proper we can find $z \in H$ such that

$$d(z, V) := \inf\{\|v - z\| | v \in V\} > 0.$$

We can then choose a sequence $\{v_m\}_{m=1}^{\infty}$ in V such that $\|v_m - z\| \to d(z, V)$. We claim that $\{v_m\}_{m=1}^{\infty}$ is a Cauchy sequence. To see this, observe that

$$\left\|\frac{v_k + v_m}{2} - z\right\|^2 = \frac{1}{4}\|(v_k - z) + (v_m - z)\|^2 = \frac{1}{4}\left(\|v_k - z\|^2 + \|v_m - z\|^2 + 2\langle v_k - z, v_m - z\rangle\right)$$

while

$$\|v_k - v_m\|^2 = \|(v_k - z) - (v_m - z)\|^2 = \|v_k - z\|^2 + \|v_m - z\|^2 - 2\langle v_k - z, v_m - z\rangle.$$

Adding four times the first equation to the second equation shows that

$$\|v_k - v_m\|^2 + 4\left\|\frac{v_k + v_m}{2} - z\right\|^2 = 2(\|v_k - z\|^2 + \|v_m - z\|^2),$$

so since $\left\|\frac{v_k+v_m}{2} - z\right\|^2 \geq d(z, V)^2$ we obtain

$$\|v_k - v_m\|^2 \leq 2(\|v_k - z\|^2 + \|v_m - z\|^2) - 4d(z, V)^2.$$

Since $\|v_k - z\|^2 \to d(z, V)^2$ as $k \to \infty$ it follows that $\|v_k - v_m\|^2 \to 0$ as $k, m \to \infty$, *i.e.* that $\{v_m\}_{m=1}^{\infty}$ is a Cauchy sequence.

Since H is complete and $V \leq H$ is a closed subspace it follows that $v_k \to v$ for some $v \in V$, necessarily with $\|v - z\| = d(z, V)$. We then see that for all $w \in V$ we have $\langle v - z, w\rangle = 0$, for $\|v + \epsilon w - z\|^2 = \|v - z\|^2 + 2\epsilon\langle w, v - z\rangle + \epsilon^2\|w\|^2$ so if $\langle v - z, w\rangle \neq 0$ we could find a small (positive or negative) ϵ such that $\|v + \epsilon w - z\| < \|v - z\| = d(z, V)$, contradicting the definition of $d(z, V)$, This proves that $v - z \in V^\perp$, and clearly we have $v - z \neq 0$ since we assumed that $z \notin V$. Then $z_0 = \frac{v-z}{\|v-z\|}$ satisfies the required property. □

COROLLARY 1.4.2. *If $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space and $W \leq H$ is a closed subspace then $(W^\perp)^\perp = W$.*

PROOF. It is trivial that $W \subset (W^{\perp})^{\perp}$. Note that $(W^{\perp})^{\perp}$ is clearly closed in H, since it is the intersection of the kernels of the bounded linear functionals $\langle x, \cdot \rangle$ as x varies through $W^{\perp}$. So $((W^{\perp})^{\perp}, \langle \cdot, \cdot \rangle)$ is a Hilbert space. If the inclusion $W \subset (W^{\perp})^{\perp}$ were proper then we could apply the preceding proposition with $V = W$ and $H = (W^{\perp})^{\perp}$ to get $z_0 \in W^{\perp} \cap (W^{\perp})^{\perp}$ with $\|z_0\| = 1$. But this is nonsense since an element of $W^{\perp} \cap (W^{\perp})^{\perp}$ would be orthogonal to itself and hence would be zero. □

THEOREM 1.4.3 (Riesz Representation Theorem). *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $\phi \colon H \to \mathbb{R}$ be a linear map which is bounded in the sense that, for some $M \in \mathbb{R}$, $|\phi(x)| \leq M\|x\|$ for all $x \in H$. Then there is $a_{\phi} \in H$ such that $\phi(x) = \langle a_{\phi}, x \rangle$ for all $x \in H$.*

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PROOF. If ϕ is identically zero we can simply take $a_{\phi} = 0$, so we assume that ϕ is not identically zero. Then since ϕ is evidently continuous, $\ker(\phi)$ is a closed subspace of H, which is proper since ϕ is not identically zero and so Proposition 1.4.1 gives $z_0 \in H$ such that $\|z_0\| = 1$ and $\langle z_0, v \rangle = 0$ for all $v \in \ker(\phi)$. I claim that the theorem is satisfied by $a_{\phi} = \phi(z_0)z_0$. Indeed, notice that, for all $x \in H$, we have $\phi(x)z_0 - \phi(z_0)x \in \ker(\phi)$ by the linearity of ϕ, and so

$$0 = \langle \phi(x)z_0 - \phi(z_0)x, z_0 \rangle = \phi(x) - \langle \phi(z_0)z_0, x \rangle$$

from which the claim follows immediately. □

In general for a Banach space $(B, \|\cdot\|)$ one lets B^* denote the space of bounded linear functionals $\phi \colon B \to \mathbb{R}$, and endows B^* with the operator norm $\|\phi\|^* = \sup_{\|x\| \leq 1} |\phi(x)|$. If $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space (and hence in particular a Banach space) then we have a map $H \to H^*$ defined by $x \mapsto \langle x, \cdot \rangle$; this map is obviously injective and (by the Cauchy-Schwarz inequality) norm-preserving, and then Theorem 1.4.3 says that the map is an isomorphism of Banach spaces.

For a Banach space B we have one topology on B^* induced by the operator norm (and in the case of a Hilbert space this coincides under the isomorphism $H \cong H^*$ with the original norm on H), but there is another topology on B^* called the *weak* topology. Here one says that a sequence $\{\phi_m\}_{m=1}^{\infty}$ *converges weakly* to $\phi \in B^*$ and writes $\phi_m \rightharpoonup \phi$ if for every $x \in B$ it holds that $\phi_m(x) \to \phi(x)$. For example if $\{e_m\}_{m=1}^{\infty}$ is an infinite orthonormal sequence in a Hilbert space it is not hard to see that the linear functionals $\langle e_m, \cdot \rangle$ converge weakly to the zero functional.

THEOREM 1.4.4. *Let B be a Banach space which contains a countable dense subset, and let $\{\phi_m\}_{m=1}^{\infty}$ be a sequence in B^* such that we have a uniform bound $\|\phi_m\|^* \leq C$ where $C > 0$ is independent of m. Then there is $\phi \in B^*$ and a subsequence $\{\phi_{m_k}\}_{k=1}^{\infty}$ of $\{\phi_m\}_{m=1}^{\infty}$ such that $\phi_{m_k} \rightharpoonup \phi$.*

REMARK 1.4.5. In fact the hypothesis that B has a countable dense subset is not needed at least if one is willing to work with nets instead of sequences, as follows from what is known as the Banach-Alaoglu theorem [**F**, Theorem 5.18]. The assumption of a countable dense subset will hold in the cases that we are interested in, and allows for a simpler proof that does not rely on Tychonoff's theorem.

PROOF. The proof is rather similar to the standard proof of the Arzelà-Ascoli theorem. Let $\{x_i\}_{i=1}^{\infty}$ be a countable dense sequence in B. Let us inductively construct a sequence of subsequences of $\{\phi_m\}_{m=1}^{\infty}$, denoted $\{\{\phi_{i,k}\}_{k=1}^{\infty}\}_{i=1}^{\infty}$, such that $\{\phi_{1,k}(x_1)\}_{k=1}^{\infty}$ is convergent (as is possible since $\{\phi_m(x_1)\}_{m=1}^{\infty}$ is a sequence in the bounded set $[-C\|x_1\|, C\|x_1\|] \subset \mathbb{R}$), and such that, for $i \geq 1$, $\{\phi_{i+1,k}\}_{k=1}^{\infty}$ is a subsequence of $\{\phi_{i,k}\}_{k=1}^{\infty}$ such that $\{\phi_{i+1,k}(x_{i+1})\}_{k=1}^{\infty}$ is convergent (as is possible since $\{\phi_{i,k}(x_{i+1})\}_{k=1}^{\infty}$ is a sequence in the bounded set $[-C\|x_{i+1}\|, C\|x_{i+1}\|]$).

Then by construction, each sequence $\{\phi_{i,k}\}_{k=1}^{\infty}$ is, for each $j \leq i$, a subsequence of $\{\phi_{j,k}\}_{k=1}^{\infty}$, and therefore has the property that $\phi_{i,k}(x_j)$ converges as $k \to \infty$ for all $j \leq i$.

Now let $\phi_{m_k} = \phi_{k,k}$. Note that $\{\phi_{k,k}\}_{k\geq i}$ is a subsequence of $\{\phi_{i,k}\}_{k=1}^{\infty}$, so in particular $\phi_{k,k}(x_i)$ converges as $k \to \infty$ for each $i \in \mathbb{N}$.

In fact we claim that $\phi_{k,k}(x)$ converges as $k \to \infty$ for each $x \in B$. Indeed, by the denseness of $\{x_i\}$ we can find i such that $\|x - x_i\| < \frac{\epsilon}{3C}$, and then we have

$$\|\phi_{k,k}(x)-\phi_{l,l}(x)\| \leq \|\phi_{k,k}(x)-\phi_{k,k}(x_i)\|+\|\phi_{k,k}(x_i)-\phi_{l,l}(x_i)\|+\|\phi_{l,l}(x_i)-\phi_{l,l}(x)\| < \frac{2\epsilon}{3}+\|\phi_{k,k}(x_i)-\phi_{l,l}(x_i)\|$$

where we have used the uniform boundedness hypothesis on the ϕ_m. For K so large that $k,l \geq K \Rightarrow \|\phi_{k,k}(x_i) - \phi_{l,l}(x_i)\| < \frac{\epsilon}{3}$ we will have $k,l \geq K \Rightarrow \|\phi_{k,k}(x) - \phi_{l,l}(x)\| < \epsilon$. So $\{\phi_{k,k}(x)\}_{k=1}^{\infty}$ is a Cauchy sequence for each $x \in B$.

So we can define, for each $x \in B$, $\phi(x) = \lim_{k\to\infty} \phi_{k,k}(x)$. Since $|\phi_{k,k}(x)| \leq C\|x\|$ for all k, x we have $|\phi(x)| \leq C\|x\|$ for all x. Also ϕ is linear:

$$\phi(cx + y) = \lim_{k\to\infty} \phi_k(cx + y) = \lim_{k\to\infty} (c\phi_{k,k}(x) + \phi_{k,k}(y)) = c\phi(x) + y.$$

So $\phi \in B^*$, and the fact that $\phi_{k,k}(x) \to \phi(x)$ for each x precisely means that $\phi_{m_k} = \phi_{k,k} \rightharpoonup \phi$. $\square$

Combining this with the isomorphism $x \mapsto \langle x, \cdot\rangle$ given by Theorem 1.4.3 we obtain:

COROLLARY 1.4.6. *If $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space which contains a countable dense subset and $\{x_m\}_{m=1}^{\infty}$ is a bounded sequence in H then there is a subsequence x_{m_k} and an element $x \in H$ such that $\langle x_{m_k}, y\rangle \to \langle x, y\rangle$ for all $y \in H$.*

For example, $H = L^2(\mathbb{R}^n)$ contains a countable dense subset (for instance one could take the rational linear combinations of characteristic functions of rectangles with rational coordinates), so if $\|f_m\|_2 \leq C$ then there is $f \in L^2(\mathbb{R}^n)$ such that, for some subsequence $\{f_{m_k}\}_{k=1}^{\infty}$, $\int_{\mathbb{R}^n} fg\,dV = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_{m_k} g\,dV$ for all $g \in L^2(\mathbb{R}^n)$.

REMARK 1.4.7. A similar statement works with $f, f_{m_k} \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, using the identification of $L^p(\mathbb{R}^n)$ with the dual of $L^q(\mathbb{R}^n)$ ([**F**, Theorem 6.15]—this is significantly harder than the $p = 2$ case which falls out from Theorem 1.4.3, and it does not extend to the case $p = 1$ since L^1 is not the dual of L^{∞}).

While the foregoing may seem a bit abstract, it yields a useful criterion for the existence of weak derivatives. We first introduce notation for *difference quotients*: for $i \in \{1,\dots,n\}$, $|h| > 0$, and $u\colon \mathbb{R}^n \to \mathbb{R}$ let us define

$$D_i^h u\colon \mathbb{R}^n \to \mathbb{R} \quad \text{by} \quad D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}.$$

THEOREM 1.4.8. *Let $u \in L^2(\mathbb{R}^n)$ and let $C > 0$. The following are equivalent:*

(i) *There is $h_0 > 0$ such that for each h with $0 < |h| < h_0$ and each $i \in \{1,\dots,n\}$ we have $\|D_i^h u\|_2 \leq C$.*

(ii) *$u \in W^{1,2}(\mathbb{R}^n)$, and for each i we have $\frac{\partial u}{\partial x_i} \overset{\mathrm{W}}{=} f_i$ where $f_i \in L^2(\mathbb{R}^n)$ has $\|f_i\|_2 \leq C$.*

PROOF. We first prove that (i)⇒(ii). By Corollary 1.4.6 (and the remark immediately thereafter), the assumption that $\|D_i^h u\|_2 \leq C$ for $h < h_0$ implies that we can find a sequence $h_k \to 0$ and a function $f_i \in L^2(\mathbb{R}^n)$ such that $D_i^{h_k} u \rightharpoonup f_i$. In particular $\|f_i\|_2^2 = \langle f_i, f_i\rangle = \lim_{k\to\infty}\langle D_i^{h_k} u, f_i\rangle \leq C\|f_i\|_2$, so $\|f_i\|_2 \leq C$. We claim that $\frac{\partial u}{\partial x_i} \overset{\mathrm{W}}{=} f_i$.

To see this, if $\phi \in C_0^\infty(\mathbb{R}^n)$, by construction we have $\langle f, \phi\rangle = \lim_{k\to\infty}\langle D_i^{h_k} u, \phi\rangle$. So, changing variables as appropriate, we have

$$\begin{aligned}\int_{\mathbb{R}^n} f\phi dV &= \lim_{k\to\infty}\int_{\mathbb{R}^n}\left(\frac{u(x+h_k e_i)-u(x)}{h_k}\right)\phi(x)dV_x \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^n} u(x)\left(\frac{\phi(x-h_k e_i)-\phi(x)}{h_k}\right)dV_x = -\int_{\mathbb{R}^n} u(x)\frac{\partial\phi}{\partial x_i}(x)dV_x\end{aligned}$$

where in the final equality we hae used the Dominated Convergence Theorem, which applies since $\frac{\phi(x-h_k e_i)-\phi(x)}{h_k}$ admits a uniform bound by the Lipschitz constant of the (smooth and compactly supported) function ϕ. The above formula shows that indeed $\frac{\partial u}{\partial x_i} \overset{\mathrm{W}}{=} f_i$, completing the proof that (i)⇒(ii).

The reverse inclusion (ii)⇒(i) is more elementary. Suppose first that $u \in C_0^\infty(\mathbb{R}^n)$. Observe that, using the Schwarz inequality, for any $x \in \mathbb{R}^n, h \neq 0$ we have

$$|u(x+he_i)-u(x)|^2 = \left|\int_0^1 h\frac{\partial u}{\partial x_i}(x+the_i)dt\right|^2 \le h^2\left(\int_0^1\left|\frac{\partial u}{\partial x_i}(x+the_i)\right|^2 dt\right)\left(\int_0^1 1^2 dt\right),$$

i.e.,

$$|D_i^h u(x)|^2 \le \int_0^1\left|\frac{\partial u}{\partial x_i}(x+the_i)\right|^2 dt.$$

So

$$\begin{aligned}\int_{\mathbb{R}^n}|D_i^h u(x)|^2 dV_x &= \int_{\mathbb{R}^n}\int_0^1\left|\frac{\partial u}{\partial x_i}(x+the_i)\right|^2 dt dV_x \\ &= \int_0^1\int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}(x+the_i)\right|^2 dV_x dt = \int_0^1\int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_i}(y)\right|^2 dV_y dt = \left\|\frac{\partial u}{\partial x_i}\right\|_2^2,\end{aligned}$$

where we have simply made the change of variables $y = x + the_i$ in the second-to-last equality. So we have shown that for any $u \in C_0^\infty(\mathbb{R}^n)$ and any $h > 0$ we have $\|D_i^h u\|_2 \le \left\|\frac{\partial u}{\partial x_i}\right\|_2$. If now $v \in W^{1,2}(\mathbb{R}^n)$, we can find a sequence $\{u_m\}_{m=1}^\infty$ with $\|u_m - v\|_{1,2} \to 0$. If $\frac{\partial v}{\partial x_i} \overset{\mathrm{W}}{=} f_i$, then we will have $\left\|\frac{\partial u_m}{\partial x_i} - f_i\right\|_2 \to 0$. Meanwhile, for any fixed $h > 0$, it is immediate from the fact that $u_m \to v$ in L^2 that likewise $\|D_i^h u_m - D_i^h v\|_2 \to 0$. So since $\|D_i^h u_m\|_2 \le \left\|\frac{\partial u_m}{\partial x_i}\right\|_2$ it follows that $\|D_i^h v\|_2 \le \|f_i\|_2$, completing the proof that (ii)⇒(i). □

REMARK 1.4.9. Using the results mentioned in Remark 1.4.7, the above theorem can equally well be proven for any p with $1 < p < \infty$.

We close this section by recording a couple of simple facts about difference quotients that will be used later—these are finitary versions of the product rule and integration by parts.

PROPOSITION 1.4.10. *Let $f, g\colon \mathbb{R}^n \to \mathbb{R}$ be two functions and $h \in \mathbb{R}\setminus\{0\}$. Then:*

(i) *Defining $f^h(x) = f(x+he_i)$, we have*

$$D_i^h(fg) = f^h D_i^h g + (D_i^h f)g.$$

(ii) *If $f, g \in L^2(\mathbb{R}^n)$ then*

$$\int_{\mathbb{R}^n}(D_i^h f)g dV = -\int_{\mathbb{R}^n} f(D_i^{-h} g)dV.$$

PROOF. (i) follows by simply writing

$$\frac{(fg)(x+he_i)-(fg)(x)}{h} = f(x+he_i)\frac{g(x+he_i)-g(x)}{h} + g(x)\frac{f(x+he_i)-f(x)}{h}.$$

For (ii), we have

$$\int_{\mathbb{R}^n}(D_h^i f)g\,dV = \frac{1}{h}\left(\int_{\mathbb{R}^n} f(x+he_i)g(x)dV_x - \int_{\mathbb{R}^n} f(x)g(x)dV_x\right) = \frac{1}{h}\left(\int_{\mathbb{R}^n} f(y)g(y-he_i)dV_y - \int_{\mathbb{R}^n} f(x)g(x)dV_x\right)$$
$$= -\int_{\mathbb{R}^n} f(x)\frac{g(x-h_e i)-g(x)}{-h}dV_x = -\int_{\mathbb{R}^n} f(D_i^{-h}g)dV.$$

□

1.5. The main regularity theorem

To prove Theorem 1.1.12 we must show that the equation $\Delta\omega = \theta$ can be solved for a smooth differential form ω whenever θ is a smooth differential form that lies in the orthogonal complement of the space of harmonic forms. This will involve two steps: we will argue that the equation has a weak solution in an appropriate (differential forms-based) version of the Sobolev space $W^{1,2}$, and we will show that any such weak solution is smooth. In this section we develop the PDE theory necessary for the second step. When written out in local coordinates, the equation $\Delta\omega = \theta$ belongs to a class of linear second-order elliptic equations $Lu = f$ which, as we will see, has the general feature that if the right-hand side f belongs to the Sobolev space $W^{k,2}$, then the solution u belongs to $W^{k+2,2}$.

Since the differential equations that we study are for local coordinate representations of functions defined on a manifold, the functions involved will be defined on some open subset $U \subset \mathbb{R}^n$ rather than all of $\mathbb{R}^n$. So our prior work on Sobolev spaces should be adapted to the context of functions defined only on open subsets of $\mathbb{R}^n$. If $U \subset \mathbb{R}^n$ is open and $u\colon U \to \mathbb{R}^n$ is locally integrable we can define weak derivatives analogously to before: for $f \in L^1_{loc}(U)$ we say that $D^\alpha u \overset{\mathrm{W}}{=} f$ if and only if, for every $\phi \in C_0^\infty(U)$, we have $\int_U uD^\alpha\phi\,dV = (-1)^{|\alpha|}\int_U f\phi\,dV$. There are two natural Sobolev spaces to consider. We can let

$$W^{k,p}(U) = \left\{u \in L^p(U) \,\middle|\, (\forall 0 \le |\alpha| \le k)(\exists f_\alpha \in L^p(U))(D^\alpha u \overset{\mathrm{W}}{=} f_\alpha)\right\}$$

and equip $W^{k,p}(U)$ with the norm $\|\cdot\|_{W^{k,p}(U)}$ given by $\|u\|^p_{W^{k,p}(U)} = \sum_{0\le|\alpha|\le k}\int_U |f_\alpha|^p dV$ if $D^\alpha u \overset{\mathrm{W}}{=} f_\alpha$. We can also consider the space $W_0^{k,p}(U)$ given as the completion of $C_0^\infty(U)$ with respect to the norm $\|\cdot\|_{W^{k,p}(U)}$. So $W_0^{k,p}(U) \subset W^{k,p}(U)$, but in contrast to the case $U = \mathbb{R}^n$ we should not expect equality—for instance if $k > n/p$ any element of $W_0^{k,p}(U)$ extends to a continuous function which vanishes along ∂U, whereas elements of $W^{k,p}(U)$ can be large near ∂U. Meanwhile since $C_0^\infty(U)$ naturally embeds (via extension by zero) into $C_0^\infty(\mathbb{R}^n)$, and since the norms $\|\cdot\|_{W^{k,p}(U)}$ and $\|\cdot\|_{k,p}$ correspond under this embedding, we can also see $W_0^{k,p}(U)$ as a subset of $W^{k,p}(\mathbb{R}^n)$.

It's easy to see (using Exercise 1.3.2 and Proposition 1.3.8) that if $\zeta \in C_0^\infty(U)$ and if $u \in W^{k,p}(U)$ then $\zeta u \in W_0^{k,p}(U)$.

Let us use the notation $V \Subset U$ to signify that V, U are open subsets of $\mathbb{R}^n$ with $\bar{V}$ compact and $\bar{V} \subset U$. The proof of the main regularity theorem will involve showing that certain elements $g \in L^2(U)$ in fact belong to $W^{1,2}(V)$ whenever $V \Subset U$.

DEFINITION 1.5.1. If $V \Subset U$, a VU**-cutoff** is a function $\zeta \in C_0^\infty(U)$ which is equal to 1 on some neighborhood of $\bar{V}$.

PROPOSITION 1.5.2. *Let $V \Subset U$, let ζ be a VU-cutoff, and let $g \in L^2(U)$. Suppose that there is a constant $C > 0$ such that we have $\int_U \zeta^2 |D_i^h g|^2 dV \le C^2$ for all $i \in \{1, \dots, n\}$ and all sufficiently small $h > 0$. Then $g \in W^{1,2}(V)$, and for some constant B (depending only on ζ, not on g) there is a bound $\|g\|_{W^{1,2}(V)} \le B(C + \|g\|_{L^2(U)})$.*

PROOF. Since $\zeta g|_V = g|_V$, we have $g \in W^{1,2}(V)$ iff $\zeta g \in W^{1,2}(V)$. Note that ζg extends by zero to a function in $L^2(\mathbb{R}^n)$, so Theorem 1.4.8 can be applied with $u = \zeta g$.

Choose $h \neq 0$ with $|h|$ so small that for every $x \in supp(\zeta)$ we have $B_{2h}(x) \subset U$. Consider the difference quotients $D_i^h(\zeta g)$ (where ζg has been extended by zero outside of U). If x fails to have $B_h(x) \subset U$ our hypothesis on h shows that ζ vanishes both at x and at $x + he_i$, so $D_i^h(x) = 0$. Meanwhile if $B_h(x) \subset U$ (so in particular $g(x+he_i)$ and $g(x)$ are both defined) then by Proposition 1.4.10 we have

$$D_i^h(\zeta g)(x) = g^h(x) D_i^h \zeta(x) + \zeta(x) D_i^h g(x)$$

where $g^h(x) = g(x + he_i)$. So if A is an upper bound for the Lipschitz constant of ζ (which exists since ζ is smooth and compactly supported) we have

$$|D_i^h(\zeta g)(x) - \zeta(x) D_i^h g(x)| \le A|g^h(x)|.$$

Hence integrating over x (and using translation invariance of the integral) gives

$$\int_U \left| D_i^h(\zeta g) - \zeta D_i^h g \right|^2 dV \le A^2 \int_U |g|^2 dV,$$

i.e. $\|D_i^h(\zeta g) - \zeta D_i^h g\|_{L^2(U)} \le A\|g\|_{L^2(U)}$. So the L^2 bound on $\zeta D_i^h g$ in the hypothesis gives an L^2 bound (independent of sufficiently small h) on $D_i^h(\zeta g)$ by $C + A\|g\|_{L^2(U)}$. So applying Theorem 1.4.8 implies that $\zeta g \in W^{1,2}(\mathbb{R}^n)$ with $\|\zeta g\|_{1,2} \le B(C + \|g\|_{L^2(U)})$ for an appropriate constant B. Since ζg coincides with g on V the result follows immediately. □

We will consider the general class of equations, for a function $u \colon U \to \mathbb{R}$ where $U \subset \mathbb{R}^n$ is open, of the form

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) = f, \tag{23}$$

where the functions $a_{ij} \colon U \to \mathbb{R}^n$ are assumed to be bounded, and to satisfy $a_{ij} = a_{ji}$ and a *uniform ellipticity condition*

$$\sum_{i,j=1}^n a_{ij}(x) v_i v_j \ge \theta \sum_{i=1}^n v_i^2 \quad \text{for all } x \in U, (v_1, \dots, v_n) \in \mathbb{R}^n \text{ and some } \theta > 0. \tag{24}$$

In other words, we are assuming that the matrices $A(x) = (a_{ij}(x))$ are symmetric and positive definite, with all eigenvalues bounded below by $\theta > 0$ independently of x. The function f will be considered as given. The Poisson equation $\Delta u = f$ is the special case that $A(x)$ is the identity for all x.

DEFINITION 1.5.3. A **weak solution** to (23) on U is a function $u \in W^{1,2}(U)$ such that, for every $v \in W_0^{1,2}(U)$, we have

$$\sum_{i,j=1}^n \int_U a_{ij} u_{x_i} v_{x_j} dV = \int_U f v dV. \tag{25}$$

(Here for $g \in W^{1,2}(U)$ we denote by g_{x_k} the weak derivative of g with respect to x_k.)

By considering approximations of v by functions in $C_0^\infty(U)$ it's easy to see that it is equivalent to just require that (25) hold for $v \in C_0^\infty(U)$. The motivation for the definition is that, if all functions involved were assumed to be smooth (or even just C^1), one could obtain (25) from (23) by multiplying both sides by v and integrating by parts.

The following theorem is the main step in proving regularity for solutions to the various equations that appear throughout these notes. The proof will frequently (and sometimes without comment) use the following inequality for real numbers:

$$\text{For } x, y \in \mathbb{R},\ \delta > 0, \quad |xy| \le \frac{\delta}{2}x^2 + \frac{1}{2\delta}y^2. \tag{26}$$

It is easy to prove this: just use the fact that $\left(\sqrt{\delta}|x| - \frac{1}{\sqrt{\delta}}|y|\right)^2 \ge 0$. Typically we will choose δ to be relatively small, so that $|xy|$ is bounded by a small constant times x^2 plus a large constant times y^2.

THEOREM 1.5.4. *Assume that the functions $a_{ij}\colon U \to \mathbb{R}$ are bounded, Lipschitz, and satisfy $a_{ij} = a_{ji}$ and (24). Assume also that $f \in L^2(U)$, and that $u \in W^{1,2}(U)$ is a weak solution to (23). Then for every $V \Subset U$ we have $u \in W^{2,2}(V)$, and there is a constant C depending on a_{ij}, U, V but not on u or f such that*

$$\|u\|_{W^{2,2}(V)} \le C(\|u\|_{W^{1,2}(U)} + \|f\|_{L^2(U)}). \tag{27}$$

PROOF. Let $\zeta \in C_0^\infty(U)$ be a VU-cutoff. Since $u \in W^{1,2}(U)$ we have $\zeta u \in W_0^{1,2}(U)$. Also, if $h \neq 0$ is sufficiently small that $B_{2|h|}(x) \subset U$ for all $x \in supp(\zeta)$, then for each $k \in \{1, \dots, n\}$ we have both $\zeta^2 D_k^h u \in W_0^{1,2}(U)$ (after extension by zero at points where $x + he_k \notin U$, since $\zeta = 0$ at all such points) and $D_k^{-h}(\zeta^2 D_k^h u) \in W_0^{1,2}(U)$. We will apply (25) with $v = -D_k^{-h}(\zeta^2 D_k^h u)$. Proposition 1.4.10(ii) then yields

$$\sum_{i,j=1}^n \int_U \left(D_k^h(a_{ij}u_{x_i})\right)(\zeta^2 D_k^h u)_{x_j} dV = -\int_U f D_k^{-h}(\zeta^2 D_k^h u) dV. \tag{28}$$

In view of Proposition 1.5.2, it will suffice to prove appropriate upper bounds for the quantity $\|\zeta D_k^h \nabla u\|_{L^2(U)}^2 = \int_U \zeta^2 |D_k^h \nabla u|^2 dV$.

Let us rewrite and estimate the left-hand side of (28), using the product rule and its finitary version from Proposition 1.4.10(i) and also using (24). We have, for some constant C depending on ζ and on the a_{ij}:

$$\begin{aligned}
\sum_{i,j=1}^n \int_U \left(D_k^h(a_{ij}u_{x_i})\right)(\zeta^2 D_k^h u)_{x_j} dV &= \int_U \left(\zeta^2 \sum_{i,j=1}^n a_{ij}^h D_k^h u_{x_i} D_k^h u_{x_j}\right) dV + \sum_{i,j=1}^n \int_U \left(2\zeta\zeta_{x_j} D_k^h(a_{ij}u_{x_i}) D_k^h u + \zeta^2 (D_k^h a_{ij}) u_{x_i} D_k^h u_{x_j}\right) dV \\
&\ge \theta \int_U \zeta^2 |D_k^h \nabla u|^2 dV + \sum_{i,j=1}^n \left(\int_U 2\zeta_{x_j}(D_h^k a_{ij}) u_{x_i} \zeta D_k^h u dV + \int_U 2\zeta a_{ij}^h D_k^h u_{x_i} \zeta_{x_j} D_k^h u dV + \int_U (\zeta(D_k^h a_{ij}) u_{x_i} \zeta(D_k^h u_{x_j}) dV\right) \\
&\ge \theta \|\zeta D_k^h \nabla u\|_{L^2(U)}^2 - C\left(\|\nabla u\|_{L^2(U)} \|\zeta D_h^k u\|_{L^2(U)} + \|\zeta D_h^k \nabla u\|_{L^2(U)} \|\zeta_{x_j} D_k^h u\|_{L^2(U)} + \|\nabla u\|_{L^2(U)} \|\zeta D_k^h \nabla u\|_{L^2(U)}\right).
\end{aligned} \tag{29}$$

Recall that h is chosen to be close enough to zero that if $x \in supp(\zeta)$ then $B_{2|h|}(x) \subset U$. If $\chi \in C_0^\infty(U)$ is a function which is identically equal to one on a radius-h neighborhood of the support of ζ, then $D_k^h(\chi u)$ will coincide with $D_k^h(u)$ on the support of ζ; on the other hand χu extends by zero to a function in $W^{1,2}(\mathbb{R}^n)$, so the implication (ii)⇒(i) in Theorem 1.4.8 gives a bound $\|D_k^h(\chi u)\|_2 \le \|\chi u\|_{1,2}$. Now $\|\chi u\|_{1,2} \le A\|u\|_{W^{1,2}(U)}$ for some constant A depending on χ but not on u. So the expression $\|\zeta D_k^h u\|_{L^2(U)}$ that appears in (29) is equal to $\|\zeta D_k^h(\chi u)\|_{L^2(U)}$, which is bounded above by a constant (depending on ζ) times $\|u\|_{W^{1,2}(U)}$. A similar remark applies to the expression $\|\zeta_{x_j} D_k^h u\|_{L^2(U)}$ in (29). So (29) and (26) give, for some constant C' and all $\delta > 0$:

$$\sum_{i,j=1}^{n} \int_U \left(D_k^h(a_{ij}u_{x_i})\right)(\zeta^2 D_k^h u)_{x_j} dV \geq \theta\|\zeta D_k^h \nabla u\|_{L^2(U)}^2 - C'\left(\|u\|_{W^{1,2}(u)}^2 + \|u\|_{W^{1,2}(U)}\|\zeta D_k^h \nabla u\|_{L^2(U)}\right)$$

$$\geq \theta\|\zeta D_k^h \nabla u\|_{L^2(U)}^2 - \frac{C'\delta}{2}\|\zeta D_k^h \nabla u\|_{L^2(U)}^2 - C'\left(1 + \frac{1}{2\delta}\right)\|u\|_{W^{1,2}(U)}^2.$$

If we choose $\delta = \frac{\theta}{C'}$, the first two terms on the right above combine to give $\frac{\theta}{2}\|\zeta D_k^h \nabla u\|_{L^2(U)}^2$. So based on (28) we obtain

$$\frac{\theta}{2}\|\zeta D_k^h \nabla u\|_{L^2(U)}^2 \leq -\int_U f D_k^{-h}(\zeta^2 D_k^h u) dV + C''\|u\|_{W^{1,2}(U)}^2 \tag{30}$$

where $C'' = C'(1 + \frac{1}{2\delta}) = C'(1 + \frac{C'}{2\theta})$.

Now for any h in our allowed range, the function $\zeta^2 D_k^h u = \frac{1}{h}(\zeta^2 u^h - \zeta^2 u)$ is a class-$W^{1,2}$ function on U whose support is contained in that of ζ, and hence extends by zero to give a function in $W^{1,2}(\mathbb{R}^n)$. So Theorem 1.4.8 applies to show that, for all sufficiently small nonzero h,

$$\|D_k^{-h}(\zeta^2 D_k^h u)\|_{L^2(U)} \leq \left\|\frac{\partial}{\partial x_k}(\zeta^2 D_k^h u)\right\|_{L^2(U)}.$$

So by the product rule and our earlier observation that bounded $\|\zeta D_k^h u\|_{L^2(U)}$ by a constant times $\|u\|_{W^{1,2}(U)}$ we obtain, for some constant B,

$$\|D_k^{-h}(\zeta^2 D_k^h u)\|_{L^2(U)} \leq B\|u\|_{W^{1,2}(U)} + \|\zeta^2 \nabla(D_k^h u)\|_{L^2(U)}.$$

For the last term above, note that $\zeta^2 \leq \zeta$ (since $0 \leq \zeta \leq 1$) and that $\nabla \circ D_k^h = D_k^h \circ \nabla$, so this term is at most our familiar quantity $\|\zeta D_k^h \nabla u\|_{L^2(U)}$. So (30), the Schwarz inequality, and (26) give, for some constant C_0 and any $\epsilon > 0$:

$$\begin{aligned}
\frac{\theta}{2}\|\zeta D_k^h \nabla u\|_{L^2(U)}^2 &\leq \|f\|_{L^2(U)}\|D_k^{-h}(\zeta^2 D_k^h u)\|_{L^2(U)} + C''\|u\|_{W^{1,2}(U)}^2 \\
&\leq \|f\|_{L^2(U)}(B\|u\|_{W^{1,2}(U)} + \|\zeta D_k^h \nabla u\|_{L^2(U)}) + C''\|u\|_{W^{1,2}(U)}^2 \\
&\leq \left(\frac{B}{2} + \frac{1}{2\epsilon}\right)\|f\|_{L^2(U)}^2 + C_0\|u\|_{W^{1,2}(U)}^2 + \frac{\epsilon}{2}\|\zeta D_k^h \nabla u\|_{L^2(U)}^2.
\end{aligned}$$

Choosing $\epsilon = \frac{\theta}{2}$ and rearranging finally gives an inequality

$$\frac{\theta}{4}\|\zeta D_k^h \nabla u\|_{L^2(U)}^2 \leq C_1(\|u\|_{W^{1,2}(U)}^2 + \|f\|_{L^2(U)}^2),$$

with all constants independent of u and of sufficiently small h. So applying Proposition 1.5.2 with g equal to the various (weak) partial derivatives u_{x_j} of u shows that each u_{x_j} belongs to $W^{1,2}(V)$, with $\|u_{x_j}\|_{W^{1,2}(V)}$ bounded above by a constant times $\|u\|_{W^{1,2}(U)} + \|f\|_{L^2(U)}$. Thus we indeed have $u \in W^{2,2}(V)$ with

$$\|u\|_{W^{2,2}(V)} \leq \tilde{C}(\|u\|_{W^{1,2}(U)} + \|f\|_{L^2(U)}).$$

The constant $\tilde{C}$ depends on the a_{ij} and on the cutoff function ζ (and also on the cutoff function χ that appeared in the middle of the proof, but this is constructed based on ζ); ζ in turn may be constructed just based on the pair of sets V and U, so our constant indeed only depends on a_{ij}, U, V (and not on u and f). □

Results such as Theorem 1.5.4 are often stated for a superficially larger class of equations:

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right)+\sum_{i=1}^{n}b_i(x)\frac{\partial u}{\partial x_i}+c(x)u=f \tag{31}$$

for given functions $a_{ij}b_i, c, f$. Here $u\in W^{1,2}(U)$ is said to be a weak solution of (31) provided that, for every $v\in W_0^{1,2}(U)$ we have

$$\sum_{i,j=1}^{n}\int_U a_{ij}u_{x_i}v_{x_j}dV+\sum_{i=1}^{n}\int_U bu_{x_i}v dV+\int_U cuvdV=\int_U f v dV.$$

However at least under reasonable hypotheses on the coefficient functions b_i, c our regularity result Theorem 1.5.4 formally implies regularity for (31):

COROLLARY 1.5.5. *If $a_{ij}, b_i, c\colon U\to\mathbb{R}$ are bounded functions with each a_{ij} Lipschitz, $a_{ij}=a_{ji}$, and such that (24) holds, and if $V\Subset U$, then there is a constant C depending on a_{ij}, b_i, c, U, V but not on u and f such that if $u\in W^{1,2}(U)$ and $f\in L^2(U)$ with u a weak solution to (31), then $u\in W^{2,2}(V)$ and*

$$\|u\|_{W^{2,2}(V)}\le C(\|u\|_{W^{1,2}(V)}+\|f\|_{L^2(V)}).$$

PROOF. Simply note that if $u\in W^{1,2}(U)$ is a weak solution to (31), then if we put $g=f-\sum_{i=1}^n b_iu_{x_i}-cu$, the boundedness of b_i, c implies that $g\in L^2(U)$, and evidently u is a weak solution to $-\sum_{i,j}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)=g$. Moreover we have $\|g\|_{L^2(U)}\le\|f\|_{L^2(U)}+C'\|u\|_{W^{1,2}(U)}$ for some constant C' depending on b_i, c. So applying Theorem 1.5.4 (with f replaced by g) immediately implies the result. □

Now we wish to show that a solution to an equation like (31) is not just of class $W^{2,2}$, but is of class $W^{k,2}$ for all k (at least assuming that the coefficient functions a_{ij}, b_i, c and the function f on the right-hand side are all smooth). This can be done simply by noticing that taking a partial derivative $\frac{\partial}{\partial x_m}$ of both sides of (31) leads to an equation of the same form for $\frac{\partial u}{\partial x_m}$. To be formal about this:

PROPOSITION 1.5.6. *Let $u\in W^{1,2}(U)$ be a weak solution of (31), where $a_{ij}, b_i, c, f\colon U\to\mathbb{R}$ are as in Theorem 1.5.5 and where we moreover assume that each of a_{ij}, b_i, c are of class C^2 and that $f\in W^{1,2}(U)$. Let $V\Subset U$. Then each weak derivative $g=u_{x_m}$ of u belongs to $W^{1,2}(V)$ and is a weak solution of the equation*

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial g}{\partial x_i}\right)+\sum_{i=1}^{n}b_i(x)\frac{\partial g}{\partial x_i}+c(x)g=\tilde{f} \tag{32}$$

where $\|\tilde{f}\|_{L^2(V)}\le B(\|f\|_{W^{1,2}(U)}+\|u\|_{W^{1,2}(U)})$ for some constant B independent of u, f (but depending on a_{ij}, b_i, c, U, V). Also, if a_{ij}, b_i, c are of class C^{l+2} and if $u\in W^{l+2,2}(V)$ and $f\in W^{l+1,2}(V)$ then $\tilde{f}\in W^{l,2}(V)$ with $\|\tilde{f}\|_{W^{l,2}(V)}\le\|f\|_{W^{l+1,2}(V)}+A_k\|u\|_{W^{l+2,2}(V)}$ for some constant A_k indepedent of f and u.

PROOF. Corollary 1.5.5 says that $u \in W^{2,2}(V)$ and hence that $u_{x_m} \in W^{1,2}(V)$. Integrating by parts (or, perhaps more accurately, using the definition of a weak derivative) and using the assumption that u is a weak solution of (31) we have, for any $v \in C_0^\infty(V)$,

$$\begin{aligned}
&\sum_{i,j=1}^n \int_V a_{ij} u_{x_m x_i} v_{x_j} dV + \sum_{i=1}^n \int_V b_i u_{x_m x_i} v dV + \int_V c u_{x_m} v dV \\
&= -\left(\sum_{i,j=1}^n \int_V u_{x_i} \frac{\partial}{\partial x_m} \left(a_{ij} v_{x_j} \right) dV + \int_V u_{x_i} \frac{\partial}{\partial x_m} (b_i v)\, dV + \int_V u \frac{\partial}{\partial x_m} (cv) dV \right) \\
&= -\int_V \left(\sum_{i,j=1}^n a_{ij} u_{x_i} v_{x_m x_i} + \sum_{i=1}^n b_i u_{x_i} v_{x_m} + c u v_{x_m} \right) dV - \int_V \left(\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_m} u_{x_i} v_{x_j} + \sum_{i=1}^n \frac{\partial b_i}{\partial x_m} u_{x_i} v + \frac{\partial c}{\partial x_m} u v \right) dV \\
&= -\int_V f v_{x_m} dV + \int_V \left(\sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(\frac{\partial a_{ij}}{\partial x_m} u_{x_i} \right) - \sum_{i=1}^n \frac{\partial b_i}{\partial x_i} u_{x_i} - \frac{\partial c}{\partial x_m} u \right) v dV \\
&= \int_V \tilde{f} v dV
\end{aligned}$$

where

$$\tilde{f} = \frac{\partial f}{\partial x_m} + \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(\frac{\partial a_{ij}}{\partial x_i} u_{x_i} \right) - \sum_{i=1}^n \frac{\partial b_i}{\partial x_i} u_{x_i} - \frac{\partial c}{\partial x_m} u. \tag{33}$$

Note that since $u \in W^{2,2}(V)$ and since a_{ij}, b_i, c are C^2 functions on the compact set $\bar{V}$ containing V (so their derivatives up to order two are bounded on V) the function $\tilde{f}$ in (33) belongs to $L^2(V)$, with a bound $\|\tilde{f}\|_{L^2(V)} \le \|f\|_{W^{1,2}(V)} + A\|u\|_{W^{2,2}(V)}$ for some constant A depending on a_{ij}, b_i, c. So by (27) we have $\|\tilde{f}\|_{L^2(V)} \le B(\|f\|_{W^{1,2}(U)} + \|u\|_{W^{1,2}(U)})$ for an appropriate constant B. Likewise, if a_{ij}, b_i, c are assumed to be of class C^{k+2} and $u \in W^{k+2,2}(V)$ then the formula for $\tilde{f}$ readily shows that $\tilde{f} \in W^{k,2}(V)$ with $\|\tilde{f}\|_{W^{k,2}(V)} \le \|f\|_{W^{k+1,2}(V)} + A_k\|u\|_{W^{k+2,2}(V)}$ for some constant A_k.

Thus we have $u_{x_m} \in W^{1,2}(V)$, $\tilde{f} \in L^2(V)$, and our calculation above shows that, for every $v \in C_0^\infty(V)$, we have

$$\sum_{i,j=1}^n \int_V a_{ij} u_{x_m x_i} v_{x_j} dV + \sum_{i=1}^n \int_V b u_{x_m x_i} v dV + \int_V c u_{x_m} v dV = \int_V \tilde{f} v dV.$$

If instead we just assume that $v \in W_0^{1,2}(V)$, then by taking a sequence $\{v_r\}_{r=1}^\infty$ in $C_0^\infty(V)$ and taking the limit as $r \to \infty$ of the versions of the above equation with v replaced by v_r, we obtain that the same equation holds for v. Thus $g = u_{x_m}$ is indeed a weak solution to (32). □

We can now prove the following, which can be seen as a vast generalization of Weyl's Lemma 1.2.12 (though with a slightly stronger regularity hypothesis on u).

THEOREM 1.5.7. *Let $U \subset \mathbb{R}^n$ be a bounded open subset, and let $a_{ij}, b_i, c\colon U \to \mathbb{R}$ be C^∞ functions, with $a_{ij} = a_{ji}$, such that (24) holds. Suppose that $f \in W^{m,2}(U)$ ($m \ge 0$), and that $u \in W^{1,2}(U)$ is a weak solution to*

$$-\sum_{i,j=1} \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x) u = f.$$

Then $u \in W^{m+2,2}(U)$, and we have a bound

$$\|u\|_{W^{m+2,2}(V)} \le C_k(\|u\|_{W^{1,2}(U)} + \|f\|_{W^{m,2}(U)}) \tag{34}$$

where C_k is a constant depending on k, a_{ij}, b_i, c, V, U but not on f and u. In particular, if $f \in C^\infty(U)$ then $u \in C^\infty(U)$.

PROOF. Let us abbreviate by L the operator

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c,$$

so our theorem concerns (weak) solutions to the equation $Lu = f$.

Let $V \Subset U$ and choose a sequence of open sets $U = V_0 \supset V_1 \supset \cdots \supset V_k \supset \cdots V_{m+1} = V$, with each $V_{i+1} \Subset V_i$. We will show inductively that $u \in W^{k+1,2}(V_k)$ with a bound $\|u\|_{W^{k+1,2}(V_k)} \le C_k(\|u\|_{W^{1,2}(U)} + \|f\|_{W^{k-1,2}(U)})$ for all $1 \le k \le m+1$. For $k = 1$ this is simply Corollary 1.5.5. In general assume the result proven for a value $k \le m$ and choose V_k' with $V_{k+1} \Subset V_k' \Subset V_k$. By sequentially applying Proposition 1.5.6 to derivatives $D^\alpha u = (D^\beta u)_{x_m}$ with $|\beta| = |\alpha| - 1 \le k-1$, we see that each $D^\alpha u$ with $|\alpha| \le k$ is, on V_k', a class-$W^{1,2}$ weak solution to an equation $Lg = f_\alpha$, where $f_\alpha \in W^{k-|\alpha|,2}(V_k')$ with

$$\begin{aligned}\|f_\alpha\|_{W^{k-|\alpha|,2}(V_k')} &\le A(\|f\|_{W^{k,2}(V_k')} + \|D^\beta u\|_{W^{k-|\alpha|+2}(V_k')})\\ &\le A(\|f\|_{W^{k,2}(V_k)} + \|u\|_{W^{k+1,2}(V_k)})\end{aligned}$$

for some constant A. But then Corollary 1.5.5 shows that, for all α with $|\alpha| = k$, we have $D^\alpha u \in W^{2,2}(V_{k+1})$, with a bound

$$\begin{aligned}\|D^\alpha u\|_{W^{2,2}(V_{k+1})} &\le C'(\|f_\alpha\|_{L^2(V_k')} + \|D^\alpha u\|_{W^{1,2}(V_k')})\\ &\le C''(\|f\|_{W^{k,2}(V_k)} + \|u\|_{W^{k+1,2}(V_k)})\end{aligned}$$

for appropriate constants C', C''. This combines with the inductive hypothesis on $\|u\|_{W^{k+1,2}(V_k)}$ to give the desired bound $\|u\|_{W^{k+2,2}(V_{k+1})} \le C_k(\|u\|_{W^{1,2}(U)} + \|f\|_{W^{k,2}(U)})$. This completes the induction; the final inductive step (leading to the assertion that the inductive hypothesis holds for $k = m+1$) gives that $u \in W^{m+2,2}(V)$ and that (34) holds.

For the final sentence, the assumption that $f \in C^\infty(U)$ implies that $f \in W^{k,2}(V)$ for any $V \Subset U$. So what we have shown establishes that, for all k and all $W \Subset V \Subset U$ we have $u \in W^{k+2,2}(W)$. This implies that u is of class C^∞ on every subset W with $W \Subset U$, and hence on all of U since smoothness is a local condition. □

REMARK 1.5.8. The fact that $Lu = f$ with $f \in W^{m,2}$ is enough to imply that $u \in W^{m+2,2}$ (modulo passing to an arbitrary precompact open subset) is perhaps surprising, since the equation $Lu = f$ only imposes a condition on a very particular combination of (some of) the second partial derivatives of u, while (34) controls all of them separately. This nice behavior of "gaining two derivatives" doesn't quite work out if one uses the more straightforward spaces $C^k(U)$ in place of the Sobolev spaces $W^{k,2}(U)$: the function $u(x,y) = (x^2 - y^2)\sqrt{-\log(x^2+y^2)}$ on the open unit disk has Δu continuous but isn't C^2.

1.6. Proof of the Hodge theorem

We now finally have enough background on Sobolev spaces and PDE's to return to the context of differential forms on a smooth manifold and to prove Theorem 1.1.12. To formulate the argument we should first define Sobolev spaces of differential forms on a compact smooth manifold M. We will do this in a somewhat simple-minded way, by summing over suitable coordinate charts. To avoid various difficulties we will assume that our coordinate charts are taken from an atlas satisfying the following definition.

DEFINITION 1.6.1. If M is a smooth manifold, a *bounded atlas* for M is a **finite** collection $\mathscr{A} = \{\phi_\alpha \colon V_\alpha \to \mathbb{R}^n | \alpha = 1, \dots, N\}$ where:

(i) Each V_α is an open subset of M, with $M = \cup_{\alpha=1}^N V_\alpha$;
(ii) Each $\phi_\alpha \colon V_\alpha \to \mathbb{R}^n$ is a smooth coordinate chart (*i.e.*, a diffeomorphism to its image, which is open in $\mathbb{R}^n$);
(iii) Each closure $\bar{V}_\alpha$ is compact, and for each α there is an open set U_α with $\bar{V}_\alpha \subset U_\alpha$ such that ϕ_α extends to a coordinate chart defined on U_α.

It is easy to see that a smooth manifold M admits a bounded atlas if and only if M is compact.

Notice that if $\mathscr{A} = \{\phi_\alpha \colon V_\alpha \to \mathbb{R}^n\}$ and $\mathscr{B} = \{\psi_\beta \colon W_\beta \to \mathbb{R}^n\}$ are both bounded atlases for M, then each transition function $\psi_\beta \circ \phi_\alpha^{-1} \colon \phi_\alpha(V_\alpha) \to \psi_\beta(W_\beta)$ extends smoothly to a neighborhood of the compact set $\overline{\phi_\alpha(V_\alpha)}$, and hence the derivatives of all orders of $\psi_\beta \circ \phi_\alpha^{-1}$ are bounded (on $\phi_\alpha(V_\alpha)$). Using the chain rule, it then readily follows that for any k, p there is a constant C such that, if $f \in C^\infty(V_\alpha \cap W_\beta)$, then

$$\|f \circ \psi_\beta^{-1}\|_{W^{k,p}(\psi_\beta(W_\beta))} \leq C \|f \circ \phi_\alpha^{-1}\|_{W^{k,p}(\phi_\alpha(V_\alpha))}.$$

By a straightforward approximation argument, this implies that, for any function $f : M \to \mathbb{R}$, we have $f \circ \psi_\beta^{-1} \in W^{k,p}(\psi_\beta(W_\beta))$ if and only if $f \circ \phi_\alpha^{-1} \in W^{k,p}(\phi_\alpha(V_\alpha))$.

In view of this it makes sense to define Sobolev spaces of differential forms in the following way. If $\omega \in \Omega^l(M)$ and if $\mathscr{A} = \{\phi_\alpha \colon V_\alpha \to \mathbb{R}^n : \alpha = 1, \dots, N\}$ is a bounded atlas, then we can write, for each α,

$$\omega|_{V_\alpha} = (\phi_\alpha^{-1})^* \left(\sum_{I=(i_1,\dots,i_l)} f_{I,\alpha} dx_{i_1} \wedge \cdots \wedge dx_{i_l} \right)$$

where each $f_{I,\alpha} \in C^\infty(\phi_\alpha(V_\alpha))$ and the sum is over $I = (i_1, \dots, i_l) \in \{1, \dots, n\}^l$ having $i_1 < \cdots < i_l$. We then define

$$\|\omega\|_{k,p,\mathscr{A}} = \left(\sum_{\alpha=1}^N \sum_I \|f_{I,\alpha}\|^p_{W^{k,p}(\phi_\alpha(V_\alpha))} \right)^{1/p}. \tag{35}$$

If $\mathscr{B} = \{\psi_\beta \colon W_\beta \to \mathbb{R}^n : \beta = 1, \dots N'\}$ is another bounded atlas, and if we write

$$\omega|_{W_\beta} = (\psi_\beta^{-1})^* \left(\sum_{J=(j_1,\dots,j_l)} g_{J,\beta} dx_{j_1} \wedge \cdots \wedge dx_{j_l} \right)$$

then the boundedness of the derivatives of the transition functions $\psi_\beta \circ \phi_\alpha^{-1}$ shows that we have bounds of the form $\|g_{J,\beta}\|_{W^{k,p}(\psi_\beta(U_\alpha \cap V_\beta))} \leq C \sum_{I,\alpha} \|f_{I,\alpha}\|_{W^{k,p}(\phi_\alpha(U_\alpha \cap V_\beta))}$ where C only depends on $\mathscr{A}$ and $\mathscr{B}$. Now since the U_α cover M there is a trivial bound $\|g_{J,\beta}\|_{W^{k,p}(\psi_\beta(V_\beta))} \leq \sum_\alpha \|g_{J,\beta}\|_{W^{k,p}(\psi_\beta(U_\alpha \cap V_\beta))}$, so we have

$$\|g_{J,\beta}\|_{W^{k,p}(\psi_\beta(V_\beta))} \leq C \sum_{I,\alpha} \|f_{I,\alpha}\|_{W^{k,p}(\phi_\alpha(U_\alpha))}.$$

So summing over (the finitely many) β and J proves:

PROPOSITION 1.6.2. *If the norm $\|\cdot\|_{k,p,\mathscr{A}}$ on $\Omega^l(M)$ is defined as in (35), then for any two bounded atlases $\mathscr{A}$, $\mathscr{B}$ there is a constant $C_{\mathscr{A},\mathscr{B}}$ such that $\|\omega\|_{k,p,\mathscr{B}} \leq C_{\mathscr{A},\mathscr{B}} \|\omega\|_{k,p,\mathscr{A}}$ for all $\omega \in \Omega^l(M)$.*

Of course this proposition is symmetric in $\mathscr{A}$ and $\mathscr{B}$, so the norms $\|\cdot\|_{k,p,\mathscr{A}}$ and $\|\cdot\|_{k,p,\mathscr{B}}$ are uniformly equivalent. Accordingly we may define:

DEFINITION 1.6.3. If M is a compact smooth manifold, $k, l \in \mathbb{N}$, and $1 \le p < \infty$, we define $W^{k,p}(\Omega^l(M))$ to be the completion of $\Omega^l(M)$ with respect to the norm $\|\cdot\|_{k,p,\mathscr{A}}$ for any bounded atlas $\mathscr{A}$ on M.

Proposition 1.6.2 shows that the space $W^{k,p}(\Omega^l(M))$ defined in this way is independent of the choice of $\mathscr{A}$ (since uniformly equivalent norms admit the same Cauchy sequences). The norm $\|\cdot\|_{k,p,\mathscr{A}}$ then extends canonically to the completion $W^{k,p}(\Omega^l(M))$; different choices of atlas $\mathscr{A}$ give different (but uniformly equivalent) norms.

More concretely, approximation arguments like those used in Proposition 1.3.8 readily show that an element $\omega \in W^{k,p}(\Omega^l(M))$ can be written in local coordinates with respect to a bounded atlas $\mathscr{A} = \{\phi_\alpha \colon V_\alpha \to \mathbb{R}^n\}$ as

$$\omega|_{V_\alpha} = (\phi_\alpha^{-1})^* \left(\sum_{I=(i_1,\dots,i_l)} f_{I,\alpha} dx_{i_1} \wedge \cdots \wedge i_l \right)$$

where now the functions $f_{I,\alpha}$ just belong to $W^{k,p}(\phi_\alpha(V_\alpha))$; conversely if ω is a "pointwise differential form" (*i.e.* a choice of element of $\Lambda^l T_x^* M$ for each $x \in M$, initially without any condition on smoothness with respect to x) that is locally represented in the above way then ω belongs to $W^{k,p}(\Omega^l(M))$.

With this definition, the exterior derivative $d \colon \Omega^l(M) \to \Omega^{l+1}(M)$ (acting on smooth forms) extends by continuity to an exterior derivative $d \colon W^{k,p}(\Omega^l(M)) \to W^{k-1,p}(\Omega^{l+1}(M))$, and similarly (given a Riemannian metric on M) the Hodge star operator $\star$ from Section 1.1.4 extends to an operator $\star \colon W^{k,p}(\Omega^l(M)) \to W^{k,p}(\Omega^{n-l}(M))$. Combining these gives yields the operator $d^* = (-1)^{n(l-1)+1} \star d \star \colon W^{k,p}(\Omega^l(M)) \to W^{k-1,p}(\Omega^l(M))$, and then the Hodge Laplacian $\Delta = d^*d + dd^*$, now viewed as a map $W^{k,p}(\Omega^l(M)) \to W^{k-2,2}(\Omega^l(M))$. We will now deduce from our main regularity theorem that the (extended) operator Δ has the property that, roughly, if $\Delta\omega \in W^{m,2}(\Omega^l(M))$ then $\omega \in W^{m+2,2}(\Omega^l(M))$; indeed, consistently with Theorem 1.5.7, we will only need to assume that $\omega \in W^{1,2}(\Omega^l(M))$ is a weak solution (to be defined below in Definition 1.6.4) to an equation $\Delta\omega = \theta$ with $\theta \in W^{m,2}(\Omega^l(M))$ to obtain this conclusion. In particular, in the case that $\theta \in \Omega^l(M)$ (*i.e.* that θ is a smooth differential form), then it will follow that $\omega \in \Omega^l(M)$. This reduces the problem of finding solutions in $\Omega^l(M)$ to equations $\Delta\omega = \theta$ to the problem of finding class-$W^{1,2}$ weak solutions to this equation.

For the rest of the section we assume that (M, g) is a compact oriented Riemannian manifold.

Fix a chart $\phi \colon V \to \mathbb{R}^n$ coming from a bounded atlas for M (so that in particular $\bar{V}$ is compact and ϕ extends to a coordinate chart on a neighborhood U of $\bar{V}$). By applying the Gram-Schmidt procedure (with respect to the Riemannian metric g) to the frame of vector fields $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ we obtain a set $\{e_1, \dots, e_n\}$ of vector fields on U which, when evaluated at any $x \in U$, give an oriented orthonormal basis for $T_x U$. Let $\{e^1, \dots, e^n\}$ be the dual basis of one-forms, and for $I = (i_1, \dots, i_l)$ with $i_1 < \cdots < i_l$ write $e^I = e^{i_1} \wedge \cdots \wedge e^{i_l}$. Note that for some smooth matrix-valued function $P \colon U \to GL(n; \mathbb{R})$ we can write $e^j = \sum_{i=1}^n P_{ij} dx_i$. In particular the derivatives of all orders of P and of P^{-1} are bounded on V (since $\bar{V} \subset U$ is compact). By replacing U by a smaller set that still contains $\bar{V}$ we may as well also assume that $\bar{U}$ is compact and that P is defined on a neighborhood of $\bar{U}$, so that the derivatives of all orders of P and of P^{-1} are bounded on U. Consequently the $W^{k,p}$ differential forms on U (resp. on V) are precisely the expressions $\sum_{I=(i_1,\dots,i_l)} f_I e^I$ where each $f_I \in W^{k,p}(U)$ (resp. $f_I \in W^{k,p}(V)$).

Based on Proposition 1.2.1, the Hodge Laplacian is given locally on U by

$$\Delta\left(\sum_I f_I e^I\right) = \sum_I \left(-\sum_{j=1}^n \nabla_{e_j}(\nabla_{e_j} f_I) \right) e^I + \sum_{j=1}^n \sum_{I,J} (\nabla_{e_j} f_I) \beta_{IJ}^i e^J + \sum_{I,J} \gamma_{IJ} f_I e^J.$$

Thus if $\omega, \theta \in \Omega^l(M)$ have $\omega|_U = \sum_I f_I e^I$ and $\theta|_U = \sum_J \theta_J e^J$, by taking the e^J component of the above expression for each J we see that $(\Delta\omega)|_U = \theta|_U$ if and only if, for all $J = (j_1, \ldots, j_l)$ with $j_1 < \cdots < j_l$,

$$\left(-\sum_{j=1}^{n} \nabla_{e_j}(\nabla_{e_j} f_J)\right) + \sum_{i=1}^{n}\sum_{I} \beta^i_{IJ} \nabla_{e_i} f_I + \sum_{I,J} \gamma_{IJ} f_I = \theta_J. \tag{36}$$

Now if we write $e_j = \sum_j Q_{ij} \frac{\partial}{\partial x_i}$ (so Q is the inverse transpose of the matrix P in the second-to-last paragraph) then

$$\begin{aligned} -\sum_{j=1}^{n} \nabla_{e_j}(\nabla_{e_j} f) &= -\sum_{i,j,k} Q_{ij} \frac{\partial}{\partial x_i}\left(Q_{kj} \frac{\partial f_J}{\partial x_k}\right) \\ &= -\sum_{i,k} \frac{\partial}{\partial x_i}\left(\sum_j Q_{ij} Q_{kj} \frac{\partial f_J}{\partial x_k}\right) + \sum_{i,j,k} \frac{\partial Q_{ij}}{\partial x_i} Q_{kj} \frac{\partial f_J}{\partial x_k} \\ &= -\sum_{i,k} \frac{\partial}{\partial x_i}\left((QQ^T)_{ik} \frac{\partial f_J}{\partial x_k}\right) + \sum_{i,j,k} \frac{\partial Q_{ij}}{\partial x_i} Q_{kj} \frac{\partial f_J}{\partial x_k} \end{aligned}$$

Now since at each point $x \in U$, $Q(x)$ is an invertible matrix, it follows that each of the $Q(x)Q(x)^T$ are positive definite, and so (since we have reduced to the case where $\bar{U}$ is compact and Q is defined on a neighborhood of $\bar{U}$, yielding a positive lower bound on the lowest eigenvalue of $Q(x)Q(x)^T$) it follows that the coefficient functions $(QQ^T)_{ik}$ appearing above satisfy the uniform ellipticity requirement in the hypothesis of Theorem 1.5.7.

DEFINITION 1.6.4. If $\theta \in L^2(\Omega^l(M))$ and $\omega \in W^{1,2}(\Omega^l(M))$, we say that ω is a weak solution to $\Delta\omega = \theta$ provided that, for all $\eta \in W^{1,2}(\Omega^l(M))$, we have

$$\langle d\omega, d\eta\rangle + \langle d^*\omega, d^*\eta\rangle = \langle \theta, \eta\rangle.$$

REMARK 1.6.5. If $\omega \in W^{2,2}(\Omega^l(M))$ (so that $\Delta\omega$ is a well-defined element of $L^2(\Omega^l(M))$), then for each $\eta \in W^{1,2}(\Omega^l(M))$ we have

$$\langle d\omega, d\eta\rangle + \langle d^*\omega, d^*\eta\rangle = \langle d^*d\omega, \eta\rangle + \langle dd^*\omega, \eta\rangle = \langle \Delta\omega, \eta\rangle,$$

so ω is a weak solution to $\Delta\omega = \theta$ if and only if $\langle \Delta\omega, \eta\rangle = \langle \theta, \eta\rangle$ for all $\eta \in W^{1,2}(\Omega^l(M))$, which (as one can see by choosing $\eta \in \Omega^l(M)$ to be L^2-close to $\Delta\omega - \theta$ if the latter is nonzero) holds if and only if $\Delta\omega = \theta$. Thus a class-$W^{2,2}$ weak solution to $\Delta\omega = \theta$ is genuinely a solution to this partial differential equation.

THEOREM 1.6.6. *For any $m \geq 0$, let $\theta \in W^{m,2}(\Omega^l(M))$ and let $\omega \in W^{1,2}(\Omega^l(M))$ be a weak solution to $\Delta\omega = \theta$. Then $\omega \in W^{m+2,2}(\Omega^l(M))$ and $\Delta\omega = \theta$. In particular if $\theta \in \Omega^l(M)$ then also $\omega \in \Omega^l(M)$ after possibly redefining ω on a set of measure zero*[12].

PROOF. Let us cover M by the domains of finitely many charts $\phi_\alpha \colon V_\alpha \to \mathbb{R}^n$, each having the property that there are open sets U_α, W_α with $\bar{V}_\alpha \subset U_\alpha \subset \bar{U}_\alpha \subset W_\alpha$ where $\bar{U}_\alpha$ (hence also $\bar{V}_\alpha$) is compact and ϕ_α extends to a coordinate chart (still denoted ϕ_α) defined on all of W_α. For any $j \in \mathbb{N}$, $\omega \in W^{j,2}(\Omega^l(M))$ if and only if, writing $\omega|_{U_\alpha} = \phi^*_\alpha\left(\sum_I f_{I,\alpha} e^I\right)$, we have $f_{I,\alpha} \in W^{j,2}(\phi_\alpha(U_\alpha))$ for each I and each α, which in turn (using the boundedness of the derivatives of the transition functions between the atlases $\{U_\alpha\}$ and $\{V_\alpha\}$) holds if and only if $f_{I,\alpha} \in W^{j,2}(\phi_\alpha(V_\alpha))$ for each I and each α. So it suffices to show inductively that, for each α, if $1 \leq k \leq m+1$ and if each $f_{I,\alpha} \in W^{k,2}(\phi_\alpha(U_\alpha))$,

[12] A set of measure zero in a smooth manifold is by definition a set whose image under each coordinate chart in some atlas has measure zero in $\mathbb{R}^n$.

then each $f_{I,\alpha} \in W^{k+1,2}(\phi_\alpha(V_\alpha))$. We now fix a choice of α and prove this latter statement; for convenience let us delete α from the notation, writing $\phi = \phi_\alpha, U = U_\alpha, V = V_\alpha, f_I = f_{I,\alpha}$.

We may apply the definition of a weak solution to the form η obtained by setting $\eta|_U = \phi^*(he_J)$ where $h \in W_0^{1,2}(\phi(U))$ is arbitrary, and setting η equal to zero outside of U. Now if $\gamma \in W^{2,2}(\Omega^l(M))$ has $\gamma|_U = \phi^*\left(\sum_I \tilde{f}_I e_I\right)$, with this choice of η we have $\langle d\gamma, d\eta\rangle + \langle d^*\gamma, d^*\eta\rangle = \langle \Delta\gamma, \eta\rangle$, and then by using (36) and the computation below it we see that for certain smooth functions $a_{ij}, b_{i,I,J}, c_{IJ} \colon U \to \mathbb{R}$ having all derivatives bounded and with a_{ij} satisfying the uniform ellipticity requirement (24), we have

$$\langle \Delta\gamma, \eta\rangle = \int_{\phi(U)} \left(-\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial \tilde{f}_J}{\partial x_i}\right) + \sum_{i,I} b_{i,I,J}\frac{\partial \tilde{f}_I}{\partial x_i} + \sum_I c_{IJ}\tilde{f}_I\right) h dV,$$

so integrating by parts gives

$$\begin{aligned} &\langle d\gamma, d\eta\rangle + \langle d^*\gamma, d^*\eta\rangle = \langle \Delta\gamma, \eta\rangle \\ &= \int_{\phi(U)} \left(\sum_{i,j} a_{ij}\frac{\partial \tilde{f}_J}{\partial x_i}\frac{\partial h}{\partial x_j} + \sum_{i,I} b_{i,I,J}\frac{\partial \tilde{f}_I}{\partial x_i}h + \sum_I c_{IJ}\tilde{f}_I h\right) dV \end{aligned} \tag{37}$$

While we deduced the above formula under the assumption that $\gamma \in W^{2,2}(\Omega^l(M))$, if we instead just assume that $\gamma \in W^{1,2}(\Omega^l(M))$ it continues to hold because $W^{2,2}(\Omega^l(M))$ is dense in $W^{1,2}(\Omega^l(M))$ and both sides of (37) are continuous with respect to the $W^{1,2}$ topology. Applying this with $\gamma = \omega$ where $\omega \in W^{1,2}(\Omega^l(M))$ is our given weak solution to $\Delta\omega = \theta$, if we write $\theta|_U = \phi^*\left(\sum_I \theta_I e^I\right)$ then since $\langle \theta, \eta\rangle = \int_{\phi(U)} \theta_J h dV$ the fact that $\langle d\omega, d\eta\rangle + \langle d^*\omega, d^*\eta\rangle = \langle \theta, \eta\rangle$ shows that

$$\int_{\phi(U)} \left(\sum_{i,j} a_{ij}\frac{\partial f_J}{\partial x_i}\frac{\partial h}{\partial x_j} + \sum_{i,I} b_{i,I,J}\frac{\partial f_I}{\partial x_i}h + \sum_I c_{IJ} f_I h\right) dV = \int_{\phi(U)} \theta_J h dV.$$

The above holds for an arbitrary choice of multi-index $J = (j_1, \ldots, j_l)$ and an arbitrary $h \in W_0^{1,2}(U)$, so, for each J, f_J is a weak solution to the equation

$$-\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial f_J}{\partial x_i}\right) = \theta_J - \sum_{i,I} b_{i,I,J}\frac{\partial f_I}{\partial x_i} - \sum_I c_{IJ} f_I = \theta_J.$$

The inductive hypothesis that each $f_I \in W^{k,2}(\phi(U))$, together with the fact that $\theta \in W^{m,2}(\Omega^l(M))$ where $m + 1 \geq k$, shows that the right-hand side above belongs to $W^{k-1,2}(\phi(U))$, and so since $\phi(V) \Subset \phi(U)$ we have $f_J \in W^{k+1,2}(\phi(V))$ by Theorem 1.5.7. This holds for all J (and moreover for all charts V as above) completing the proof of the inductive step that if $\omega \in W^{k,2}(\Omega^l(M))$ with $k \leq m+1$ then $\omega \in W^{k+1,2}(\Omega^l(M))$. Thus any class-$W^{1,2}$ weak solution ω to $\Delta\omega = \theta$ belongs to $W^{m+2,2}(\Omega^l(M))$ (and in particular to $W^{2,2}(\Omega^l(M))$).

The additional conclusions in the theorem are easy to prove. Since $\omega \in W^{2,2}(\Omega^l(M))$, $\Delta\omega$ is defined, and Remark 1.6.5 shows that in fact $\Delta\omega = \theta$. If $\theta \in \Omega^l(M)$ (*i.e.*, if θ is a smooth differential form) then $\theta \in W^{m,2}(\Omega^l(M))$ for all m, so by what we have shown we have $\omega \in W^{m+2,2}(\Omega^l(M))$ for all m, and so (after possibly redefining ω on a set of measure zero) $\omega \in \Omega^l(M))$ by Corollary 1.3.22. $\square$

We now begin work on finding weak solutions in the sense of Definition 1.6.4. Define a map $W^{1,2}(\Omega^l(M)) \times W^{1,2}(\Omega^l(M)) \to \mathbb{R}$ by

$$B(\omega, \eta) = \langle d\omega, d\eta\rangle + \langle d^*\omega, d^*\eta\rangle,$$

so that a weak solution ω to $\Delta\omega = \theta$ is precisely an element of $W^{1,2}(\Omega^l(M))$ such that $B(\omega, \eta) = \langle \omega, \eta\rangle$ for all $\eta \in W^{1,2}(\Omega^l(M))$.

Clearly B is a symmetric bilinear form on $W^{1,2}(\Omega^l(M))$. It will be useful to compare this symmetric bilinear form to another one on the same space, namely the inner product $\langle\cdot,\cdot,\rangle_{1,2,\mathscr{A}}$ defined by choosing a bounded atlas $\{\phi_\alpha\colon V_\alpha \to \mathbb{R}^n | \alpha = 1,\ldots,N\}$ and setting

$$\langle\omega,\eta\rangle_{1,2,\mathscr{A}} = \sum_{\alpha,I}\int_{\phi_\alpha(V_\alpha)} \big(f_{I,\alpha}g_{I,\alpha} + (\nabla f_{I,\alpha})\cdot(\nabla g_{I,\alpha})\big)\,dV$$

if ω,η are written locally as

$$\omega|_{V_\alpha} = \phi_\alpha^*\left(\sum_I f_{I,\alpha}dx_{i_1}\wedge\cdots\wedge dx_{i_l}\right) \qquad \eta|_{V_\alpha} = \phi_\alpha^*\left(\sum_I g_{I,\alpha}dx_{i_1}\wedge\cdots\wedge dx_{i_l}\right).$$

So in particular $\langle\omega,\omega\rangle_{1,2,\mathscr{A}} = \|\omega\|^2_{1,2,\mathscr{A}}$ where the latter is defined in (35), and so $(W^{1,2}(\Omega^l(M)),\langle\cdot,\cdot\rangle_{1,2,\mathscr{A}})$ is a Hilbert space. By writing the definition of B out in local coordinates (as is done for instance in the proof of Lemma 1.6.7 below)it is not hard to see that B is a bounded bilinear form on this Hilbert space, in the sense that there is $C_1 > 0$ such that

$$B(\omega,\omega) \le C_1\langle\omega,\omega\rangle_{1,2,\mathscr{A}} \text{ for all } \omega\in W^{1,2}(\Omega^l(M)). \tag{38}$$

In general this inequality cannot be reversed, but the following weaker version of a reversal of (38) will be very important to us:

LEMMA 1.6.7. *There are constants $A_1,A_2 > 0$ such that for all $\omega\in W^{1,2}(\Omega^l(M))$ we have*

$$B(\omega,\omega) \ge A_1\langle\omega,\omega\rangle_{1,2,\mathscr{A}} - A_2\langle\omega,\omega\rangle$$

REMARK 1.6.8. We emphasize that the last term $\langle\omega,\omega\rangle$ is just the standard L^2-norm $\int_M \omega\wedge\star\omega$ that was introduced at the start of these notes; in particular the derivatives of the local coordinate representations of ω do not appear in local coordinate formulas for $\langle\omega,\omega\rangle$, whereas they do of course appear in local coordinate formulas for $\langle\omega,\omega\rangle_{1,2,\mathscr{A}}$.

PROOF. Fix α and fix an extension of the coordinate chart $\phi_\alpha\colon V_\alpha \to \mathbb{R}$ to an open set U_α containing $\bar{V}_\alpha$, and for $I = (i_1,\ldots,i_l)$ let $e^I = e^{i_1}\wedge\cdots\wedge e^{i_l}$ be the l-form constructed from an orthonormal frame on U_α as before. Without loss of generality we assume that the orthonormal frame is compatible with the orientation, so that $\phi_\alpha^*(e^1\wedge\cdots\wedge e^n)$ is a positive top-degree form on U_α. We will sometimes abuse notation and regard I as a set rather than an ordered tuple, so that $i\in I$ means that $i\in\{i_1,\ldots,i_l\}$. Also recall the notation I° for the ordered $(n-l)$-tuple with underlying set $\{1,\ldots,n\}\setminus\{i_1,\ldots,i_l\}$, and recall that $e^I\wedge\star e^I = e^1\wedge\cdots\wedge e^n = \epsilon(I)e^I\wedge e^{I^\circ}$ for an appropriate choice of sign $\epsilon(I)\in\{\pm1\}$, while $e^I * e^J = 0$ for $I\ne J$. These facts readily imply that $\beta\wedge\star\beta$ is equal to a nonnegative function times an oriented volume form on M for any $\beta\in L^2(\Omega^l(M))$; in particular $\int_M\beta\wedge\star\beta \ge \int_U \beta\wedge\star\beta$ for any subset $U\subset M$.

By definition, $B(\omega,\omega) = \langle d\omega,d\omega\rangle + \langle\star d\star\omega,\star d\star\omega\rangle$ (since $d^* = (-1)^{n(l-1)+1}\star d\star$). To estimate the first term, if $\omega|_{V_\alpha} = \phi_\alpha^*\big(f_{I,\alpha}e^I\big)$, we see that

$$\begin{aligned}
\langle d\omega,d\omega\rangle &= \int_M (d\omega)\wedge\star(d\omega) \ge \int_{V_\alpha}(d\omega)\wedge\star(d\omega)\\
&= \int_{\phi_\alpha(V_\alpha)}\left(\sum_I\left(\sum_{i\notin I}(\nabla_{e_i}f_{I,\alpha}e^i\wedge e^I) + f_{I,\alpha}d(e^I)\right)\right)\wedge\star\left(\sum_I\left(\sum_{i\notin I}(\nabla_{e_i}f_{I,\alpha}e^i\wedge e^I) + f_{I,\alpha}d(e^I)\right)\right)dV\\
&\ge \sum_{(i,I):i\notin I}\int_{\phi_\alpha(V_\alpha)}|\nabla_{e_i}f_{I,\alpha}|^2e^1\wedge\cdots\wedge e^n - C_\alpha\sum_{I,J}\|\nabla f_{I,\alpha}\|_{L^2(\phi_\alpha(V_\alpha))}\|\|f_{J,\alpha}\|_{L^2(\phi_\alpha(V_\alpha))} - C'_\alpha\sum_I\|f_{I,\alpha}\|^2_{L^2(\phi_\alpha(V_\alpha))}
\end{aligned}$$

for certain constants C_α, C'_α depending only on the coordinate chart and the basis forms e^I (relating primarily to the various derivatives $d(e^I)$, the coefficient functions of which are bounded since they are smooth and extend to U_α).

As for the other term appearing in $B(\omega,\omega)$, namely $\langle \star d \star \omega, \star d \star \omega\rangle$, first of all note that in general (since $\star\star = (-1)^{q(n-q)}$ acting on q forms) $\star\eta \wedge \star\star\zeta = \zeta \wedge \star\eta$ when ζ and η are forms of complementary degree, so in fact $\langle \star d \star \omega, \star d \star \omega\rangle = \langle d \star \omega, d \star \omega\rangle$. Now

$$\star\omega|_{V_\alpha} = \phi_\alpha^*\left(\sum_I \epsilon(I) f_{I,\alpha} e^{I^\circ}\right) \quad \text{where } \epsilon(I) \in \{\pm 1\},$$

so an identical calculation as above shows that, for certain constants D_α, D'_α, we have

$$\langle d\omega, d\omega\rangle \geq \sum_{(i,I):i\notin I^\circ} \int_{\phi_\alpha(V_\alpha)} |\nabla_{e_i} f_{I,\alpha}|^2 e^1\wedge\cdots\wedge e^n - D_\alpha \sum_{I,J} \|\nabla f_{I,\alpha}\|_{L^2(\phi_\alpha(V_\alpha))} \| \|f_{J,\alpha}\|_{L^2(\phi_\alpha(V_\alpha))} - D'_\alpha \sum_I \|f_{I,\alpha}\|^2_{L^2(\phi_\alpha(V_\alpha))}.$$

If we add the two inequalities that we have just obtained, the sum over $\{(i,I) : i \notin I\}$ combines with the sum over $\{(i,I) : i \notin I^\circ\}$ to just give a sum over all possible i and I. This yields (for any choice of α):

$$B(\omega,\omega) \geq \sum_I \|\nabla f_{I,\alpha}\|^2_{L^2(\phi_\alpha(V_\alpha))} - (C_\alpha + D_\alpha)\sum_{I,J} \|\nabla f_{I,\alpha}\|_{L^2(\phi_\alpha(V_\alpha))} \| \|f_{J,\alpha}\|_{L^2(\phi_\alpha(V_\alpha))} - (C'_\alpha + D'_\alpha)\sum_I \|f_{I,\alpha}\|^2_{L^2(\phi_\alpha(V_\alpha))}.$$

We can now use the same trick that we used throughout the proof of Theorem 1.5.4 to observe that, for each I

$$\sum_J \|\nabla f_{I,\alpha}\|_{L^2(\phi_\alpha(V_\alpha))} \| \|f_{J,\alpha}\|_{L^2(\phi_\alpha(V_\alpha))} \leq \frac{1}{2(C'_\alpha + D'_\alpha)} \|\nabla f_{I,\alpha}\|^2_{L^2(\phi_\alpha(V_\alpha))} + \frac{C'_\alpha + D'_\alpha}{2}\left(\sum_J \|f_{J,\alpha}\|_{L^2(\phi_\alpha(V_\alpha))}\right)^2.$$

It then follows that, for some constant A_α,

$$B(\omega,\omega) \geq \frac{1}{2}\sum_I \|\nabla f_{I,\alpha}\|^2_{L^2(V_\alpha)} - A_\alpha \sum_I \|f_{I,\alpha}\|^2_{L^2(V_\alpha)}. \tag{39}$$

This holds for each of our finitely many charts $\phi_\alpha \colon V_\alpha \to \mathbb{R}^n$. Continuing to denote the total number of charts in our atlas by N, we have, for some constant C depending on the transition functions relating the e^I to the $dx_{i_1} \wedge \cdots \wedge dx_{i_l}$,

$$\begin{aligned} \langle \omega,\omega\rangle_{1,2,\mathscr{A}} &\leq C\sum_{\alpha,I}\left(\|\nabla f_{I,\alpha}\|^2_{L^2(\phi_\alpha(V_\alpha))} + \|f_\alpha\|^2_{L^2(\phi_\alpha(V_\alpha))}\right) \\ &\leq 2NCB(\omega,\omega) + \sum_{\alpha,I}(C + 2A_\alpha)\|f_\alpha\|^2_{L^2(\phi_\alpha(V_\alpha))} \end{aligned}$$

where the second inequality follows by rearranging (39) and summing over α. The last term on the right is bounded above by a constant times the standard L^2 norm $\langle\omega,\omega\rangle$, and so rearranging the above inequality proves the result. □

COROLLARY 1.6.9. *If $\lambda \geq A_2$ where A_2 is the constant from Lemma 1.6.7, then the map $B_\lambda \colon W^{1,2}(\Omega^l(M)) \times W^{1,2}(\Omega^l(M)) \to \mathbb{R}$ defined by*

$$B_\lambda(\omega,\eta) = B(\omega,\eta) + \lambda\langle\omega,\eta\rangle$$

is an inner product on $W^{1,2}(\Omega^l(M))$, such that for some constant $L > 0$ we have

$$L^{-2}\langle\omega,\omega\rangle_{1,2,\mathscr{A}} \leq B_\lambda(\omega,\omega) \leq L^2\langle\omega,\omega\rangle_{1,2,\mathscr{A}}.$$

In particular B_λ induces on $W^{1,2}(\Omega^l(M))$ a Hilbert space structure with the property that, for $\omega_m, \omega \in W^{1,2}(\Omega^l(M))$ we have $\omega_m \to \omega$ with respect to B_λ if and only if $\omega_m \to \omega$ with respect to $\langle\cdot,\cdot\rangle_{1,2,\mathscr{A}}$.

PROOF. Clearly B_λ is symmetric and bilinear. Combining (38) and Lemma 1.6.7 we have uniform estimates

$$A_1\langle\omega,\omega\rangle_{1,2,\mathscr{A}} \leq B_\lambda(\omega,\omega) \leq (C_1+\lambda)\langle\omega,\omega\rangle_{1,2,\mathscr{A}}.$$

and in particular $B_\lambda(\omega,\omega) \geq 0$ with equality only if $\omega = 0$ and the desired estimate holds with $L^2 = \max\{A_1^{-1}, C_1+\lambda\}$. The bounds imply that a sequence is Cauchy (resp. convergent) with respect with respect to B_λ if and only if it is Cauchy (resp. convergent) with respect to $\langle\cdot,\cdot\rangle_{1,2,\mathscr{A}}$, so since $\langle\cdot,\cdot\rangle_{1,2,\mathscr{A}}$ makes $W^{1,2}(\Omega^l(M))$ into a Hilbert space the result follows immediately. $\square$

COROLLARY 1.6.10. *Let $\lambda = A_2$ where A_2 is the constant from Lemma 1.6.7. Then for each $\theta \in L^2(\Omega^l(M))$ there is a unique $\omega_\theta \in W^{1,2}(\Omega^l(M))$ such that $B_\lambda(\omega_\theta,\eta) = \langle\theta,\eta\rangle$ for all $\eta \in W^{1,2}(\Omega^l(M))$. Moreover, for some constant C independent of θ, we have a uniform estimate*

$$\|\omega_\theta\|_{1,2,\mathscr{A}} \leq C\|\theta\|_{L^2(\Omega^l(M))}. \tag{40}$$

PROOF. The Schwarz inequality readily implies an estimate $\langle\theta,\eta\rangle \leq C_0\|\theta\|_{0,2,\mathscr{A}}\|\eta\|_{0,2,\mathscr{A}}$, so since $\|\cdot\|_{0,2,\mathscr{A}} \leq \|\cdot\|_{1,2,\mathscr{A}}$ we see that $\langle\theta,\cdot\rangle$ is a bounded linear functional on the Hilbert space $(W^{1,2}(\Omega^l(M)), B_\lambda)$ having norm at most $LC_0\|\theta\|_{0,2,\mathscr{A}}$ where L is the constant from the previous corollary. So the Riesz Representation Theorem 1.4.3 shows that there is $\omega_\theta \in W^{1,2}(\Omega^l(M))$ such that $B_\lambda(\omega,\cdot) = \langle\theta,\cdot\rangle$. Clearly there can only be one such ω_θ, since for any other choice ω'_θ we would have $B_\lambda(\omega_\theta - \omega'_\theta,\cdot) = 0$, forcing $\omega_\theta - \omega'_\theta = 0$ since B_λ is an inner product. Also,

$$L^{-1}\|\omega_\theta\|^2_{1,2,\mathscr{A}} \leq B_\lambda(\omega_\theta,\omega_\theta) = \langle\theta,\omega_\theta\rangle \leq LC_0\|\theta\|_{0,2,\mathscr{A}}\|\|\omega_\theta\|_{1,2,\mathscr{A}},$$

so (40) holds with $C = L^2C_0$. $\square$

In view of the uniqueness of ω_θ, the assignment $\theta \mapsto \omega_\theta$ is clearly linear since $B_\lambda(c\omega_{\theta_1} + \omega_{\theta_2},\cdot) = \langle c\theta_1+\theta_2,\cdot\rangle$. Let us now define a map

$$\begin{aligned} K\colon L^2(\Omega^l(M)) &\to L^2(\Omega^l(M)) \\ \theta &\mapsto \lambda\omega_\theta. \end{aligned}$$

The proof of Theorem 1.1.12 will rest on several properties of this map. Note that we treat the codomain of K as being $L^2(\Omega^l(M))$ even though by construction we always have $K\theta \in W^{1,2}(\Omega^l(M))$.

PROPOSITION 1.6.11. *The map $K\colon L^2(\Omega^l(M)) \to L^2(\Omega^l(M))$ is a* ***compact*** *operator in the sense that if $\{\theta_m\}_{m=1}^\infty$ is a sequence satisfying $\|\theta_m\|_{0,2,\mathscr{A}} \leq C$ (where C is independent of m) then $\{K\theta_m\}_{m=1}^\infty$ has a subsequence which converges in $L^2(\Omega^l(M))$.*

PROOF. By construction and (40) the hypothesis implies that for each m we have $K\theta_m \in W^{1,2}(\Omega^l(M))$ with $\|K\theta_m\| \leq LC_0C\lambda$. If we fix a partition of unity $\{\chi_\alpha\}$ subordinate to the cover $\{V_\alpha\}$ given by our finite atlas $\mathscr{A}$, this implies a uniform upper bound on the $W^{1,2}$ norms of the coefficient functions appearing in the local coordinate representations of the various forms $\chi_\alpha K\theta_m$; hence Theorem 1.3.23 implies that, for some subsequence $\{\theta_{m_j}\}_{j=1}^\infty$, each of the $\{\chi_\alpha K\theta_{m_j}\}_{j=1}^\infty$ converge in L^2, and hence the $K\theta_{m_j} = \sum_\alpha \chi_\alpha K\theta_{m_j}$ converge in L^2. $\square$

PROPOSITION 1.6.12. *The map $K\colon L^2(\Omega^l(M)) \to L^2(\Omega^l(M))$ is* ***self-adjoint*** *in the sense that, for all $\theta,\phi \in L^2(\Omega^l(M))$, we have*

$$\langle\theta,K\phi\rangle = \langle K\theta,\phi\rangle.$$

PROOF. We see that, by the definition of K and the fact that B_λ and $\langle\cdot,\cdot\rangle$ are inner products,

$$\begin{aligned} \langle\theta,K\phi\rangle &= B_\lambda(\lambda^{-1}K\theta,K\phi) = B_\lambda(\lambda^{-1}K\phi,K\theta) \\ &= \langle\phi,K\theta\rangle = \langle K\theta,\phi\rangle. \end{aligned}$$

$\square$

We now connect the operator K to our differential equation $\Delta\omega = \theta$. By definition, for any $\theta, \eta \in L^2(\Omega^l(M))$, we have

$$\begin{aligned}\langle\theta,\eta\rangle &= B_\lambda(\lambda^{-1}K\theta,\eta)\\ &= \lambda^{-1}\left(\langle dK\theta, d\eta\rangle + \langle d^*K\theta, d^*\eta\rangle\right) + \langle K\theta,\eta\rangle.\end{aligned}$$

Rearranging this shows that, for any $\theta \in L^2(\Omega^l(M))$,

$$K\theta \text{ is a weak solution to } \Delta(K\theta) = \lambda(\theta - K\theta). \tag{41}$$

In particular it follows from this (and Theorem 1.6.6) that we in fact have $K\theta \in W^{2,2}(\Omega^l(M))$ for all $\theta \in L^2(M)$.

Let us now consider the operator $I - K\colon\ L^2(\Omega^l(M)) \to L^2(\Omega^l(M))$, where I is the identity. It is an immediate consequence of (41) that

$$\mathrm{Im}(I-K) \subset \mathrm{Im}(\Delta\colon\ W^{2,2}(\Omega^l(M)) \to L^2(\Omega^k(M))).$$

As for the kernel of $I-K$, if $\omega \in \ker(I-K)$, *i.e.* if $K\omega = \omega$, then automatically $\omega \in W^{2,2}(\Omega^l(M))$, and by definition

$$\langle\omega,\cdot\rangle = B_\lambda(\lambda^{-1}\omega,\cdot) = \lambda^{-1}\langle\Delta\omega,\cdot\rangle + \langle\omega,\cdot\rangle,$$

so that $\Delta\omega = 0$. Conversely if $\Delta\omega = 0$ then $B_\lambda(\lambda^{-1}\omega,\cdot) = \langle\omega,\cdot\rangle$ so $K\omega = \omega$. Thus

$$\ker(I-K) = \ker(\Delta\colon\ W^{2,2}(\Omega^l(M)) \to L^2(\Omega^l(M))) = \mathscr{H}^l(M)$$

where $\mathscr{H}^l(M)$ is as defined in (11) and where we have used Theorem 1.6.6 to deduce that any class-$W^{2,2}$ element of $\ker(\Delta)$ is in fact smooth and hence belongs to $\mathscr{H}^l(M)$.

Theorem 1.1.12 now follows from what we have done together with the following general proposition:

PROPOSITION 1.6.13. *Let H be a Hilbert space, let $K\colon\ H \to H$ be a self-adjoint compact operator, and let I be the identity. Then* $\ker(I-K)$ *is finite-dimensional, and* $\mathrm{Im}(I-K) = \ker(I-K)^\perp$.

In fact we'll prove this proposition based on a still more general lemma, formulated for later use in other contexts:

LEMMA 1.6.14. *Let H_0, H_1, H_2 be three Hilbert spaces with associated norms $\|\cdot\|_0, \|\cdot\|_1, \|\cdot\|_2$, and suppose that $D\colon\ H_0 \to H_1$ and $K\colon\ H_0 \to H_2$ are bounded linear maps such that K is compact and such that there is $C > 0$ such that for all $x \in H_0$ we have*

$$\|x\|_0 \le C(\|Dx\|_1 + \|Kx\|_2). \tag{42}$$

Then $\mathrm{Im}(D)$ *is closed, and* $\ker(D)$ *is finite-dimenisonal.*

PROOF OF PROPOSITION 1.6.13, ASSUMING LEMMA 1.6.14. Let $H_0 = H_1 = H_2 = H$ and $D = I-K$. Then (42) holds with $C = 1$ by the triangle inequality. So Lemma 1.6.14 asserts that $\ker(I-K)$ is finite-dimensional and that $\mathrm{Im}(I-K)$ is closed. Now since K is self-adjoint we have, for any $x, y \in H$

$$\langle(I-K)x, y\rangle = \langle x, (I-K)y\rangle.$$

So if $z = (I-K)y$ and $x \in \ker(I-K)$ then $\langle x, z\rangle = \langle(I-K)x, y\rangle = 0$, proving that $\mathrm{Im}(I-K) \subset \ker(I-K)^\perp$. To prove the reverse inclusion, first note that if $x \in \mathrm{Im}(I-K)^\perp$ then the above equation shows that $\langle(I-K)x, y\rangle = 0$ for all $y \in H$ and hence that $x \in \ker(I-K)$. Thus $\mathrm{Im}(I-K)^\perp \subset \ker(I-K)$, from which it directly follows that $\ker(I-K)^\perp \subset (\mathrm{Im}(I-K)^\perp)^\perp$. But because $\mathrm{Im}(I-K)$ is closed, we have $(\mathrm{Im}(I-K)^\perp)^\perp = \mathrm{Im}(I-K)$ by Corollary 1.4.2. So $\ker(I-K)^\perp \subset \mathrm{Im}(I-K)$. □

PROOF OF LEMMA 1.6.14. First we prove that $\ker(D)$ is finite-dimensional. If this were not the case, then we could find an infinite orthonormal sequence $\{x_m\}_{m=1}^{\infty}$ in $\ker(D) \subset H_0$. Since K is a compact operator and each $\|x_m\|_0 = 1$, it follows that there is a subsequence $\{x_{m_j}\}_{j=1}^{\infty}$ such that $\{Kx_{m_j}\}_{j=1}^{\infty}$ is Cauchy. But since each $x_m \in \ker(D)$, (42) says that, for each j_1, j_2, $\|x_{m_{j_1}} - x_{m_{j_2}}\|_0 \le C\|Kx_{m_{j_1}} - Kx_{m_{j_2}}\|_2$, so $\{x_{m_j}\}_{j=1}^{\infty}$ is Cauchy. But this is nonsense since the fact that the x_m are orthonormal shows that $\|x_{m_{j_1}} - x_{m_{j_2}}\|_0 = \sqrt{2}$ for $j_1 \neq j_2$. So indeed $\ker(D)$ must be finite-dimensional.

To prove that $\mathrm{Im}(D)$ is closed, let $y_m \in \mathrm{Im}(D)$ with $y_m \to y \in H_1$. We may write $y_m = Dx_m$ where $x_m \in \ker(D)^{\perp}$ by projecting an initial choice of preimage of y_m orthogonally to $\ker(D)$. If the norms of the x_m are bounded, then some subsequence $\{x_{m_j}\}_{j=1}^{\infty}$ has Kx_{m_j} convergent, and then since also $Dx_{m_j} \to y$ it follows from (42) that $\{x_{m_j}\}_{j=1}^{\infty}$ is a Cauchy sequence, converging say to $x \in H$. But then $Dx = \lim_{j\to\infty} Dx_{m_j} = \lim_{j\to\infty} y_{m_j} = y$, proving that $y \in \mathrm{Im}(D)$ if the norms of the x_m are bounded. On the other hand if the norms of the x_m are unbounded then we can find a subsequence $\{x_{m_j}\}_{j=1}^{\infty}$ such that $\|x_{m_j}\| \to \infty$ and hence $D\left(\frac{x_{m_j}}{\|x_{m_j}\|}\right) \to 0$. Using the compactness of K, a further subsequence of the $\frac{x_{m_j}}{\|x_{m_j}\|}$ would then converge to an element $z \in \ker(D)$, necessarily with norm 1. But since each $x_{m_j} \in \ker(D)^{\perp}$ we will also have $z \in \ker(D)^{\perp}$, contradicting the obvious fact that $\ker(D) \cap \ker(D)^{\perp}$ is trivial. Thus $\mathrm{Im}(D)$ is closed. $\square$

We have thus shown that

$$\mathscr{H}^l(M) = \ker(I-K) \text{ is finite-dimensional}$$

and that

$$\ker(I-K)^{\perp} = \mathrm{Im}(I-K) \subset \mathrm{Im}(\Delta\colon W^{2,2}(\Omega^l(M)) \to L^2(\Omega^l(M)).$$

The Hodge theorem asserts that $\mathscr{H}^l(M)$ is finite-dimensional, and that the orthogonal complement $\mathscr{H}^l(M)^{\perp}$ of $\mathscr{H}^l(M)$ *within the space of smooth forms* is equal to the image of the operator Δ acting on smooth forms. Now $\mathscr{H}^l(M)^{\perp} = \ker(I-K)^{\perp} \cap \Omega^l(M)$, so the above shows that $\mathscr{H}^l(M)^{\perp} \subset \mathrm{Im}(\Delta\colon W^{2,2}(\Omega^l(M)) \to L^2(\Omega^l(M))$. But Theorem 1.6.6 shows that if $\Delta\omega \in \Omega^l(M)$ then $\omega \in \Omega^l(M)$, so in fact $\mathscr{H}^l(M)^{\perp} \subset \mathrm{Im}(\Delta|_{\Omega^l(M)})$. The reverse inclusion is trivial (as already mentioned in Remark 1.1.13) so the proof of the Hodge Theorem is complete.

REMARK 1.6.15. In fact, Lemma 1.6.14 continues to hold if we just assume that H_0, H_1, H_2 are Banach spaces rather than Hilbert spaces, as will be useful later in the proof of Theorem 3.4.3. The proof of this is a slight modification of the one given above; the point is that the proof goes through if we replace references to orthogonal complements and orthogonal projections by appeals to the following fact: if B is a Banach space and if $A \le B$ is a closed subspace, then for any $x \in B \setminus A$ there is $a_x \in A$ such that $\|x - a_x\| > \frac{1}{2}\|x-a\|$ for all $a \in A$. It is easy to prove this fact: just use that since A is closed the quantity $\inf\{\|x-a\| | a \in A\}$ is positive.

Given this fact, it's easy to see that any infinite-dimensional subspace V of a Banach space B must admit a sequence $\{v_m\}_{m=1}^{\infty}$ with $\|v_m\| = 1$ and $\|v_m - v_n\| \ge 1/2$ for all m, n: inductively, having chosen $v_1, \ldots, v_n \in V$, apply the previous paragraph to the subspace A spanned by $v_1, \ldots, v_n$ and to some choice of $x \in V \setminus V_n$, and let v_{n+1} be an appropriate rescaling of $x - a_x$. Hence in particular an estimate (42) still implies the finite-dimensionality of $\ker(D)$ when the H_i are just Banach spaces, since the compactness of K shows that the unit ball in $\ker(D)$ is sequentially compact.

To prove that D has closed range, we follow the proof in the Hilbert space case almost word-for-word except that, instead of taking the x_m to be in $\ker(D)^{\perp}$ (which in the Banach space case has no meaning) we take them to have the property that $\|x_m\|_0 > \frac{1}{2}\|x_m - a\|_0$ for all $a \in \ker(D)$. Just as before, if the x_m are bounded then they converge to a preimage of y, while if the $\|x_m\|_0$ are unbounded then, after passing to a subsequence, the elements $z_m = \frac{z_m}{\|z_m\|_0}$ have $Dz_m \to 0$ and Kz_m

Cauchy, whence $z_m \to z$ with $Dz = 0$, whereas the condition on the x_m forces $\|z_m - a\|_0 > 1/2$ for all $a \in \ker(D)$, a contradiction.

CHAPTER 2

Hodge theory on complex manifolds

2.1. The Hodge theorem with complex coefficients

The goal in this chapter will be to establish a different version of the Hodge theorem on complex manifolds; in the case of a Kähler manifold (a complex manifold equipped with a Riemannian metric that is compatible with the complex structure in a certain sense) this will have a simple relation to Theorem 1.1.10 and will lead to interesting topological consequences. In Hodge theory on complex manifolds one works with differential forms with complex coefficients, so in this section we will introduce the linear algebra needed to do this and show that our earlier formulations of the Hodge theorem (Theorems 1.1.10 and 1.1.12) extend trivially to complex-coefficient differential forms. (In this section our manifold will not be assumed to be complex.)

The starting point is the following definition:

DEFINITION 2.1.1. If V is a vector space over $\mathbb{R}$, the *complexification* of V is the complex vector space

$$V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = \{v_1 + iv_2 | v_1, v_2 \in \mathbb{R}\}.$$

Of course, the scalar multiplication for the complex vector space structure on $V_{\mathbb{C}}$ is given by $(x+iy)(v_1+iv_2) = (xv_1-yv_2)+i(xv_2+yv_1)$. An obvious but important point is that complexification can be seen as a *functor* from the category of real vector spaces to the category of complex vector spaces: not only do we obtain a complex vector space $V_{\mathbb{C}}$ for every real vector space V, but to each $\mathbb{R}$-linear map $f: V \to W$ we obtain an induced $\mathbb{C}$-linear map $f_{\mathbb{C}}: V_{\mathbb{C}} \to W_{\mathbb{C}}$ defined by $f_{\mathbb{C}}(v_1+iv_2) = f(v_1) + if(v_2)$, and one has identities $(f \circ g)_{\mathbb{C}} = f_{\mathbb{C}} \circ g_{\mathbb{C}}$ and $I_{\mathbb{C}} = I$. Usually I will abuse notation and just write $f_{\mathbb{C}}$ as f unless the redundant notation is likely to cause confusion.

The complexification $V_{\mathbb{C}}$ of a real vector space V comes with a $\mathbb{R}$-linear conjugation map $v \mapsto \bar{v}$ defined by $\overline{v_1 + iv_2} = v_1 - iv_2$ for $v_1, v_2 \in V$.

If $(\cdot,\cdot): V \times V \to \mathbb{R}$ is an inner product on the real vector space V, there is a canonically-determined (Hermitian) inner product $\langle\cdot,\cdot\rangle: V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ on $V_{\mathbb{C}}$ defined by

$$\langle v_1 + iv_2, w_1 + iw_2\rangle = ((v_1, v_2) + (w_1, w_2)) + i((v_2, w_1) - (v_1, w_2)) \tag{43}$$

(our convention is that a Hermitian inner product is complex-linear in its first argument and conjugate-linear in its second argument). Trivially, if $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $(V, (\cdot,\cdot))$ then $\{e_1, \ldots, e_n\}$ is also an orthonormal basis for $\big(V_{\mathbb{C}}, \langle\cdot,\cdot\rangle\big)$. We now make the following simple observation:

PROPOSITION 2.1.2. *Suppose that $(V, (\cdot,\cdot)_V)$ and $(W, (\cdot,\cdot)_W)$ are real inner product spaces, and that $A: V \to W$ and $A^*: W \to V$ are linear maps obeying the identity $(Av, w)_W = (v, A^*w)_V$ for all $v \in V, w \in W$. Then where $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$ the Hermitian inner products on $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ constructed using (43), we likewise have $\langle A_{\mathbb{C}}v, w\rangle_W = \langle v, (A^*)_{\mathbb{C}}w\rangle_V$ for all $v \in V_{\mathbb{C}}, w \in W_{\mathbb{C}}$.*

(In other words, the complexification of an adjoint to a real-linear operator is a Hermitian adjoint to the complexification of that operator.)

PROOF. For $v = v_1 + iv_2 \in V_\mathbb{C}$ and $w = w_1 + iw_2 \in W_\mathbb{C}$ we have:

$$\begin{aligned}\langle A_\mathbb{C} v, w\rangle_W &= \langle Av_1 + iAv_2, w_1 + iw_2\rangle_W = (Av_1, w_1)_W + (Av_2, w_2)_W + i(Av_2, w_1)_W - i(Av_1, w_2)_W \\ &= (v_1, A^* w_1)_V + i(v_2, A^* w_1)_V + (v_2, A^* w_2)_V - i(v_1, A^* w_2)_V = \langle v_1 + iv_2, A^* w_1\rangle_W - i\langle v_1 + iv_2, A^* w_2\rangle_V \\ &= \langle v_1 + iv_2, A^*(w_1 + iw_2)\rangle_V = \langle v, (A^*)_\mathbb{C} w\rangle_V.\end{aligned}$$

□

2.1.1. Complexifying the Hodge theorem. Now let us assume that M is a compact n-dimensional oriented Riemannian manifold. We have a Hodge star operator $\star\colon \Omega^k(M) \to \Omega^{n-k}(M)$, which then complexifies to give an operator $\star = \star_\mathbb{C}\colon \Omega^k(M)_\mathbb{C} \to \Omega^{n-k}(M)_\mathbb{C}$. In (9, we defined an inner product $(\cdot,\cdot)$ on $\Omega^k(M)$ by $(\alpha, \beta) = \int_M \alpha \wedge \star\beta$. We can then as usual use (43) to obtain a Hermitian inner product $\langle\cdot,\cdot\rangle$ on $\Omega^k(M)_\mathbb{C}$ by, for $\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2 \in \Omega^k(M)_\mathbb{C}$,

$$\begin{aligned}\langle\alpha, \beta\rangle &= (\alpha_1, \beta_1) + i(\alpha_2, \beta_1) - i(\alpha_1, \beta_2) + (\alpha_2, \beta_2) \\ &= \int_M \alpha_1 \wedge \star\beta_1 + i\int_M \alpha_2 \wedge \star\beta_1 - i\int_M \alpha_1 \wedge \star\beta_2 + \int_M \alpha_2 \wedge \star\beta_2 \\ &= \int_M (\alpha_1 + i\alpha_2) \wedge \star\beta_1 + \int_M (\alpha_1 + i\alpha_2) \wedge \star(-i\beta_2) = \int_M (\alpha_1 + i\alpha_2) \wedge \star(\beta_1 - i\beta_2) \\ &= \int_M \alpha \wedge \star\bar{\beta}. \qquad (44)\end{aligned}$$

Now we have en exterior derivative $d\colon \Omega^{k-1}(M) \to \Omega^k(M)$, with adjoint $d^* = (-1)^{n(k-1)+1} \star d \star\colon \Omega^k(M) \to \Omega^{k-1}(M)$. Proposition 2.1.2 then shows that the complexified map $d_\mathbb{C}\colon \Omega^{k-1}(M)_\mathbb{C} \to \Omega^k(M)_\mathbb{C}$ has adjoint (with respect to (44)) given by the complexification $d^*_\mathbb{C}$ of d^*. Trivially, a complexified form $\omega = \omega_1 + i\omega_2 \in \Omega^k(M)_\mathbb{C}$ obeys $d_\mathbb{C}\omega = 0$ if and only if $d\omega_1 = d\omega_2 = 0$, so $\ker(d_\mathbb{C}) = \ker(d)_\mathbb{C}$ and likewise $\ker(d^*_\mathbb{C}) = \ker(d^*)_\mathbb{C}$. Similarly $\mathrm{Im}(d_\mathbb{C}) = \mathrm{Im}(d)_\mathbb{C}$ since $d_\mathbb{C}$ is just given by acting on real and imaginary parts separately by d. In view of this the *complex de Rham cohomology*

$$H^k(M;\mathbb{C}) = \frac{\ker(d_\mathbb{C}\colon \Omega^k(M)_\mathbb{C} \to \Omega^{k+1}(M)_\mathbb{C})}{\mathrm{Im}(d_\mathbb{C}\colon \Omega^{k-1}(M)_\mathbb{C} \to \Omega^k(M)_\mathbb{C})}$$

is naturally identified with the complexification $H^k(M)_\mathbb{C}$ of the original de Rham cohomology $H^k(M)$: indeed the fact that $\ker(d_\mathbb{C}) = \ker(d)_\mathbb{C}$ implies that $\omega_1 + i\omega_2 \mapsto [\omega_1] + i[\omega_2]$ is a well-defined and surjective map $\ker(d_\mathbb{C}) \to H^k(M)_\mathbb{C}$, and the fact that $\mathrm{Im}(d_\mathbb{C}) = \mathrm{Im}(d)_\mathbb{C}$ implies that this map has kernel exactly equal to $\mathrm{Im}(d_\mathbb{C})$.

We can then form the complexified Hodge Laplacian $\Delta_\mathbb{C} = d_\mathbb{C} d^*_\mathbb{C} + d^*_\mathbb{C} d_\mathbb{C}$, and the fact that $d^*_\mathbb{C}$ is adjoint to $d_\mathbb{C}$ shows that, for any $\omega \in \Omega^k(M)_\mathbb{C}$,

$$\langle\Delta_\mathbb{C}\omega, \omega\rangle = \langle d_\mathbb{C}\omega, d_\mathbb{C}\omega\rangle + \langle d^*_\mathbb{C}\omega, d^*_\mathbb{C}\omega\rangle$$

and so just as in the real case we have $\Delta_\mathbb{C}\omega = 0$ if and only if $d_\mathbb{C}\omega = d^*_\mathbb{C}\omega = 0$. Evidently $\ker(\Delta_\mathbb{C}) = \ker(\Delta)_\mathbb{C} = \mathscr{H}^k(M)_\mathbb{C}$. The Hodge Theorem 1.1.10 shows that the map $\mathscr{H}^k(M) \to H^k(M)$ given by $\omega \mapsto [\omega]$ is an isomorphism, so it follows immediately that:

PROPOSITION 2.1.3. *If M is a compact oriented Riemannian manifold and $\Delta = d^*d + dd^*$ is the Hodge Laplacian, we have isomorphisms*

$$\ker(\Delta_\mathbb{C}\colon \Omega^k(M)_\mathbb{C} \to \Omega^k(M)_\mathbb{C}) \cong H^k(M;\mathbb{C}) \cong H^k(M)_\mathbb{C}$$

defined respectively by $\omega_1 + i\omega_2 \mapsto [\omega_1 + i\omega_2]$ and $[\omega_1 + i\omega_2] \mapsto [\omega_1] + i[\omega_2]$.

2.1.2. More about the complexified Hodge star operator. We now return to the context of linear algebra. If $(V, (\cdot,\cdot), \mathfrak{o})$ is an oriented n-dimensional inner product space over $\mathbb{R}$, we have constructed the inner product $\langle\cdot,\cdot\rangle$ on $V_{\mathbb{C}}$ via (43), and we have a complexified Hodge star operator $\star\colon (\Lambda^k V^*)_{\mathbb{C}} \to (\Lambda^{n-k}V^*)_{\mathbb{C}}$. The operator $\star$ was designed to have the property that if $\{e_1,\dots,e_n\}$ is any oriented basis (over $\mathbb{R}$) for V then $\star(e^1 \wedge \cdots e^k) = e^{k+1}\wedge\cdots\wedge e^n$. Now if $\{e_1,\dots,e_n\}$ is a basis over $\mathbb{R}$ for V then it is also a basis over $\mathbb{C}$ for $V_{\mathbb{C}}$. It will soon be useful to us to consider the action of $\star$ on orthonormal bases for $V_{\mathbb{C}}$ whose elements do not necessarily belong to V (*i.e.* which might have nontrivial "imaginary part"), and we will work out how to do this presently. (One issue here is that we do not immediately have a notion of what it means for such a basis to be "oriented.")

First let us consider the domain $(\Lambda^k V^*)_{\mathbb{C}}$ of the complexified Hodge star operator $\star$. By definition, $\Lambda^k V^*$ consists of alternating k-linear maps $\eta\colon V^k \to \mathbb{R}$, while $(\Lambda^k V^*)_{\mathbb{C}}$ consists of elements $\eta_1 + i\eta_2$ where $\eta_1,\eta_2 \in \Lambda^k V^*$; equivalently $(\Lambda^k V^*)_{\mathbb{C}}$ consists of k-linear maps $\eta = \eta_1 + i\eta_2\colon V^k \to \mathbb{C}$. (Here of course "$k$-linear" in this context means "k-linear over $\mathbb{R}$" since we are only assuming V to be a real vector space.)

One could also consider the space $\Lambda^k_{\mathbb{C}} V^*_{\mathbb{C}}$, consisting of alternating *complex*-k-linear maps $\eta\colon (V_{\mathbb{C}})^k \to \mathbb{C}$. This is of course a complex vector space, and since any such η is uniquely determined as soon as one gives arbitrary values to $\eta(e_{j_1},\dots,e_{j_k})$ for a complex basis $\{e_1,\dots,e_n\}$ for $V_{\mathbb{C}}$, as $(j_1,\dots,j_k)$ varies over increasing k-tuples in $\{1,\dots,n\}$, we have $\dim_{\mathbb{C}} \Lambda^k_{\mathbb{C}} V^*_{\mathbb{C}} = \binom{n}{k}$. Likewise, $\dim_{\mathbb{C}}(\Lambda^k V^*)_{\mathbb{C}} = \dim_{\mathbb{R}} \Lambda^k V^* = \binom{n}{k}$. In fact these two spaces are naturally isomorphic: since $V \subset V_{\mathbb{C}}$, we have a restriction map

$$\begin{aligned} r\colon \Lambda^k_{\mathbb{C}} V^*_{\mathbb{C}} &\to (\Lambda^k V^*)_{\mathbb{C}} \\ \eta &\mapsto \eta|_{V^k}. \end{aligned}$$

If $\eta \in \Lambda^k_{\mathbb{C}} V^*_{\mathbb{C}}$ has $\eta|_{V^k} = 0$ then the complex-k-linearity of η readily implies that $\eta(v_1 + iw_1,\dots,v_k + iw_k) = 0$ for all $v_j, w_j \in V$ and hence that $\eta = 0$. Thus the restriction map r is an injective linear map, and hence a linear isomorphism since its domain and codomain have the same dimension. The inverse is given by extending an arbitrary alternating k-linear $\eta\colon V^k \to \mathbb{C}$ to a complex linear map $(V_{\mathbb{C}})^k \to \mathbb{C}$ in the obvious way, *e.g.* if $k = 2$ then

$$(r^{-1}\eta)(v_1 + iw_1, v_2 + iw_2) = \eta(v_1,v_2) + i\eta(v_1,w_2) + i\eta(v_2,w_1) - \eta(w_1,w_2).$$

As a rather special case, for any real vector space W, recalling that $\Lambda^1 W^* = W^*$ we get an isomorphism $r\colon (W_{\mathbb{C}})^{*_{\mathbb{C}}} \to (W^*)_{\mathbb{C}}$ (where the superscript $*_{\mathbb{C}}$ means the complex dual space) given by restriction of complex-linear maps $W_{\mathbb{C}} \to \mathbb{C}$ to $W \subset W_{\mathbb{C}}$.

Recall from (5) that $\star\colon \Lambda^k V^* \to \Lambda^{n-k}V^*$ is given as a composition

$$\Lambda^k V^* \xrightarrow{(\Lambda^k \ell)^{-1}} \Lambda^k V \xrightarrow{\phi} (\Lambda^{n-k}V)^* \xrightarrow{\iota^{-1}} \Lambda^{n-k}V^* \tag{45}$$

where $\ell\colon V \to V^*$ is given by $v \mapsto (\cdot, v)$, where $(\phi\alpha)(\beta)$ is the value $t \in \mathbb{R}$ for which $\alpha\wedge\beta = t\omega_V$ where ω_V is the canonical generator for $\Lambda^n V$, and where $(\iota\theta)(f_1\wedge\cdots\wedge f_{n-k}) = \theta(f_1,\dots,f_{n-k})$ for any $f_1,\dots,f_{n-k} \in V$. So the complexified operator $\star = \star_{\mathbb{C}}\colon (\Lambda^k V^*)_{\mathbb{C}} \to (\Lambda^{n-k}V^*)_{\mathbb{C}}$ can be computed as the composition of the complexifications of the three maps in (45). These complexifications are perhaps easier to understand in terms of the various restriction isomorphisms $r\colon \Lambda^l_{\mathbb{C}} W^*_{\mathbb{C}} \to (\Lambda^l W^*)_{\mathbb{C}}$. (Recall that $\Lambda^k V$ is defined in these notes by using the canonical identification of V with V^{**} and interpreting $\Lambda^k V$ as $\Lambda^k V^{**}$, so in particular one special case of the restriction isomorphisms is an isomorphism $\Lambda^k_{\mathbb{C}} V_{\mathbb{C}} \to (\Lambda^k V)_{\mathbb{C}}$.) So consider the diagram

$$
\begin{tikzcd}
(\Lambda^k V^*)_{\mathbb{C}} \arrow[r, "(\Lambda^k \ell)_{\mathbb{C}}^{-1}"] & (\Lambda^k V)_{\mathbb{C}} \arrow[r, "\phi_{\mathbb{C}}"] & ((\Lambda^{n-k} V)^*)_{\mathbb{C}} \arrow[r, "\iota_{\mathbb{C}}^{-1}"] & (\Lambda^{n-k} V^*)_{\mathbb{C}} \\
\Lambda^k_{\mathbb{C}} V^*_{\mathbb{C}} \arrow[u, "r"] \arrow[r, "\psi_1"] & \Lambda^k_{\mathbb{C}} V_{\mathbb{C}} \arrow[u, "r"] \arrow[r, "\psi_2"] & ((\Lambda^{n-k} V)_{\mathbb{C}})^{*_{\mathbb{C}}} \arrow[u, "r"] \arrow[r, "\psi_3"] & \Lambda^{n-k}_{\mathbb{C}} V^*_{\mathbb{C}} \arrow[u, "r"]
\end{tikzcd}
\tag{46}
$$

where the maps on the bottom row are chosen to make the diagram commute (as is possible in a unique way since the vertical maps r are all isomorphisms).

Note that the usual formula for wedge product (in terms of a sum over permutations) extends trivially either to alternating $\mathbb{R}$-multilinear maps $V^k \to \mathbb{C}$ (*i.e.* to $(\Lambda^k V^*)_{\mathbb{C}}$) or to alternating $\mathbb{C}$-multilinear maps $(V_{\mathbb{C}})^k \to \mathbb{C}$ (*i.e.* to $\Lambda^k_{\mathbb{C}} V^*_{\mathbb{C}}$); we use these extensions below.

Continue to denote by ω_V the canonical generator for $\Lambda^n V$ associated to our oriented inner product space $(V, (\cdot,\cdot), \mathfrak{o})$ (given by $\omega_V = e_1 \wedge \cdots \wedge e_n$ for an arbitrary oriented orthornomal basis for V). Then ω_V includes trivially into $(\Lambda^n V)_{\mathbb{C}}$ (with zero imaginary part), and corresponds under the restriction isomorphism $r\colon \Lambda^n_{\mathbb{C}} V_{\mathbb{C}} \to (\Lambda^n V)_{\mathbb{C}}$ to the unique element $\omega_V^{\mathbb{C}} \in \Lambda_{\mathbb{C}} V_{\mathbb{C}}$ (*i.e.* the unique complex n-linear map $((V^*)_{\mathbb{C}})^n \to \mathbb{C}$) that restricts to $(V^*)^n$ as ω_V. With this preparation, we can state the following:

PROPOSITION 2.1.4. *Let $(V, (\cdot,\cdot), \mathfrak{o})$ be an oriented inner product space, let $\{e_1, \ldots, e_n\}$ be a complex basis for $V_{\mathbb{C}}$ that is orthonormal for the Hermitian inner product (43), and let $\{e^1, \ldots, e^n\}$ be the complex-dual basis for $V_{\mathbb{C}}$. Assume moreover that $e_1 \wedge \cdots \wedge e_n = \omega_V^{\mathbb{C}}$. Then*

$$\star\left(r\left(\langle\cdot, \bar{e}_1\rangle \wedge \cdots \wedge \langle\cdot, \bar{e}_k\rangle\right)\right) = r(e^{k+1} \wedge \cdots \wedge e^n). \tag{47}$$

REMARK 2.1.5. Note that the maps $\langle\cdot, \bar{e}_j\rangle\colon V_{\mathbb{C}} \to \mathbb{C}$ are complex-linear; so $\langle\cdot, \bar{e}_1\rangle \wedge \cdots \wedge \langle\cdot, \bar{e}_k\rangle$ is a well-defined element of $\Lambda^k_{\mathbb{C}} V^*_{\mathbb{C}}$ and the left-hand side is well-defined. By contrast the maps $\langle e_j \cdot\rangle$ are of course not complex-linear. In our applications the $\bar{e}_j$ will also be members of the basis under consideration, and so the $\langle\cdot, \bar{e}_j\rangle$ will be dual basis elements.

PROOF. Referring to (46), we are to show that the maps ψ_1, ψ_2, ψ_3 that make that diagram commute obey $\psi_3(\psi_2(\psi_1(\langle\cdot, \bar{e}_1\rangle \wedge \cdots \wedge \langle\cdot, \bar{e}_k\rangle))) = e^{k+1} \wedge \cdots \wedge e^n$.

First of all we claim that $\psi_1(\langle\cdot, \bar{e}_1\rangle \wedge \cdots \wedge \langle\cdot, \bar{e}_k\rangle) = e_1 \wedge \cdots \wedge e_k$, *i.e.* that $(\Lambda^k \ell)_{\mathbb{C}}(r(e_1 \wedge \cdots \wedge e_k)) = r\left(\langle\cdot, \bar{e}_1\rangle \wedge \cdots \wedge \langle\cdot, \bar{e}_k\rangle\right)$. Now, as usual interpreting various terms by using the identification of V with V^{**},

$$
\begin{aligned}
r(e_1 \wedge \cdots \wedge e_k) &= (e_1 \wedge \cdots \wedge e_k)|_{(V^*)^k} \\
&= (e_1)|_{V^*} \wedge \cdots \wedge (e_k)|_{V^*} \\
&= (f_1 + i g_1) \wedge \cdots \wedge (f_k + i g_k)
\end{aligned}
$$

where we write the various e_j as $e_j = f_j + i g_j$ where each $f_j, g_j \in V = V^{**}$. Hence

$$
\begin{aligned}
(\Lambda^k \ell)_{\mathbb{C}}(r(e_1 \wedge \cdots \wedge e_k)) &= \ell(f_1 + i g_1) \wedge \cdots \wedge \ell(f_k + i g_k) \\
&= (\ell(f_1) + i\ell(g_1)) \wedge \cdots \wedge (\ell(f_k) + i\ell(g_k)) \\
&= ((\cdot, f_1) + i(\cdot, g_1)) \wedge \cdots \wedge ((\cdot, f_k) + i(\cdot, g_k)) \\
&= (\langle\cdot, f_1 - i g_1\rangle|_V) \wedge \cdots \wedge (\langle\cdot, f_k - i g_k\rangle|_V) \\
&= r\left(\langle\cdot, \bar{e}_1\rangle \wedge \cdots \wedge \langle\cdot, \bar{e}_k\rangle\right),
\end{aligned}
$$

confirming the claim.

Next we consider the element $\psi_2(e_1 \wedge \cdots \wedge e_k) \in (\Lambda^{n-k}_{\mathbb{C}} V_{\mathbb{C}})^{*_{\mathbb{C}}}$. The space $\Lambda^{n-k}_{\mathbb{C}} V_{\mathbb{C}}$ is generated by elements $e_J = e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$ as J varies over increasing $(n-k)$-tuples $(j_1, \ldots, j_{n-k})$. Since wedge product commutes with restriction of multilinear maps from $V_{\mathbb{C}}$ to V, and since all elements of $(\Lambda^n V_{\mathbb{C}})^*$ are multiples of $\omega_V^{\mathbb{C}}$, the values $(\psi_2(e_1 \wedge \cdots \wedge e_k))(e_J)$ are just the complex numbers t_J satisfying $e_1 \wedge \cdots \wedge e_k \wedge e_J = t_J \omega_V^{\mathbb{C}}$. For $J \neq \{k+1, \ldots, n\}$ this value is zero since then $j_l \in \{1, \ldots, k\}$ for some l, while the hypothesis of the proposition asserts that $e_1 \wedge \cdots \wedge e_k \wedge e_{(k+1,\ldots,n)} = \omega_V^{\mathbb{C}}$ so that $t_{(k+1,\ldots,n)} = 1$. Thus

$$\psi_2(e_1 \wedge \cdots \wedge e_k)(e_J) = \begin{cases} 1 & J = (k+1, \ldots, n) \\ 0 & \text{otherwise} \end{cases}$$

Now the map $\psi_3 \colon (\Lambda_{\mathbb{C}}^{n-k} V)^{*_{\mathbb{C}}} \to \Lambda_{\mathbb{C}}^{n-k} V_{\mathbb{C}}^*$ in (46) is the inverse of the isomorphism $\iota^{\mathbb{C}} \colon \Lambda_{\mathbb{C}}^{n-k} V_{\mathbb{C}}^* \to (\Lambda_{\mathbb{C}}^{n-k} V)^{*_{\mathbb{C}}}$ defined by letting $(\iota^{\mathbb{C}}\theta)(f_1 \wedge \cdots \wedge f_{n-k}) = \theta(f_1, \ldots, f_{n-k})$ (that $\iota^{\mathbb{C}}$ is well-defined follows just as in the discussion of (6)). Indeed, if we set $\psi_3 = (\iota^{\mathbb{C}})^{-1}$ then the right square of (46) will commute, and since the other maps in (46) are all isomorphisms it follows that $\psi_3 = (\iota^{\mathbb{C}})^{-1}$. But it is immediate from the definition that, for any increasing $(n-k)$-tuple J,

$$\iota^{\mathbb{C}}(e^{k+1} \wedge \cdots \wedge e^n)(e_J) = \begin{cases} 1 & J = (k+1, \ldots, n) \\ 0 & \text{otherwise} \end{cases} .$$

Thus $\psi_3(\psi_2(e_1 \wedge \cdots \wedge e_k)) = e^{k+1} \wedge \cdots \wedge e^n$. Together with the calculation at the start of the proof, this establishes the result. □

Here is a somewhat easier-to-use version of Proposition 2.1.4. Dually to $\omega_V \in \Lambda^n V$, given an oriented inner product space V we define $\mathrm{vol}_V \in \Lambda^n V^* \subset (\Lambda^n V^*)_{\mathbb{C}}$ by putting $\mathrm{vol}_V = e^1 \wedge \cdots \wedge e^n$ where $\{e^1, \ldots, e^n\}$ is the dual basis to an arbitrary oriented orthonormal basis for V (as with ω_V, this is independent of the choice of such a basis because the relevant basis change matrix has determinant one). Likewise we let $\mathrm{vol}_V^{\mathbb{C}}$ be the unique element of $\Lambda_{\mathbb{C}}^n V_{\mathbb{C}}^*$ mapping to vol_V under the restriction isomorphism r.

COROLLARY 2.1.6. *Let $(V, (\cdot,\cdot), \mathfrak{o})$ be an oriented inner product space and let $\{e_1, \ldots, e_n\}$ be a complex basis for $V_{\mathbb{C}}$ that is orthonormal with respect to the Hermitian inner product (43), and let $\{e^1, \ldots, e^n\}$ be the dual basis for $V_{\mathbb{C}}^{*_{\mathbb{C}}}$. Assume that $e^1 \wedge \cdots \wedge e^n = \alpha \mathrm{vol}_V^{\mathbb{C}}$. Then for $0 < l < n$,*

$$\star\left(r(e^1 \wedge \cdots \wedge e^l)\right) = \alpha r\left(\langle \cdot, \bar{e}_{l+1} \rangle \wedge \cdots \wedge \langle \cdot, \bar{e}_n \rangle\right).$$

PROOF. By definition if we take an orthonormal (real) basis $\{f_1, \ldots, f_n\}$ for V, then this basis is also an orthonormal basis over $\mathbb{C}$ for $V_{\mathbb{C}}$, and $\mathrm{vol}_V^{\mathbb{C}} = f^1 \wedge \cdots \wedge f^n$ for the dual basis $\{f^1, \ldots, f^n\}$. Now the change-of-basis matrix relating our given basis $\{e_1, \ldots, e_n\}$ to $\{f_1, \ldots, f_n\}$ is unitary, so its determinant has modulus 1; thus the parameter α in the hypothesis of the corollary obeys $|\alpha| = 1$. In particular the basis $\{e_1, \ldots, e_{n-1}, \alpha e_n\}$ is still orthonormal, with dual basis $\{e^1, \ldots, e^{n-1}, \alpha^{-1} e^n\}$. Since $\langle \cdot, \overline{\alpha e_n} \rangle = \alpha \langle \cdot, \bar{e}_n \rangle$, this reduces us to the case that $\alpha = 1$, which we assume from now on.

That $\alpha = 1$ implies that the change-of-basis matrix from the previous paragraph has determinant 1 (as α is the reciprocal of this determinant), and hence that $e_1 \wedge \cdots \wedge e_n = \omega_V^{\mathbb{C}}$. Now apply the previous proposition with $k = n - l$ to the basis $\{(-1)^{l(n-l)} e_{l+1}, e_{l+2}, \ldots, e_n, e_1, \ldots, e_l\}$ (the sign is used to ensure that the wedge product of the elements in this new basis is still equal to $\omega_V^{\mathbb{C}}$) to find that

$$(-1)^{l(n-l)} \star (r(\langle \cdot, \bar{e}_{l+1} \rangle \wedge \cdots \wedge \langle \cdot \bar{e}_n \rangle)) = r(e^1 \wedge \cdots \wedge e^l).$$

Now apply $\star$ to both sides and recall that $\star\star$ acts on $\Lambda^l V^*$ by $(-1)^{l(n-l)}$ to obtain the result. □

EXAMPLE 2.1.7. *Let $V = \mathbb{R}^2$ with its standard inner product and orientation, so that $V_{\mathbb{C}} = \mathbb{C}^2$ with its standard inner product $\langle (z_1, z_2), (w_1, w_2) \rangle = z_1 \bar{w}_1 + z_2 \bar{w}_2$. Let $\theta = (\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}})$, so $\bar{\theta} = (\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}})$, so in terms of the standard basis for $\mathbb{R}^2$ we have $\theta = \frac{1}{\sqrt{2}}(e_1 + ie_2)$ and $\bar{\theta} = \frac{1}{\sqrt{2}}(e_1 - ie_2)$.*

If $\{\phi, \bar{\phi}\}$ is the dual basis to $\{\theta, \bar{\theta}\}$, we find that

$$\phi = \frac{1}{\sqrt{2}}(e^1 - ie^2) \qquad \bar{\phi} = \frac{1}{\sqrt{2}}(e^1 + ie^2)$$

and hence that $\phi \wedge \bar{\phi} = ie^1 \wedge e^2$. Hence Corollary 2.1.6 shows that

$$\star(\phi) = i\langle \cdot, \overline{\bar{\theta}}\rangle = i\langle \cdot, \theta\rangle = i\phi.$$

Similarly since $\bar{\phi} \wedge \phi = -ie^1 \wedge e^2$ one finds $\star(\bar{\phi}) = -i\bar{\phi}$.

Of course these identities can also be checked directly; however the approach given by Corollary 2.1.6 will significantly simplify similar higher-dimensional computations, as we will see below.

2.2. Complexification and complex structures

DEFINITION 2.2.1. If V is a vector space over $\mathbb{R}$, a *complex structure* on V is a linear map $J\colon V \to V$ such that $J^2 = -I$ (where I always denotes the identity).

If J is a complex structure on V, then we can view V as a vector space over $\mathbb{C}$ using the scalar multiplication $(a+ib)v = av + Jbv$. However this is not usually the perspective that we will take. Instead, we will consider the complexification $V_{\mathbb{C}}$, which is a complex vector space on which we have an induced complex-linear map $J = J_{\mathbb{C}}\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ that still satisfies $J^2 = -I$. Evidently the possible eigenvalues of J are $\pm i$. We denote the corresponding eigenspaces by

$$V_{1,0} = \ker(iI - J) \qquad V_{0,1} = \ker(-iI - J).$$

Here are some simple facts about these spaces:

PROPOSITION 2.2.2. *Let V be an m-dimensional real vector space with a complex structure $J\colon V \to V$. Then:*

(i) *We have a direct sum decomposition of complex vector spaces $V_{\mathbb{C}} = V_{1,0} \oplus V_{0,1}$.*
(ii) *The complex conjugation map $\bar{\ }\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ restricts to an isomorphism of real vector spaces $V_{1,0} \to V_{0,1}$. Hence m is even, and $\dim_{\mathbb{C}} V_{1,0} = \dim_{\mathbb{C}} V_{0,1} = \frac{m}{2}$.*
(iii) *We have*

$$V_{1,0} = \{v - iJv | v \in V\} \qquad V_{0,1} = \{v + iJv | v \in V\}.$$

PROOF. Of course $V_{1,0}$ and $V_{0,1}$, being eigenspaces of a complex-linear operator on the complex vector space $V_{\mathbb{C}}$, are both complex subspaces of $V_{\mathbb{C}}$. Obviously $V_{1,0} \cap V_{0,1} = \{0\}$ since an element $v \in V_{1,0} \cap V_{0,1}$ would obey $iv = Jv = -iv$. The fact that $J\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ is complex-linear with $J^2 = -I$ immediately implies identities

$$(iI - J)(iI + J) = (iI + J)(iI - J) = 0, \tag{48}$$

so for any $v \in V_{\mathbb{C}}$ the formula

$$v = \frac{1}{2}(I - iJ)v + \frac{1}{2}(I + iJ)v = -\frac{i}{2}(iI + J)v - \frac{i}{2}(iI - J)v$$

expresses v as $v = v_{1,0} + v_{0,1}$ where $v_{1,0} = \frac{1}{2}(v - iJv) \in V_{1,0}$ and $v_{0,1} = \frac{1}{2}(v + iJv) \in V_{0,1}$. So indeed $V_{\mathbb{C}} = V_{1,0} \oplus V_{0,1}$.

For (ii), if $v_1, v_2 \in V$ we see that $v_1 + iv_2 \in V_{1,0}$ (*i.e.* $J(v_1 + iv_2) = i(v_1 + iv_2)$) iff $Jv_1 = -v_2$ and $Jv_2 = v_1$ (actually the second statement follows from the first and the fact that $J^2 = -I$), while $v_1 + iv_2 \in V_{0,1}$ iff $Jv_1 = v_2$ and $Jv_2 = -v_1$. Thus $v_1 + iv_2 \in V_{1,0}$ if and only if $v_1 - iv_2 = \overline{v_1 + iv_2} \in V_{0,1}$. So since the conjugation map is $\mathbb{R}$-linear and is its own inverse this proves that the conjugation map sends $V_{1,0}$ isomorphically (as vector spaces over $\mathbb{R}$) to $V_{0,1}$. So $\dim_{\mathbb{R}} V_{1,0} = \dim_{\mathbb{R}} V_{0,1}$. So since

$\dim_{\mathbb{R}} V_{\mathbb{C}} = 2m$ it follows from (i) that $\dim_{\mathbb{R}} V_{1,0} = \dim_{\mathbb{R}} V_{0,1} = m$. So since $V_{1,0}$ and $V_{0,1}$ are complex vector spaces, it follows that m is even and that $\dim_{\mathbb{C}} V_{1,0} = \dim_{\mathbb{C}} V_{0,1} = \frac{m}{2}$.

Now to prove (iii), the restriction of the map $I - iJ\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ to V is clearly injective (since it acts trivially on real parts), and since $I - iJ = -i(iI + J)$, (48) shows that $I - iJ$ has image contained in $V_{1,0}$. So $(I - iJ)|_V$ is an injective linear map from V to $V_{1,0}$, and we have just shown that these spaces have the same dimension, so indeed $V_{1,0} = \{(I - iJ)v | v \in V\}$. Taking complex conjugates shows that likewise $V_{0,1} = \{(I + iJ)v | v \in V\}$. □

If $\dim_{\mathbb{R}} V = m = 2n$, let $\{v_1, \ldots, v_n\}$ be a basis over $\mathbb{C}$ for $V_{1,0}$. Then $\{\bar{v}_1, \ldots, \bar{v}_n\}$ is evidently a basis for $V_{0,1}$, and so we get a basis $\{v_1, \bar{v}_1, \ldots, v_n, \bar{v}_n\}$ for $V_{\mathbb{C}} = V_{1,0} \oplus V_{0,1}$.

We can then take a dual basis $\{v^1, \bar{v}^1, \ldots, v^n, \bar{v}^n\}$ for the dual space $V_{\mathbb{C}}^{\mathbb{C}}$. Here and elsewhere, for $\phi \in V_{\mathbb{C}}^{*_{\mathbb{C}}}$, the complex conjugate $\bar{\phi}$ is defined by

$$\bar{\phi}(v) = \overline{\phi(\bar{v})} \quad (\phi \in V_{\mathbb{C}}^{*_{\mathbb{C}}},\, v \in V_{\mathbb{C}}).$$

Under the restriction isomorphism $r\colon V_{\mathbb{C}}^{*_{\mathbb{C}}} \to (V^*)_{\mathbb{C}}$ discussed in the previous section, this conjugation operator on $V_{\mathbb{C}}^{*_{\mathbb{C}}}$ corresponds to the original complex conjugation $\phi_1 + i\phi_2 \mapsto \phi_1 - i\phi_2$ on $(V^*)_{\mathbb{C}}$. (One needs to take the conjugate of the input to $\bar{\phi}$ above in order for $\bar{\phi}$ to be complex-linear.) With this said, from now on we will implicitly identify $V_{\mathbb{C}}^{*_{\mathbb{C}}}$ with $(V_{\mathbb{C}})^*$, and likewise $\Lambda^k_{\mathbb{C}} V^*_{\mathbb{C}}$ with $(\Lambda^k V^*)_{\mathbb{C}}$, omitting the restriction isomorphism r from the notation.

Let

$$V^{1,0} = \{\phi \in V_{\mathbb{C}}^{*_{\mathbb{C}}} | \phi|_{V_{0,1}} = 0\}, \qquad V^{0,1} = \{\phi \in V_{\mathbb{C}}^{*_{\mathbb{C}}} | \phi|_{V_{1,0}} = 0\}.$$

Evidently our dual basis $\{v^1, \bar{v}^1, \ldots, v^n, \bar{v}^n\}$ has $v^1, \ldots, v^n \in V^{1,0}$ and $\bar{v}^1, \ldots, \bar{v}^n \in V^{0,1}$. Hence $V_{\mathbb{C}}^{*_{\mathbb{C}}} = V^{1,0} \oplus V^{0,1}$.

Similarly, for $0 \le p \le k$ we may consider the subspace $\Lambda^{p,k-p} V^*$ of $\Lambda^k_{\mathbb{C}} V^*_C$ defined by

$$\Lambda^{p,k-p} V^* = \left\{ \phi \in \Lambda^k_{\mathbb{C}} V^*_{\mathbb{C}} \,\middle|\, \phi|_{(V^{1,0})^r \times (V^{0,1})^{k-r}} = 0 \text{ for all } r \neq p \right\}.$$

More concretely, given our basis $\{v^1, \bar{v}^1, \ldots, v^n, \bar{v}^n\}$ as above, we have, for $0 \le p, q \le n$,

$$\Lambda^{p,q} V^* = span\{v^{i_1} \wedge \cdots \wedge v^{i_p} \wedge \bar{v}^{j_1} \wedge \cdots \wedge \bar{v}^{j_q} | 1 \le i_1 < \cdots < i_p \le n,\, 1 \le j_1 < \ldots < j_q \le n\}.$$

From this it is clear that

$$\Lambda^k_{\mathbb{C}} V^* = \bigoplus_{p+q=k} \Lambda^{p,q} V^*.$$

Note that

$$\text{If } \omega \in \Lambda^{p,q} V^* \text{ then } \bar{\omega} \in \Lambda^{q,p} V^* \tag{49}$$

as follows from directly from the fact that the conjugation operator on $\Lambda^k_{\mathbb{C}} V^*_{\mathbb{C}}$ is given by $\bar{\omega}(v_1, \ldots, v_k) = \overline{\omega(\bar{v}_1, \ldots, \bar{v}_k)}$. Also

$$\text{If } \omega \in \Lambda^{p,q} V^*, \theta \in \Lambda^{r,s} V^* \text{ then } \omega \wedge \theta \in \Lambda^{p+r,q+s} V^*. \tag{50}$$

Recalling that V can be made into a complex vector space by interpreting J as scalar multiplication by i, if $\{e_1, \ldots, e_n\}$ is a basis (over $\mathbb{C}$) for this complex vector space then $\{e_1, Je_1, \ldots, e_n, Je_n\}$ is a basis over $\mathbb{R}$ for V. This induces an orientation $\mathfrak{o}$ for V, the "complex orientation," by saying that $\{e_1, Je_1, \ldots, e_n, Je_n\}$ is an oriented ordered basis. (This orientation is independent of the choice of the e_i, essentially because $GL(n, \mathbb{C})$ is connected and hence the set of such choices is connected.)

Let us say that an inner product $(\cdot,\cdot)$ on V is *J-compatible* if $(Jv, Jw) = (v, w)$ for all $v, w \in V$. Such inner products certainly exist; for instance this will hold if we define $(\cdot,\cdot)$ by the property that some specific basis $\{e_1, Je_1, \ldots, e_n, Je_n\}$ is orthonormal. Conversely, if $(\cdot,\cdot)$ is a J-compatible inner product then one can find an orthonormal basis of the form $\{e_1, Je_1, \ldots, e_n, Je_n\}$, as follows by an easy argument by induction on $\frac{\dim_{\mathbb{R}} V}{2}$ (if $(e,e)=1$, then $\{e, Je\}$ is an orthonormal set, and we may apply the inductive hypothesis to $\{e, Je\}^{\perp}$).

PROPOSITION 2.2.3. *Let J be a complex structure on the $2n$-dimensional real vector space V, let $(\cdot,\cdot)$ be a J-compatible inner product on V, and endow $V_{\mathbb{C}}$ with the Hermitian inner product $\langle\cdot,\cdot\rangle$ be the Hermitian inner product defined by (43). Then the subspaces $V_{1,0}, V_{0,1} \leq V_{\mathbb{C}}$ are orthogonal to each other.*

PROOF. If $v = v_1 + iv_2 \in V_{1,0}$ and $w = w_1 + iw_2 \in V_{0,1}$ (where $v_1, w_1, v_2, w_2 \in V$) then the fact that $Jv = iv$ implies that $Jv_1 = -v_2$ while the fact that $Jw = -iw$ implies that $Jw_1 = w_2$. So

$$\langle v_1+iv_2, w_1+iw_2\rangle = (v_1,w_1)+(v_2,w_2)+i(v_2,w_1)-i(v_1,w_2) = (v_1,w_1)+(-Jv_1,Jw_1)-i(Jv_1,w_1)-i(v_1,Jw_1) = 0$$

(since in particular $(v_1, Jw_1) = (Jv_1, J^2w_1) = -(Jv_1, w_1)$). □

PROPOSITION 2.2.4. *Let V be a $2n$-dimensional vector space over $\mathbb{R}$ with a complex structure J and a J-compatible inner product $(\cdot,\cdot)$ and the complex orientation. Then the (complexified) Hodge star operator $\star\colon \Lambda^k_{\mathbb{C}} V^*_{\mathbb{C}} \to \Lambda^{2n-k}_{\mathbb{C}} V^*_{\mathbb{C}}$ has the property that*

$$\textit{If } \omega \in \Lambda^{p,q}V^* \textit{ then } \star\omega \in \Lambda^{n-q,n-p}V^*$$

PROOF. Any element of V^* can be written as a linear combination of elements of the form $e^1 \wedge \cdots \wedge e^p \wedge f^1 \wedge \cdots \wedge f^q$ where $\{e_1,\ldots,e_n\}$ is an orthonormal basis for $V_{1,0}$, $\{f_1,\ldots,f_n\}$ is an orthonormal basis for $V_{0,1}$, and $\{e^1,\ldots,e^n\}$ and $\{f^1,\ldots,f^n\}$ are the corresponding dual bases. Proposition 2.2.3 then shows that $\{e_1,\ldots,e_n,f_1,\ldots,f_n\}$ is an orthonormal basis for $V_{\mathbb{C}}$, so it follows from Corollary 2.1.6 (continuing to suppress the restriction isomorphism r from the notation) that, for some $\alpha \in S^1$,

$$\star(e^1 \wedge \cdots \wedge e^p \wedge f^1 \wedge \cdots \wedge f^q) = \alpha\langle\cdot,\bar{e}_{p+1}\rangle \wedge \cdots \wedge \langle\cdot,\bar{e}_n\rangle \wedge \langle\cdot,\bar{f}_{q+1}\rangle \wedge \cdots \wedge \langle\cdot,\bar{f}_n\rangle.$$

Now each $\bar{e}_j \in V_{0,1}$, so by Proposition 2.2.3 $\langle\cdot,\bar{e}_j\rangle$ vanishes on $V_{1,0}$, *i.e.* belongs to $V^{0,1}$. Likewise each $\bar{f}_j \in V_{1,0}$, so each $\langle\cdot,\bar{f}_j\rangle \in V^{1,0}$. Thus the right-hand side above is a wedge product of $n-p$ elements of $V^{0,1}$ and $n-q$ elements of $V^{1,0}$, so it belongs to $\Lambda^{n-q,n-p}V^*$. □

If $\dim V = 2n$, the formula (44) motivates consideration of the bilinear pairing

$$\begin{aligned} \Lambda^k_{\mathbb{C}}V^*_{\mathbb{C}} \times \Lambda^k_{\mathbb{C}}V^*_{\mathbb{C}} &\to \Lambda^{2n}_{\mathbb{C}}V^*_{\mathbb{C}} \\ (\alpha,\beta) &\mapsto \alpha\wedge\star\bar{\beta} \end{aligned}$$

According to (49),(50), and Proposition 2.2.4, if $\alpha \in \Lambda^{p,k-p}V^*$ and $\beta \in \Lambda^{r,k-r}V^*$ we have $\star\bar{\beta} \in \Lambda^{n-r,n-k+r}V^*$ and hence $\alpha\wedge\star\bar{\beta} \in \Lambda^{n+p-r,n+r-p}V^*$. But since $\dim V = 2n$ we have $\Lambda^{n+j,n-j}V^* = \{0\}$ unless $j=0$, so:

COROLLARY 2.2.5. *If $\alpha \in \Lambda^{p,k-p}V^*$ and $\beta \in \Lambda^{r,k-r}V^*$ then $\alpha\wedge\star\bar{\beta} = 0$ unless $p = r$.*

In other words, the decomposition $\Lambda^k_{\mathbb{C}}V^*_{\mathbb{C}} = \oplus_{p+q=k}\Lambda^{p,q}V^*$ is an **orthogonal** decomposition with respect to the pairing $(\alpha,\beta) \mapsto \alpha\wedge\star(\bar{\beta})$.

2.3. Almost complex and complex manifolds

The linear algebra constructions of the previous section can be made differential-geometric by imposing them on the tangent spaces at each point on a manifold.

DEFINITION 2.3.1. Let M be a smooth manifold.

- An *almost complex structure* J on M is a smoothly-varying[1] choice, for each $m \in M$, of a complex structure $J_m \colon T_mM \to T_mM$ on the real vector space T_mM. In this case the pair (M, J) is called an *almost complex manifold*
- An *almost Hermitian structure* on M consists of an almost complex structure J on M and a Riemannian metric g on M such that, for each $m \in M$ and $v, w \in T_mM$ we have $g_m(J_m v, J_m w) = g_m(v, w)$. In this case the triple (M, g, J) is called an *almost Hermitian manifold*, and g is said to be compatible with J.

By working locally and then using partitions of unity it is not hard to see that any almost complex manifold (M, J) admits Riemannian metrics g that are compatible with J. So any almost complex manifold can, non-canonically, be made into an almost Hermitian manifold.

If (M, J) is an almost complex manifold, then we can form the complexified tangent bundle $(TM)_{\mathbb{C}}$ (with fiber over m given by $(T_mM)_{\mathbb{C}}$, and local complex trivializations given by complexifying the local trivializations for TM—this initial step does not use J), and then we obtain a bundle endomorphism J of $TM_{\mathbb{C}}$ by acting on the fiber over m via the complexified version of J_m. This leads to a decomposition of subbundles $TM_{\mathbb{C}} = T_{1,0}M \oplus T_{0,1}M$, with (using Proposition 2.2.2 (iii) for the second equalities in each line):

$$T_{1,0}M = \{v \in (TM)_{\mathbb{C}} | Jv = iv\} = \{v - iJv | v \in TM\}$$
$$T_{0,1}M = \{v \in (TM)_{\mathbb{C}} | Jv = -iv\} = \{v + iJv | v \in TM\}$$

Dually, for any $k \in \mathbb{N}$ we may consider the complexification $\Omega^k(M)_{\mathbb{C}}$ of the space of k-forms on M. In view of the restriction isomorphism r from Section 2.1.2, an element $\omega \in \Omega^k(M)_{\mathbb{C}}$ may be regarded as a smoothly-varying choice, for every $m \in M$, of an element $\omega_m \in \Lambda^k_{\mathbb{C}}(T_mM)^*_{\mathbb{C}}$, *i.e.* as an alternating complex-multilinear k-form on $(T_mM)_{\mathbb{C}}$. Applying the decomposition from Section 2.2 pointwise gives a decomposition

$$\Omega^k(M)_{\mathbb{C}} = \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

where $\Omega^{p,q}(M)$ consists of alternating complex-multilinear k-forms which, for each $m \in M$, vanish on $(T_{1,0}M)^r_m \times (T_{1,0}M)^{k-r}_m$ for all $r \neq p$.

Suppose now that (M, J, g) is almost Hermitian, and (since Proposition 2.2.2 shows that M has even dimension) let $\dim M = 2n$. The almost complex structure J induces a smoothly-varying orientation on the various T_mM and hence an orientation on the manifold M; thus we obtain a (complexified) Hodge star operator $\star \colon \Omega^k(M)_{\mathbb{C}} \to \Omega^{2n-k}(M)_{\mathbb{C}}$, and Proposition 2.2.4 shows that $\star$ restricts to $\Omega^{p,q}(M)$ as a map $\star \colon \Omega^{p,q}(M) \to \Omega^{n-q,n-p}(M)$. If M is compact, we have a Hermitian inner product on $\Omega^k(M)_{\mathbb{C}}$ defined by $\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star\bar{\beta}$. For $\alpha \in \Omega^{p,k-p}(M)$ and $\beta \in \Omega^{r,k-r}(M)$ we have $\alpha \wedge \star\bar{\beta} \in \Omega^{n+p-r,n-r+p}(M)$, so it follows that $\langle \alpha, \beta \rangle = 0$ unless $p = r$. This proves:

PROPOSITION 2.3.2. *If (M, J) is an almost complex manifold we have a direct sum decomposition*

$$\Omega^k(M)_{\mathbb{C}} = \bigoplus_{p+q=k} \Omega^{p,q}(M),$$

and if (M, J, g) is almost Hermitian then this direct sum decomposition is orthogonal with respect to the inner product (44) on $\Omega^k(M)_{\mathbb{C}}$ induced by g.

[1] in the sense that if X is a smooth vector field on M then putting $(JX)_m = J_m X_m$ defines a smooth vector field JX on M

The collection of spaces $\Omega^{p,q}(M)$ can helpfully be visualized as follows:

$$\begin{array}{ccccccccccc}
 & & & & & \Omega^{n,n} & & & & & \\
 & & & & \Omega^{n,n-1} & & \Omega^{n-1,n} & & & & \\
 & & & \Omega^{n,n-2} & & \Omega^{n-1,n-1} & & \Omega^{n-2,n} & & & \\
 & & \iddots & \cdot & & \cdot & & \cdot & \ddots & & \\
 & \Omega^{n,1} & \cdot & \cdot & & \cdot & & \cdot & \cdot & \Omega^{1,n} & \\
\Omega^{n,0} & \Omega^{n-1,1} & \cdot & \cdot & & \cdot & & \cdot & \cdot & \Omega^{1,n-1} & \Omega^{0,n} \\
 & \Omega^{n-1,0} & \cdot & \cdot & & \cdot & & \cdot & \cdot & \Omega^{0,n-1} & \\
 & & \ddots & \cdot & & \cdot & & \cdot & \iddots & & \\
 & & & \Omega^{2,0} & & \Omega^{1,1} & & \Omega^{0,2} & & & \\
 & & & & \Omega^{1,0} & & \Omega^{0,1} & & & & \\
 & & & & & \Omega^{0,0} & & & & &
\end{array}$$

The "height" of $\Omega^{p,q}$ in the picture above is the value $k = p+q$, while the horizontal coordinate is given by $q-p$. Complex conjugation restricts as a map $\Omega^{p,q} \to \Omega^{q,p}$ and thus reflects the picture through the central vertical axis $\{q-p=0\}$ while the Hodge star operator $\Omega^{p,q} \to \Omega^{n-q,n-p}$ acts as reflection through the central horizontal axis $\{p+q=n\}$.

We now turn to differentiation. The space $\Omega^{0,0}(M) = \Omega^0(M)_{\mathbb{C}}$ is simply the space of smooth, complex-valued functions $f = f_1 + if_2 \colon M \to \mathbb{C}$ where $f_1, f_2 \in C^\infty(M)$ are real-valued. The differentiation operator $d \colon \Omega^0(M) \to \Omega^1(M)$ complexifies to an operator $d \colon \Omega^{0,0}(M) \to \Omega^1(M)_{\mathbb{C}} = \Omega^{1,0}(M) \to \Omega^{0,1}(M)$, as always by simply requiring $d(f_1+if_2) = df_1 + idf_2$. We can thus uniquely write

$$df = \partial f + \bar{\partial} f \quad \text{where} \partial f \in \Omega^{1,0}(M),\ \bar{\partial} f \in \Omega^{0,1}(M).$$

So we have operators $\partial \colon \Omega^{0,0}(M) \to \Omega^{1,0}(M)$ and $\bar{\partial} \colon \Omega^{0,0}(M) \to \Omega^{0,1}(M)$ with $d = \partial + \bar{\partial}$.

It is often useful to work locally; on some open set U (over which $T_{1,0}M$ and $T_{0,1}M$ admit trivializations) let us choose local (complexified) vector fields $\{e_1, \ldots, e_n\}$ such that, for each $m \in U$, $\{(e_1)_m, \ldots, (e_n)_m\}$ is a basis for $(T_{1,0}M)_m$. Then the conjugations $\{(\bar{e}_1)_m, \ldots, (\bar{e}_n)_m\}$ likewise form a basis for each $(T_{0,1}M)_m$, and if $\{e^1, \ldots, e^n, \bar{e}^1, \ldots, \bar{e}^n\}$ is the corresponding (pointwise) dual basis of 1-forms we will have

$$\partial f = \sum_j (\nabla_{e_j} f) e^j, \qquad \bar{\partial} f = \sum_j (\nabla_{\bar{e}_j} f) \bar{e}^j. \tag{51}$$

One can also produce simple global formulas for ∂f and $\bar{\partial} f$ in terms of df and J: specifically:

$$\partial f = \frac{1}{2}(df - idf \circ J), \qquad \bar{\partial} f = \frac{1}{2}(df + idf \circ J).$$

Indeed, the right-hand sides above obviously sum to df, and for $v \in TM$ we have

$$(df \mp idf \circ J)(v \pm iJv) = df(v) \mp idf(Jv) \pm idf(Jv) + df(J^2 v) = 0$$

which in view of Proposition 2.2.2 (iii) shows that the right-hand sides annihilate $T_{0,1}M$ and $T_{1,0}M$ respectively, and hence belong respectively to $\Omega^{1,0}(M)$ and $\Omega^{0,1}(M)$. In particular $\bar{\partial} f = 0$ if and only if $df(v) = -idf(Jv)$ for all $v \in TM$, *i.e* if and only if $df(Jv) = idf(v)$ for all v, which is to say that, for all $m \in M$, the map $(df)_m \colon T_m M \to \mathbb{C}$ is *complex linear* when $T_m M$ is made into a complex vector space using J. Thus the functions f with $\bar{\partial} f = 0$ are (under the natural definition) precisely the holomorphic functions form the almost complex manifold (M, J) to $\mathbb{C}$.

Now let us consider 1-forms. Suppose that $\omega \in \Omega^{1,0}(M)$, and as above let $\{e_1, \dots, e_n\}$ be a local frame for $T^{1,0}M$ in some open set U, with dual coframe $\{e^1, \dots, e^n\}$. We can then write

$$\omega|_U = \sum_{j=1}^{n} f_j e^j \quad \text{where } f_j \in \Omega^{0,0}(U) = C^\infty(U, \mathbb{C})$$

and so

$$d\omega|_U = \sum_{j=1}^{n} (df_j \wedge e^j + f_j de^j) = \sum_{j=1}^{n} \partial f_j \wedge e^j + \sum_{j=1}^{n} \bar{\partial} f_j \wedge e^j + \sum_{j=1}^{n} f_j de^j. \tag{52}$$

The first term on the right above evidently lies in $\Omega^{2,0}(U)$ and the second term above lies in $\Omega^{1,1}(U)$. However in this level of generality there is not much that we can say about the third term, and in particular it might have a nontrivial component in $\Omega^{2,0}(U)$. Accordingly we now restrict attention to a context where we can say something, namely where our manifold is complex and not just almost complex.

DEFINITION 2.3.3. A *complex manifold* of (complex) dimension n is a smooth manifold M of (real) dimension $2n$ equipped with an atlas of coordinate charts $\{\phi_\alpha \colon U_\alpha \to \mathbb{C}^n = \mathbb{R}^{2n}\}$ where the U_α form an open cover of M and all transition functions $\phi_\beta \circ \phi_\alpha^{-1} \colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are holomorphic.

Here if $U \subset \mathbb{C}^r$ is an open subset, a map $h \colon U \to \mathbb{C}^n$ is said to be holomorphic if and only if, for all $x \in U$, the derivative $(dh)_x$ (which *a priori* is a real-linear map from $\mathbb{C}^r$ to $\mathbb{C}^n$) is complex-linear. If your preferred definition of a holomorphic function looks different from this, then you should check that this is equivalent to your definition.

If M is a complex manifold, it can naturally be viewed as an almost complex manifold in the following way: for $m \in M$, choose a chart $\phi_\alpha \colon U_\alpha \to \mathbb{C}^n$ from the atlas in Definition 2.3.3 such that $m \in U_\alpha$, and for $v \in T_m M$ define $J_m v = \phi_{\alpha*}^{-1}(i\phi_{\alpha*} v)$. This is independent of the choice of α since for an alternative choice β the holomorphicity of the transition function says that (letting i denote the operation of scalar multiplication by i) $i \circ \phi_{\beta*}\phi_{\alpha*}^{-1} = \phi_{\beta*}\phi_{\alpha*}^{-1} \circ i$ and hence $\phi_{\beta*}^{-1} \circ i \circ \phi_{\beta*} = \phi_{\alpha*}^{-1} \circ i \circ \phi_{\alpha*}$.

In any coordinate chart $\phi \colon U \to \mathbb{C}^n$ for a complex manifold, the individual coordinates $z_1, \dots, z_n$ of ϕ define functions $z_1, \dots, z_n \colon U \to \mathbb{C}$. For $j = 1, \dots, n$, the very definition of J makes clear that $dz_j(Jv) = i dz_j(v)$ for each $v \in T_m M$ and $m \in U$ (just take the jth component of the equation $\phi_* \circ J = i\phi_*$). So $\{dz_1, \dots, dz_n\}$ restricts to each $m \in U$ as a basis for $(T^{1,0}M)_m$, and likewise $\{d\bar{z}_1, \dots, d\bar{z}_n\}$ gives a basis for $(T^{0,1}M)_m$. We now define the complexified vector fields $\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}$ on U by requiring that $\{dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n\}$ be the dual basis to $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n}\}$; if $z_j = x_j + iy_j$ and $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ denote the standard coordinate vector fields for the real coordinate chart $(x_1, \dots, x_n, y_1, \dots, y_n)$, one easily sees that

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right). \tag{53}$$

As a special case of (51), the operators ∂ and $\bar{\partial}$ on $\Omega^{0,0}(M)$ are given locally by

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j, \qquad \bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

The crucial point now is that the one-forms dz_j in our coframe are *closed* (since $dz_j = dx_j + idy_j$ and dx_j and dy_j are closed), and so when we compute the exterior derivative of a $(1,0)$-form as in (52) the third term in that formula does not appear. More generally, for a tuple $I = (i_1, \dots, i_p)$ with $i_1 < \cdots < i_p$ let us write $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $d\bar{z}_I = d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p}$. Each of these forms is closed by the Leibniz rule. Then any $\omega \in \Omega^{p,q}(M)$ has restriction to our holomorphic coordinate chart U given by

$$\omega|_U = \sum_{\#I=p,\#J=q} f_{IJ} dz_I \wedge d\bar{z}_J$$

and so by the Leibniz rule and the fact that each $d(dz_I) = d(d\bar{z}_J) = 0$,

$$d\omega|_U = \sum_{\#I=p,\#J=q} (\partial f_{IJ}) \wedge dz_I \wedge d\bar{z}_J + \sum_{\#I=p,\#J=q} (\bar{\partial} f_{IJ}) \wedge dz_I \wedge d\bar{z}_J. \tag{54}$$

COROLLARY 2.3.4. *Let M be an n-dimensional complex manifold. Then there are linear operators $\partial, \bar{\partial}$ on $\oplus_{k=0}^{n} \Omega^k(M)_{\mathbb{C}}$ such that $d = \partial + \bar{\partial}$ and such that ∂ restricts to $\Omega^{p,q}(M)$ as a map $\partial \colon \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$, while $\bar{\partial}$ restricts to $\Omega^{p,q}(M)$ as a map $\bar{\partial} \colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. Moreover we have identities*

$$\partial^2 = 0, \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0, \qquad \bar{\partial}^2 = 0.$$

PROOF. In the context of (54), each term $(\partial f_{IJ}) \wedge dz_I \wedge d\bar{z}_J$ belongs to $\Omega^{p+1,q}(U)$, and each term $(\bar{\partial} f_{IJ}) \wedge dz_I \wedge d\bar{z}_J$ belongs to $\Omega^{p,q+1}(U)$. Since M is covered by coordinate charts U such as those used in (54), this shows that $d(\Omega^{p,q}(M)) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$. So we can (and must) simply define ∂ and $\bar{\partial}$ to be, respectively, the first and second components of d with respect to this splitting.

To see that the identities at the end of the proposition are satisfied, note that if $\omega \in \Omega^{p,q}(M)$ then

$$0 = dd\omega = (\partial + \bar{\partial})(\partial\omega + \bar{\partial}\omega) = \partial\partial\omega + (\partial\bar{\partial} + \bar{\partial}\partial)\omega + \bar{\partial}\bar{\partial}\omega.$$

Now $\partial\partial\omega \in \Omega^{p+2,q}(M)$, $(\partial\bar{\partial} + \bar{\partial}\partial)\omega \in \Omega^{p+1,q+1}(M)$, and $\bar{\partial}\bar{\partial}\omega \in \Omega^{p,q+2}(M)$, so since these three spaces are complementary the fact that $dd\omega = 0$ implies that all three of $\partial\partial\omega, (\partial\bar{\partial} + \bar{\partial}\partial)\omega, \bar{\partial}\bar{\partial}\omega$ must be zero. □

We emphasize that if (M, J) were only almost complex we would not be able to conclude that $d(\Omega^{1,0}(M)) \subset \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$. In fact, the famous Newlander-Nirenberg theorem can be phrased as saying that this latter condition holds if and only if there exists a complex manifold structure on M with J as its associated almost complex structure.

Since the operator $\bar{\partial}$ on forms from Corollary 2.3.4 satisfies $\bar{\partial}^2 = 0$, we can make the following definition:

DEFINITION 2.3.5. If M is a complex manifold and $p, q \in \mathbb{N}$, the (p,q)-*Dolbeault cohomology* is the complex vector space

$$H^{p,q}_{\bar{\partial}}(M) = \frac{\ker(\bar{\partial} \colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\operatorname{Im}(\bar{\partial} \colon \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

The **Hodge numbers** of a complex manifold (M, J) are the numbers $h^{p,q}(M)$ given by

$$h^{p,q}(M) = \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(M).$$

Of course, we could do a similar thing with the operator ∂, giving groups $H^{p,q}_{\partial}(M)$, but this would not give any new information since it is easy to check that there is an identity $\overline{\partial\omega} = \bar{\partial}\bar{\omega}$ yielding an isomorphism $H^{p,q}_{\partial}(M) \cong H^{q,p}_{\bar{\partial}}(M)$.

I will mention in passing that, analogously to the de Rham theorem identifying de Rham cohomology with (for instance) Čech cohomology with coefficients in the constant sheaf $\mathbb{R}$, there is a Dolbeault theorem identifying $H^{p,q}_{\bar{\partial}}(M)$ with the cohomology $H^q(M, \Omega^p_{hol})$ where Ω^p_{hol} is the sheaf of holomorphic p-forms on M (so local sections are given in our language by $(p,0)$-forms ω satisfying $\bar{\partial}\omega = 0$).

2.4. The Dolbeault Laplacians

Let M be a compact complex manifold of complex dimension n with associated almost complex structure J, and suppose moreover that M is equipped with a Riemannian metric g that is compatible with J. (In this case we call (M,J,g) a Hermitian manifold, dropping the modifier "almost" since J arises form a genuine complex manifold structure.) By (10) and Proposition 2.1.2, the complexified exterior derivative $d\colon \Omega^k(M)_{\mathbb{C}} \to \Omega^{k+1}(M)_{\mathbb{C}}$ has an adjoint $d^*\colon \Omega^{k+1}(M)_{\mathbb{C}} \to \Omega^k(M)_{\mathbb{C}}$ with respect to the inner product $\langle\alpha,\beta\rangle = \int_M \alpha\wedge\star\bar{\beta}$, given by $d^* = -\star d\star$. (The sign is simpler than in previous formulas because M automatically has even real dimension.) In view of Proposition 2.2.4 and Corollary 2.3.4, the action of d^* on any $\Omega^{p,q}(M)$ is given by

$$d^* = -\star\partial\star - \star\bar{\partial}\star \text{ where } -\star\partial\star\colon \Omega^{p,q}(M)\to\Omega^{p,q-1}(M) \text{ and } -\star\bar{\partial}\star\colon \Omega^{p,q}(M)\to\Omega^{p-1,q}(M).$$

PROPOSITION 2.4.1. *If we set $\bar{\partial}^* = -\star\partial\star$ and $\partial^* = -\star\bar{\partial}\star$, then for all $\alpha\in\Omega^{k-1}(M)_{\mathbb{C}}$ and $\beta\in\Omega^k(M)_{\mathbb{C}}$ we have*

$$\langle\bar{\partial}\alpha,\beta\rangle = \langle\alpha,\bar{\partial}^*\beta\rangle \qquad \langle\partial\alpha,\beta\rangle = \langle\alpha,\partial^*\beta\rangle.$$

PROOF. We will prove the first equation; the second follows by an identical argument (or just by complex conjugation of the first). By the sesquilinearity of $\langle\cdot,\cdot\rangle$ it suffices to prove the statements in the case that $\alpha\in\Omega^{p,k-1-p}(M)$ and $\beta\in\Omega^{r,k-r}(M)$ for some p,r. By Proposition 2.3.2, both sides of the equation $\langle\bar{\partial}\alpha,\beta\rangle = \langle\alpha,\bar{\partial}^*\beta\rangle$ will then automatically be zero if $p\neq r$. If $p=r$, note that $\partial\alpha\in\Omega^{p+1,k-1-p}(M)$ is orthogonal to $\beta\in\Omega^{p,k-p}(M)$, and likewise $\star\bar{\partial}\star\beta\in\Omega^{p-1,k-p}(M)$ is orthogonal to $\alpha\in\Omega^{p,k-1-p}(M)$. Hence

$$\langle\bar{\partial}\alpha,\beta\rangle = \langle d\alpha,\beta\rangle = \langle\alpha,d^*\beta\rangle = \langle\alpha,-\star\partial\star\beta\rangle,$$

as desired. □

Motivated by this, we can form the Dolbeault Laplacians

$$\Box_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*, \qquad \Box_{\partial} = \partial^*\partial + \partial\partial^*.$$

By construction, each $\bar{\partial}^*$ restricts as a map $\Omega^{p,q+1}(M)\to\Omega^{p,q}(M)$ and each ∂ restricts as a map $\Omega^{p+1,q}(M)\to\Omega^{p,q}(M)$, so $\Box_{\bar{\partial}}$ and $\Box_{\partial}$ both restrict as endomorphisms of $\Omega^{p,q}(M)$.

Just as with the original Hodge Laplacian, the calculation

$$\langle\Box_{\bar{\partial}}\alpha,\alpha\rangle = \langle\bar{\partial}^*\bar{\partial}\alpha,\alpha\rangle + \langle\bar{\partial}\bar{\partial}^*\alpha,\rangle = \langle\bar{\partial}\alpha,\bar{\partial}\alpha\rangle + \langle\bar{\partial}^*\alpha,\bar{\partial}^*\alpha\rangle$$

implies that $\ker(\Box_{\bar{\partial}}) = \ker(\bar{\partial})\cap\ker(\bar{\partial}^*)$. In particular if $\alpha\in\Omega^{p,q}(M)$ lies in $\ker(\Box_{\bar{\partial}})$ then α determines a class $[\alpha]$ in the Dolbeault cohomology $H^{p,q}_{\bar{\partial}}(M) = \ker(\bar{\partial})/\operatorname{Im}(\bar{\partial})$, and the complex version of the Hodge theorem asserts that this is an isomorphism. The proof of this will rely on much the same analysis as that of the ordinary Hodge Laplacian Δ, as we will establish that $\Box_{\bar{\partial}}$ is a very similar operator to Δ, and indeed is simply directly proportional to (the complexification of) Δ under an additional assumption on the metric. Explaining this will require the following digression.

2.4.1. The fundamental 2-form and the Kähler condition.

DEFINITION 2.4.2. The **fundamental 2-form** of a Hermitian manifold (M,J,g) is the element $\omega\in\Omega^2(M)$ defined by

$$\omega(v,w) = g(Jv,w).$$

(Note that ω is indeed alternating, since the J-compatibility and symmetry of g show that $\omega(w,v) = g(JJw,Jv) = -g(w,Jv) = -\omega(v,w)$.) Of course $\Omega^2(M)$ sits inside its complexification $\Omega^2(M)_{\mathbb{C}}$, all elements of which extend (via the restriction isomorphisms $\Lambda^2_{\mathbb{C}}T^*_mM_{\mathbb{C}} \cong (\Lambda^2T^*_mM)_{\mathbb{C}}$) uniquely to complex-bilinear alternating 2-forms on each of the complexified tangent spaces $T_mM_{\mathbb{C}}$;

we will continue to denote this complexified form by ω. Denoting by $\langle\cdot,\cdot\rangle$ the Hermitian inner products on the $T_mM_{\mathbb{C}}$ induced by the Riemannian metric g together with (43), we evidently have

$$\omega(v,w) = \langle Jv, \bar{w}\rangle \quad \text{for } v,w \in T_mM_{\mathbb{C}}.$$

So it follows that if U is an open set and $\{e_1,\dots,e_n\}$ is an orthonormal frame of vector fields for $T_{1,0}M|_U$ (*i.e.*, for each $m \in U$ $\{(e_1)_m,\dots,(e_n)_m\}$ is an orthonormal basis for $(T_{1,0}M)_m$ with respect to $\langle\cdot,\cdot\rangle$), with dual coframe $\{e^1,\dots,e^n\}$, then

$$\omega|_U = i\sum_{j=1}^{n} e^j \wedge \bar{e}^j. \tag{55}$$

In particular $\omega \in \Omega^{1,1}(U)$. For an alternative description, if $(z_1,\dots,z_n)\colon U \to \mathbb{C}^n$ is a local holomorphic coordinate chart for (M,J) and if we define functions $h_{jk}\colon U \to \mathbb{C}$ by $h_{jk} = \left\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right\rangle$ (so $h_{kj} = \bar{h}_{jk}$) then

$$\omega|_U = i\sum_{j,k} h_{jk}\, dz_j \wedge d\bar{z}_k. \tag{56}$$

(Elsewhere in the literature you may see a similar formula but with $\frac{i}{2}$ in place of i; the explanation is that our h_{jk} is one-half of what many authors use: if g is the standard Euclidean metric on $\mathbb{C}^n$ our definition gives $h_{jk} = \frac{1}{2}\delta_{jk}$.)

DEFINITION 2.4.3. A Hermitian manifold (M,J,g) is said to be a **Kähler manifold** if its associated fundamental 2-form ω satisfies $d\omega = 0$.

As mentioned in the footnote in Section 1.2.1, a general construction in Riemannian geometry ("exponential coordinates") allows one to choose a coordinate chart around any point in terms of which the Riemannian metric agrees with the standard Euclidean metric up to second order near the chosen point. If the Riemannian manifold also carries a complex structure, this construction cannot always be made compatible with that structure. One interpretation of the Kähler criterion is that it is precisely what is required for such "complex exponential coordinates" to exist.

PROPOSITION 2.4.4. *A Hermitian manifold (M,J,g) is Kähler if and only if for each point $m \in M$ there is a holomorphic coordinate chart $\phi = (z_1,\dots,z_n)\colon U \to \mathbb{C}^n$ for M such that $\phi(m) = \vec{0}$ and a constant C such that, for each $j,k \in \{1,\dots,n\}$ and $x \in U$,*

$$\left|\left\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right\rangle_x - \frac{1}{2}\delta_{jk}\right| \le C\|\phi(x)\|^2.$$

PROOF. Throughout the proof we denote $h_{jk}(x) = \left\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right\rangle_x$ for any given local holomorphic chart $(z_1,\dots,z_n)\colon U \to \mathbb{C}^n$ for M. The reverse implication is straightforward: its hypothesis implies that for any $m \in M$ we can find a local holomorphic chart for which $(dh_{jk})_m = 0$, and so differentiating (56) shows that the fundamental 2-form ω satisfies $(d\omega)_m = 0$. But m is arbitrary, so $d\omega = 0$.

Conversely suppose that $d\omega = 0$. By postcomposing an arbitrary coordinate chart with a suitable complex-linear map (namely the basis change matrix corresponding to applying the Gram-Schmidt process to the basis $\{\frac{\partial}{\partial z_j}\}$ for $T_{1,0}M_m$ associated to an initial coordinate chart), we can find a local holomorphic chart $(z_1,\dots,z_n)\colon U \to \mathbb{C}^n$ mapping m to $\vec{0}$ for which $h_{jk}(m) = \frac{1}{2}\delta_{jk}$. Our task then is to modify this to a different local coordinate chart that additionally has the property that

$(dh_{jk})_m = 0$. We will take the coordinates of this chart to be given by

$$w_j = z_j + \sum_{kl} a_{jkl} z_k z_l \tag{57}$$

for some complex numbers a_{jkl} to be determined; any such choice will evidently have $(dw_j)_m = (dz_j)_m$ and hence $\langle \frac{\partial}{\partial w_j}, \frac{\partial}{\partial w_k} \rangle_m = \frac{1}{2}\delta_{jk}$. The chart will have the desired property provided that

$$\frac{i}{2}\sum_{j=1}^{n} dw_j \wedge d\bar{w}_j = \sum_{j,k} \big(h_{jk} + O(\|\vec{z}\|^2)\big) dz_j \wedge d\bar{z}_k,$$

where we use the standard notation $O(\|\vec{z}\|^2)$ for a generic quantity that is bounded above by a constant times $\sum_j |z_j|^2$ (indeed, one has $\langle \frac{\partial}{\partial w_j}, \frac{\partial}{\partial w_k} \rangle = \omega(-i\frac{\partial}{\partial w_j}, \frac{\partial}{\partial \bar{w}_k})$ and by (56) the indicated condition implies that $\omega(-i\frac{\partial}{\partial w_j}, \frac{\partial}{\partial \bar{w}_k}) = \frac{1}{2}\delta_{jk} + O(\|\vec{z}\|^2) = \frac{1}{2}\delta_{jk} + O(\|\vec{w}\|^2)$).

Now for w_j as in (57), one finds

$$\frac{i}{2}\sum_{j} dw_j \wedge d\bar{w}_j = \sum_{j,k}\left(\frac{1}{2}\delta_{j,k} + \sum_l (a_{klj} + a_{kjl}) z_l + \sum_l (\bar{a}_{jlk} + \bar{a}_{jkl})\bar{z}_l\right) dz_j \wedge d\bar{z}_k.$$

So the desired property holds provided that

$$\frac{\partial h_{jk}}{\partial z_l} = a_{klj} + a_{kjl} \qquad \frac{\partial h_{jk}}{\partial \bar{z}_l} = \bar{a}_{jlk} + \bar{a}_{jkl} \quad \text{for all } j,k,l;$$

in fact the second set of equations follows from the first by complex conjugation since $h_{jk} = \bar{h}_{kj}$. So we just need to check that we can choose complex numbers a_{jkl} such that $\frac{\partial h_{jk}}{\partial z_l} = a_{klj} + a_{kjl}$; the relevant obstruction here is that this would require $\frac{\partial h_{jk}}{\partial z_l}$ to be equal to $\frac{\partial h_{lk}}{\partial z_j}$. But $\frac{\partial h_{jk}}{\partial z_l} - \frac{\partial h_{lk}}{\partial z_j}$ is the coefficient of $dz_l \wedge dz_j \wedge d\bar{z}_k$ in $d\omega$, so the assumption that (M, J, g) is Kähler implies that $\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j}$. So we can satisfy the requirement by setting $a_{kjl} = \frac{1}{2}\frac{\partial h_{jk}}{\partial z_l} = \frac{1}{2}\frac{\partial h_{lk}}{\partial z_j}$ and defining the w_j as in (57). □

COROLLARY 2.4.5. *Let (M, J, g) be a Kähler manifold. Then for any $m \in M$ there is a neighborhood U of m and an orthonormal frame of vector fields $\{e_1, \ldots, e_n\}$ for $T_{1,0}M|_U$ such that the dual coframe $\{e^1, \ldots, e^n\}$ satisfies $(de^j)_m = 0$ for each j.*

PROOF. Let $(z_1, \ldots, z_n)\colon U \to \mathbb{C}^n$ be a coordinate chart as in Proposition 2.4.4 and apply the Gram-Schmidt process pointwise to the frame $\{\frac{\partial}{\partial z_1}, \ldots \frac{\partial}{\partial z_n}\}$ to obtain $\{e_1, \ldots, e_n\}$. The fact that $\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \rangle = \frac{1}{2}\delta_{jk}$ is easily seen to imply that, for each j, $e_j = \sqrt{2}\frac{\partial}{\partial z_j} + O(\|\vec{z}\|^2)$, and hence also that $e^j = dz_j + O(\|\vec{z}\|^2)$. But then (since the coordinate chart maps m to the origin) $(de^j)_m = (ddz_j)_m = 0$. □

2.4.2. Interior and exterior products.

Here are two simple types of operations on the (complex, for definiteness) exterior algebra of a vector space:

DEFINITION 2.4.6. Let W be a vector space over $\mathbb{C}$, let $v \in W$, and let $\alpha \in W^*$. We define:

- The *interior product with v* to be the operation $\iota_v\colon \Lambda^k_{\mathbb{C}} W^* \to \Lambda^{k-1}_{\mathbb{C}} W^*$ given by

$$(\iota_v \theta)(w_1, \ldots, w_{k-1}) = \theta(v, w_1, \ldots, w_{k-1}).$$

- The *exterior product with α* to be the operation $\epsilon_\alpha\colon \Lambda^k_{\mathbb{C}} W^* \to \Lambda^{k+1}_{\mathbb{C}} W^*$ given by

$$\epsilon_\alpha \theta = \alpha \wedge \theta.$$

PROPOSITION 2.4.7. *Let $\{v_1, \ldots, v_m\}$ be a basis for the complex vector space W, with dual basis $\{v^1, \ldots, v^m\}$. Then*

$$\epsilon_{v^j}\iota_{v_k} + \iota_{v_k}\epsilon_{v^j} = \begin{cases} I & j = k \\ 0 & j \neq k. \end{cases}$$

PROOF. For $L = (l_1, \ldots, l_r)$ let us write $v^L = v^{l_1} \wedge \cdots \wedge v^{l_r}$. It suffices to consider the action of $\epsilon_{v^j}\iota_{v_k} + \iota_{v_k}\epsilon_{v^j}$ on elements of the form v^L or $v^k \wedge v^L$ where we always assume that $k \notin \{l_1, \ldots, l_r\}$. We have $\iota_{v_k} v^L = 0$, while $\iota_{v_k}(v^k \wedge v^L) = v^L$. For $j \neq k$ we see that

$$(\epsilon_{v^j}\iota_{v_k} + \iota_{v_k}\epsilon_{v^j})(v^L) = 0 + 0 = 0,$$

while

$$(\epsilon_{v^j}\iota_{v_k} + \iota_{v_k}\epsilon_{v^j})(v^k \wedge v^L) = v^j \wedge v^L + \iota_{v_k}(v^j \wedge v^k \wedge v^L) = v^j \wedge v^L - v^j \wedge v^L = 0.$$

Meanwhile

$$(\epsilon_{v^k}\iota_{v_k} + \iota_{v_k}\epsilon_{v^k})(v^L) = 0 + \iota_{v_k}(v^k \wedge v^L) = v^L$$

and

$$(\epsilon_{v^k}\iota_{v_k} + \iota_{v_k}\epsilon_{v^k})(v^k \wedge v^L) = v^k \wedge v^L + 0$$

so $\epsilon_{v^k}\iota_{v_k} + \iota_{v_k}\epsilon_{v^k}$ is the identity. □

Now let us assume that our complex vector space W is the complexification $W = V_{\mathbb{C}}$ of an oriented real inner product space $(V, (\cdot,\cdot), \mathfrak{o})$, so that we have an induced Hermitian inner product $\langle\cdot,\cdot\rangle$ on W and a complexified Hodge star operator $\star\colon \Lambda^k_{\mathbb{C}} W^* \to \Lambda^{m-k}_{\mathbb{C}} W^*$.

PROPOSITION 2.4.8. *For each $v \in V_{\mathbb{C}}$ we have $\star \circ \iota_v = (-1)^{k-1}\epsilon_{\langle\cdot,\bar{v}\rangle} \circ \star$ as maps defined on $\Lambda^k_{\mathbb{C}} V^*_{\mathbb{C}}$.*

PROOF. Both sides are complex linear in v, and both of the maps are linear, so it suffices to consider the effects of the maps on wedge products $v^1 \wedge \cdots \wedge v^k$ where the v^j are the dual basis elements to an orthonormal basis $\{v_1, \ldots, v_m\}$ having $v_1 \wedge \cdots \wedge v_m = \omega^{\mathbb{C}}_V$, and where v is equal to either v_1 or to v_{k+1}.

If $v = v_{k+1}$, then $\iota_v(v^1 \wedge \cdots \wedge v^k) = 0$ while

$$\star(v^1 \wedge \cdots \wedge v^k) = \langle\cdot,\bar{v}\rangle \wedge \cdots \wedge \langle\cdot,\bar{v}_m\rangle$$

and hence $\epsilon_{\langle\cdot,\bar{v}\rangle}(\star(v^1 \wedge \cdots \wedge v^k)) = 0$. So in this case both maps in the proposition evaluate as zero on our element.

If $v = v_1$, then

$$\begin{aligned} \star \circ \iota_v(v^1 \wedge \cdots \wedge v^k) &= \star(v^2 \wedge \cdots \wedge v^k) \\ &= (-1)^{k-1}\langle\cdot,\bar{v}_1\rangle \wedge \langle\cdot,\bar{v}_{k+1}\rangle \wedge \cdots \langle\cdot,\bar{v}_m\rangle \\ &= (-1)^{k-1}\epsilon_{\langle\cdot,\bar{v}\rangle} \star (v^1 \wedge \cdots \wedge v^k). \end{aligned}$$

Here in the second equality we have used Corollary 2.1.6 and the fact that

$$v_2 \wedge \cdots \wedge v_k \wedge v_1 \wedge v_{k+1} \wedge \cdots \wedge v_m = (-1)^{k-1} v_1 \wedge \cdots \wedge v_m = (-1)^{k-1}\omega^{\mathbb{C}}_V.$$

□

COROLLARY 2.4.9. *If $(V, (\cdot,\cdot), \mathfrak{o})$ is an even-dimensional oriented inner product space and if $\{v_1, \ldots, v_m\}$ is an orthonormal basis for $V_{\mathbb{C}}$ with dual basis $\{v^1, \ldots, v^m\}$, then*

$$\star\epsilon_{v^1}\star = \iota_{\bar{v}_1}.$$

PROOF. Applying the preceding proposition with $v = \bar{v}_1$ (and hence $\langle\cdot,\bar{v}\rangle = v^1$) gives $\epsilon_{v^1}\star = (-1)^{k-1}\star\iota_{\bar{v}_1}$ as functions on $\Lambda^k_{\mathbb{C}} V^*_{\mathbb{C}}$. So since (using the even-dimensionality of V) $\star\star$ acts as $(-1)^{k-1}$ on $\Lambda^{k-1}_{\mathbb{C}} V^*_{\mathbb{C}}$, the result follows by applying $\star$ to both sides. □

2.4.3. The Kähler identities. As we will see in this section, the Kähler condition on a Hermitian manifold (M,J,g) leads to certain surprising identities involving the fundamental 2-form ω and the operators $\partial,\partial^*,\bar\partial,\bar\partial^*$; we will also see that in the non-Kähler case these identities still hold "modulo lower-order terms."

For the next several paragraphs, assume that we have an open subset $U\subset M$ and an orthonormal frame $\{e_1,\dots,e_n\}$ for $T_{1,0}M|_U$; thus $\omega|_U = i\sum_j e^j\wedge\bar e^j$. For $m\in U$, we will say that this frame is *adapted at m* if $(de_j)_m=0$ for all j. So Corollary 2.4.5 asserts that, if M is Kähler, then for each $m\in M$ there exists a neighborhood U of m and an orthonormal frame for $T_{1,0}M|_U$ which is adapted at m.

We will write $e^I=e^{i_1}\wedge\cdots\wedge e^{i_p}$ and $\bar e^J=\bar e^{j_1}\wedge\cdots\wedge\bar e^{j_q}$ for tuples $I=(i_1,\dots,i_p)$, $J=(j_1,\dots,j_q)$. Here is a seemingly technical but hopefully conceptually-simple definition.

DEFINITION 2.4.10. For any natural numbers k,l, a *zeroth-order operator* $T\colon\Omega^k(U)_{\mathbb C}\to\Omega^l(U)_{\mathbb C}$ is a linear map for which there exist smooth functions p_{IJKL} on U such that T is given by the formula

$$T\left(\sum_{I,J} f_{IJ}e^I\wedge\bar e^J\right)=\sum_{I,J,K,L}p_{IJKL}f_{IJ}e^K\wedge\bar e^L.$$

For $m\in U$, we say that a zeroth-order operator T *vanishes at m* if the functions p_{IJKL} all obey $p_{IJKL}(m)=0$ (or equivalently, if $(T\alpha)_m=0$ for all $\alpha\in\Omega^k(U)_{\mathbb C}$).

Similarly, a *first-order operator* $T\colon\Omega^k(U)_{\mathbb C}\to\Omega^l(U)_{\mathbb C}$ is a linear map for which there exist smooth functions $p_{IJKL},q_{IJKL,r},\tilde q_{IJKL,r}$ on U such that T is given by the formula

$$T\left(\sum_{I,J} f_{IJ}e^I\wedge\bar e^J\right)=\sum_{I,J,K,L}\left(q_{IJKL,r}\nabla_{e_r}f_{IJ}+\tilde q_{IJKL,r}\nabla_{\bar e_r}f_{IJ}+p_{I,J,K,L}f_{IJ}\right)e^K\wedge\bar e^L.$$

For $j=1,\dots,n$ let us define operators $\partial_j,\bar\partial_j\colon\Omega^k(U)_{\mathbb C}\to\Omega^k(U)_{\mathbb C}$ by:

$$\partial_j\left(\sum_{I,J} f_{IJ}e^I\wedge\bar e^J\right)=\sum_{I,J}(\nabla_{e_j}f_{IJ})e^I\wedge\bar e^J,\quad \bar\partial_j\left(\sum_{I,J} f_{IJ}e^I\wedge\bar e^J\right)=\sum_{I,J}(\nabla_{\bar e_j}f_{IJ})e^I\wedge\bar e^J.$$

So both ∂_j and $\bar\partial_j$ are first-order operators. Note also that

$$\partial(f\,e^I\wedge\bar e^J)=\sum_j\partial_j(\epsilon_{e^j}(f\,e^I\wedge\bar e^J))+f\,\partial(e^I\wedge\bar e^J),$$

and similarly for $\bar\partial$. Hence $\partial-\sum_j\partial_j\circ\epsilon_{e^j}$ and $\bar\partial-\sum_j\bar\partial_j\circ\epsilon_{\bar e^j}$ are zeroth order operators, and these operators vanish at m if the frame $\{e_1,\dots,e_n\}$ is adapted at m.

Note that the operators ∂_j commute with any operator that both is linear over $C^\infty(M,\mathbb C)$ and sends each $e^I\wedge\bar e^J$ to $ce^K\wedge\bar e^L$ for some $c\in\mathbb C$ and multi-indices K, L. Such operators include the Hodge star operator (by Corollary 2.1.6, since $\langle\cdot,\bar e_i\rangle=\bar e^i$ and $\langle\cdot,e_i\rangle=e^i$) and also the exterior products $\epsilon_{e_j},\epsilon_{\bar e_j}$ and the interior products $\iota_{e_j},\iota_{\bar e_j}$. From this we can quickly observe:

PROPOSITION 2.4.11. *The operators* $\bar\partial^*\colon\Omega^{p,q}(U)\to\Omega^{p,q-1}(U)$ *and* $\partial^*\colon\Omega^{p,q}(U)\to\Omega^{p-1,q}(U)$ *are given by*

$$\bar\partial^*=-\sum_j(\partial_j\circ\iota_{\bar e_j})+T,\qquad \partial^*=-\sum_j\bar\partial_j\circ\iota_{e_j}+T'$$

where T and T' are zeroth-order operators. Moreover T and T' both vanish at m if the frame $\{e_1,\dots,e_n\}$ is adapted at m.

PROOF. We have $\bar{\partial}^* = -\star\partial\star$, and we have shown above that ∂ differs from $\sum_j \partial_j \circ \epsilon_{e^j}$ by a zeroth order operator. So the result for $\bar{\partial}^*$ follows directly from the facts that $\star\partial_j = \partial_j\star$ and that, by Corollary 2.4.9, $\star\circ\epsilon_{e^j}\circ\star = \iota_{\bar{e}_j}$. The argument for $\partial^* = -\star\bar{\partial}\star$ is identical. □

THEOREM 2.4.12 (Kähler identities). *Define an operator $L\colon \Omega^k(U)_{\mathbb{C}} \to \Omega^{k+2}(U)_{\mathbb{C}}$ by $L(\alpha) = \omega\wedge\alpha$. Then*

$$L\bar{\partial}^* - \bar{\partial}^* L = -i\partial - S \qquad L\partial^* - \partial^* L = i\bar{\partial} - S'$$

where S, S' are zeroth-order operators. Moreover if (M, J, g) is Kähler then $S = S' = 0$.

PROOF. By (55) we have $L = i\sum_k \epsilon_{e^k}\circ\epsilon_{\bar{e}^k}$, which in particular implies that L commutes with the operators $\partial_j, \bar{\partial}_j$. Let T, T' be the zeroth-order operators from Proposition 2.4.11. We then have, using Proposition 2.4.7 in the second and third equalities:

$$L\circ(\bar{\partial}^* - T) - (\bar{\partial}^* - T)\circ L = -i\left(\sum_{j,k}\left(\epsilon_{e_k}\epsilon_{\bar{e}_k}\partial_j\iota_{\bar{e}_j} - \partial_j\iota_{\bar{e}_j}\epsilon_{e_k}\epsilon_{\bar{e}_k}\right)\right) \quad = -i\sum_{j,k}\partial_j\circ\left(\epsilon_{e_k}\epsilon_{\bar{e}_k}\iota_{\bar{e}_j} + \epsilon_{e_k}\iota_{\bar{e}_j}\epsilon_{\bar{e}_k}\right)$$
$$= -i\sum_{j,k}\partial_j\epsilon_{e_k}\delta_{jk}I = -i\sum_j \partial_j\epsilon_{e_j}$$

(where δ_{jk} is the Kronecker symbol). This right-hand side differs from $-i\partial$ by a zeroth-order operator which vanishes at m if our frame is adapted at m. So since $-LT + TL$ is likewise a zeroth order operator which vanishes at m if our frame is adapted at m, this proves that the difference $S = -i\partial^* - (L\bar{\partial}^* - \bar{\partial}^* L)$ is zeroth-order operator which vanishes at m if our frame is adapted at m.

To see that $S = 0$ if (M, J, g) is Kähler, we have shown that, for any $\alpha \in \Omega^k(U)_{\mathbb{C}}$ and any $m \in U$ we have $(L\bar{\partial}^*\alpha - \bar{\partial}^* L\alpha)_m = (-i\partial\alpha)_m + (S\alpha)_m$, and moreover if we choose a frame which is adapted at m (as is possible by Corollary 2.4.5) then $(S\alpha)_m = 0$. But the terms $(L\bar{\partial}^*\alpha - \bar{\partial}^* L\alpha)_m$ and $(-i\partial\alpha)_m$ are independent of which frame we choose, and so by choosing an adapted frame we learn that in fact $(L\bar{\partial}^*\alpha - \bar{\partial}^* L\alpha)_m = (-i\partial\alpha)_m$. This holds for all m, so indeed $L\bar{\partial}^* - \bar{\partial}^*\alpha = -i\partial$ in the Kähler case.

This completes the proof of the statements concerning the operators $\bar{\partial}^*$ and S; the statements concerning ∂^* and S' may be proven either by an identical argument or by taking the complex conjugate of the identities that we have already proven. □

We can now work more globally. Let us say that a $\mathbb{C}$-linear map $T\colon \Omega^k(M)_{\mathbb{C}} \to \Omega^l(M)_{\mathbb{C}}$ is a zeroth-order operator (resp. a first-order operator) if there is a cover of M by open sets U equipped with local orthonormal frames for $T_{1,0}M|_U$ such that whenever $supp(\alpha) \subset U$ we also have $supp(T\alpha) \subset U$, and such that there is a zeroth-order (resp. first-order) operator $T_U\colon \Omega^k(U)_{\mathbb{C}} \to \Omega^l(U)_{\mathbb{C}}$ such that $(T\alpha)|_U = T_U\alpha$ whenever $supp(\alpha) \subset U$. (In other words, a first-order operator is one which is locally given by the action of a first-order partial differential operator on the coefficient functions of a k-form, and likewise for a zeroth-order operator.) It is easy to see that the composition of two zeroth-order operators is a zeroth-order operator, and likewise the composition of a first-order operator and a zeroth-order operator (in either order) is a first-order operator. Note that in our convention a zeroth-order operator is a special case of a first-order operator (so perhaps we should say "at most first-order" instead of "first-order"). Theorem 2.4.12 then immediately implies:

COROLLARY 2.4.13. *Defining $L\colon \Omega^k(M)_{\mathbb{C}} \to \Omega^{k+2}(M)_{\mathbb{C}}$ we have $\partial = i(L\bar{\partial}^* - \bar{\partial}^* L) + S$ and $\bar{\partial} = -i(L\partial^* - \partial^* L) + S'$ where S, S' are zeroth-order operators. If (M, J, g) is Kähler then $S = S' = 0$.*

Recall our Laplacians

$$\Delta_{\mathbb{C}} = d^*d + dd^*, \quad \Box_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*, \quad \Box_{\partial} = \partial^*\partial + \partial\partial^*.$$

The Kähler identities show that these operators are closely related:

THEOREM 2.4.14. *If (M, J, g) is a Hermitian manifold, there are first-order operators T, T' such that*

$$\Box_{\bar\partial} = \frac{1}{2}\Delta_{\mathbb{C}} + T \quad \textit{and} \quad \Box_{\partial} = \frac{1}{2}\Delta_{\mathbb{C}} + T'.$$

Moreover if (M, J, g) is Kähler then $T = T' = 0$ and so $\Box_{\bar\partial} = \Box_{\partial} = \frac{1}{2}\Delta_{\mathbb{C}}$.

PROOF. With notation as in Corollary 2.4.13, we have:

$$\begin{aligned}\Box_{\bar\partial} &= \bar\partial^*\bar\partial + \bar\partial\bar\partial^* = -i\left(\bar\partial^*(L\partial^* - \partial^* L) + (L\partial^* - \partial^* L)\bar\partial^*\right) + (\bar\partial^* S' + S'\bar\partial^*)\\ &= i\left(\bar\partial^*\partial^* L - \bar\partial^* L\partial^* + \partial^* L\bar\partial^* - L\partial^*\bar\partial^*\right) + (\bar\partial^* S' + S'\bar\partial^*).\end{aligned}$$

Similarly

$$\begin{aligned}\Box_{\partial} &= \partial^*\partial + \partial\partial^* = i\left(\partial^*(L\bar\partial^* - \bar\partial^* L) + (L\bar\partial^* - \bar\partial^* L)\partial^*\right) + (\partial^* S + S\partial^*)\\ &= i\left(-\partial^*\bar\partial^* L - \bar\partial^* L\partial^* + \partial^* L\bar\partial^* + L\bar\partial^*\partial^*\right) + (\partial^* S + S\partial^*).\end{aligned}$$

Now the fact that $\partial\bar\partial + \bar\partial\partial = 0$ implies that $\bar\partial^*\partial^* = -\partial^*\bar\partial^*$, so the expressions multiplying i in the above formulas for $\Box_\partial, \Box_{\bar\partial}$ are equal to each other. Thus:

$\Box_{\bar\partial} - \Box_{\partial} = (\bar\partial^* S' + S'\bar\partial^*) - (\partial^* S + S\partial^*)$ is a first-order operator, and $\Box_{\bar\partial} = \Box_\partial$ if (M, J, g) is Kähler.

To compare $\Box_\partial$ and $\Box_{\bar\partial}$ to $\Delta_{\mathbb{C}}$, note that

$$\begin{aligned}\Delta_{\mathbb{C}} &= (\partial + \bar\partial)^*(\partial + \bar\partial) + (\partial + \bar\partial)(\partial + \bar\partial)^*\\ &= \Box_\partial + \Box_{\bar\partial} + (\partial^*\bar\partial + \bar\partial\partial^*) + (\bar\partial^*\partial + \partial\bar\partial^*).\end{aligned}$$

Now using Corollary 2.4.13 again,

$$\begin{aligned}\partial^*\bar\partial + \bar\partial\partial^* &= -i\,(\partial^*(L\partial^* - \partial^* L) + (L\partial^* - \partial^* L)\partial^*) + (\partial^* S' + S'\partial^*)\\ &= \partial^* S' + S'\partial^*\end{aligned}$$

since $\partial^*\partial^* = 0$. So $\partial^*\bar\partial + \bar\partial\partial^*$ is a first-order operator, and is zero in the Kähler case. An essentially identical calculation shows that $\bar\partial^*\partial + \partial\bar\partial^*$ is a first-order operator that vanishes in the Kähler case.

Thus $\Delta_{\mathbb{C}} - \Box_{\bar\partial} - \Box_\partial$ is a first-order operator that vanishes in the case that (M, J, g) is Kähler. Together with the fact that $\Box_{\bar\partial} - \Box_\partial$ is a first-order operator that vanishes in the case that (M, J, g) is Kähler this proves the result. □

2.5. The complex Hodge Theorem

Theorem 2.4.14 allows us to prove statements for the Dolbeault Laplacians $\Box_\partial$ and $\Box_{\bar\partial}$ along exactly the same lines as we proved them for the Hodge Laplacian Δ, leading to versions of the Hodge theorem for $\Box_\partial$ and $\Box_{\bar\partial}$. We now proceed through the relevant arguments; the basic point here is that the precise nature of the first order part of Δ was irrelevant to our proof of Theorem 1.1.12. The fact that $2\Box_{\bar\partial}$ and $2\Box_\partial$ each differ from ther complexification $\Delta_{\mathbb{C}}$ of Δ by first-order operators implies that, for any $\theta \in \Omega^k(U)_{\mathbb{C}}$, either of the equations $2\Box_\partial\alpha = \theta$ or $2\Box_{\bar\partial}\alpha = \theta$ may be written in local coordinates in the form (36). So if we define a weak solution to $\Box_{\bar\partial}\alpha = \theta$ to be a class-$W^{1,2}$ complexified form α satisfying $\langle\bar\partial\alpha, \bar\partial\eta\rangle + \langle\bar\partial^*\alpha, \bar\partial^*\eta\rangle = \langle\theta, \eta\rangle$ for all η, it follows exactly as in Theorem 1.6.6 that, if θ is of class $W^{m,2}$, then any weak solution to $\Box_{\bar\partial}\alpha = \theta$ is of class $W^{m+2,2}$. A similar statement applies to solutions to $\Box_\partial\alpha = \theta$.

If one now defines $B_{\bar\partial}(\alpha, \beta) = \langle\bar\partial\alpha, \bar\partial\beta\rangle + \langle\bar\partial^*\alpha, \bar\partial^*\beta\rangle$, just as in Lemma 1.6.7 we obtain a bound $B_{\bar\partial}(\alpha, \alpha) \geq A_1\|\alpha\|_{1,2} - A_2\langle\alpha, \alpha\rangle$ where $A_1, A_2 > 0$, in view of which setting $B^\lambda_{\bar\partial}(\alpha, \beta) = B_{\bar\partial}(\alpha, \beta) + \lambda\langle\alpha, \beta\rangle$ for any $\lambda \geq A_2$ defines an inner product on $W^{1,2}(\Omega^{p,q}(M))$ which is uniformly equivalent to the usual inner product. So just as in Corollary 1.6.10 the (complex version of the) Riesz Representation theorem gives, for any $\theta \in L^2(\Omega^{p,q}(M))$, a unique solution $\alpha_\theta \in W^{1,2}(\Omega^{p,q}(M))$ to the equation $B^\lambda_{\bar\partial}(\alpha_\theta, \cdot) = \langle \theta, \cdot \rangle$. As in (41), if we set $K\theta = \lambda \alpha_\theta$ then $K\colon L^2(\Omega^{p,q}(M)) \to L^2(\Omega^{p,q}(M))$ is a compact, self-adjoint operator such that $K\theta$ is a weak solution to the equation $\square_{\bar\partial}(K\theta) = \lambda(\theta - K\theta)$. Consequently $\mathrm{Im}(I-K) \subset \mathrm{Im}(\square_{\bar\partial}\colon W^{2,2}(\Omega^{p,q}(M)) \to L^2(\Omega^{p,q}(M)))$ and $\ker(I-K) = \ker(\square_{\bar\partial}\colon W^{2,2}(\Omega^{p,q}(M)) \to L^2(\Omega^{p,q}(M)))$.

So Proposition 1.6.13 shows that $\ker(\square_{\bar\partial}\colon W^{2,2}(\Omega^{p,q}(M)) \to L^2(\Omega^{p,q}(M)))$ is finite-dimensional and that its L^2-orthogonal complement is contained in $\mathrm{Im}(\square_{\bar\partial}\colon W^{2,2}(\Omega^{p,q}(M)) \to L^2(\Omega^{p,q}(M))$. Hence for any (smooth) element $\theta \in \Omega^{p,q}(M)$ which is L^2-orthogonal to $\ker(\square_{\bar\partial}\colon \Omega^{p,q}(M) \to \Omega^{p,q}(M))$ there is a class-$W^{2,2}$ solution α to $\square_{\bar\partial}\alpha = \theta$. But as noted earlier such a solution must be of class $W^{m+2,2}$ for all m and hence is smooth. Thus, analogously to Theorem 1.1.12:

THEOREM 2.5.1. *If (M,J,g) is a compact Hermitian manifold then for any p,q let $\mathscr{H}^{p,q}_{\bar\partial}(M) = \ker(\square_{\bar\partial}\colon \Omega^{p,q}(M) \to \Omega^{p,q}(M))$. Then $\mathscr{H}^{p,q}_{\bar\partial}(M)$ is finite-dimensional, and*

$$\mathscr{H}^{p,q}_{\bar\partial}(M)^\perp \subset \mathrm{Im}(\square_{\bar\partial}\colon \Omega^{p,q}(M) \to \Omega^{p,q}(M)).$$

As with Theorem 1.1.12 the reverse inequality is trivial: if $\alpha \in \Omega^{p,q}(M)$ and $\theta \in \ker(\square_\partial)$ then $\langle \square_{\bar\partial}\alpha, \phi\rangle = \langle \alpha, \square_{\bar\partial}\phi\rangle = 0$. One can then deduce the analogue of Theorem 1.1.10 along exactly the same lines as was done in Section 1.1.4: elements ϕ of $\mathscr{H}^{p,q}_{\bar\partial}(M)$ obey $\langle \bar\partial\phi, \bar\partial\phi\rangle + \langle \bar\partial^*\phi, \bar\partial^*\phi\rangle = \langle (\bar\partial^*\bar\partial + \bar\partial\bar\partial^*)\phi, \phi\rangle = 0$ and hence $\bar\partial\phi = \bar\partial^*\phi = 0$, so we obtain

$$\mathrm{Im}(\bar\partial) \oplus \mathrm{Im}(\bar\partial^*) \subset \mathscr{H}^{p,q}_{\bar\partial}(M)^\perp = \mathrm{Im}(\bar\partial^*\bar\partial + \bar\partial\bar\partial^*) \subset \mathrm{Im}(\bar\partial) \oplus \mathrm{Im}(\bar\partial^*),$$

so $\mathscr{H}^{p,q}_{\bar\partial}(M)^\perp = \mathrm{Im}(\bar\partial) \oplus \mathrm{Im}(\bar\partial^*)$ and we have an orthogonal direct sum decomposition $\Omega^{p,q}(M) = \mathscr{H}^{p,q}_{\bar\partial}(M) \oplus \mathrm{Im}(\bar\partial) \oplus \mathrm{Im}(\bar\partial^*)$, with $\ker(\bar\partial) = \mathscr{H}^{p,q}_{\bar\partial}(M) \oplus \mathrm{Im}(\bar\partial)$. Hence:

THEOREM 2.5.2 ($\bar\partial$ version of the Hodge theorem). *Let (M,J,g) be a compact Hermitian manifold. Then the map*

$$\begin{aligned} \mathscr{H}^{p,q}_{\bar\partial}(M) &\to H^{p,q}_{\bar\partial}(M) = \frac{\ker(\bar\partial\colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\mathrm{Im}(\bar\partial\colon \Omega^{p,q-1}(M) \to \Omega^{p,q}(M)} \\ \alpha &\mapsto [\alpha] \end{aligned}$$

is an isomorphism. In particular $H^{p,q}_{\bar\partial}(M)$ is finite-dimensional.

Of course, the exact same arguments show that we likewise have an isomorphism between $\mathscr{H}^{p,q}_{\partial}(M) = \ker(\square_\partial)$ and $H^{p,q}_\partial(M) = \frac{\ker\partial}{\mathrm{Im}\,\partial}$.

One consequence of this is an analogue of Poincaré duality which is a special case of Serre duality from algebraic geometry. Recall that complex conjugation gives a conjugate linear isomorphism from $H^{p,q}_{\bar\partial}(M)$ to $H^{q,p}_{\partial}(M)$. Meanwhile, recalling that $\partial^* = -\star\bar\partial\star$ and $\bar\partial^* = -\star\partial\star$, one sees that

$$\square_{\bar\partial} = -(\star\partial\star\bar\partial + \bar\partial\star\partial\star), \quad \square_\partial = -(\star\bar\partial\star\partial + \partial\star\bar\partial\star).$$

From this one finds that $\square_\partial(\star\alpha) = \star(\square_{\bar\partial}\alpha)$ for any α (since $\star\star$ acts on k-forms as $(-1)^k$ on our complex—and hence even-real-dimensional—manifold). Thus $\star$ defines an isomorphism $H^{p,q}_{\bar\partial}(M) \to H^{n-q,n-p}_{\partial}(M)$. Combined with the conjugation isomorphism mentioned earlier we obtain:

COROLLARY 2.5.3 (Serre duality). *If (M,J) is a compact complex manifold there is a conjugate-linear isomorphism $H^{p,q}_{\bar\partial}(M) \cong H^{n-p,n-q}_{\bar\partial}(M)$.*

Theorem 2.5.2 and Corollary 2.5.3 hold regardless of whether the compact Hermitian manifold (M,J,g) is Kähler; however in the Kähler case matters simplify significantly and we obtain other interesting results. For the rest of this section assume that (M,J,g) is a compact Kähler manifold. We then have $\Delta_{\mathbb{C}} = 2\Box_{\partial} = 2\Box_{\bar\partial}$, so taking the kernels of these operators gives (for $k = p+q$),

$$\mathscr{H}^k(M)_{\mathbb{C}} \cap \Omega^{p,q}(M) = \mathscr{H}^{p,q}_{\partial}(M) = \mathscr{H}^{p,q}_{\bar\partial}(M) \quad \text{if } (M,J,g) \text{ is Kähler.} \tag{58}$$

So Theorem 2.5.2 and its analogue for $H^{p,q}_{\partial}$ give an isomorphisms $H^{p,q}_{\bar\partial}(M) \cong H^{p,q}_{\partial}(M)$. Together with the conjugate-linear isomorphism $H^{p,q}_{\bar\partial}(M) \to H^{q,p}_{\partial}(M)$ and the Serre duality isomorphism of Corollary 2.5.3, this shows that the Hodge numbers of Definition 2.3.5 satisfy a four-fold symmetry

$$h^{p,q}(M) = h^{q,p}(M) = h^{n-p,n-q}(M) = h^{n-q,n-p}(M) \quad \text{if } (M,J,g) \text{ is Kähler.}$$

More interestingly, in the Kähler case there is a very straightforward relation between the Dolbeault and de Rham cohomologies.[2] Indeed, the fact that $\Delta_{\mathbb{C}} = 2\Box_{\bar\partial}$ shows that the map $\Delta_{\mathbb{C}} \colon \Omega^k(M)_{\mathbb{C}} \to \Omega^k(M)_{\mathbb{C}}$ restricts, for any p,q with $p+q=k$, to a map $\Delta_{\mathbb{C}} \colon \Omega^{p,q}(M) \to \Omega^{p,q}(M)$. In view of this, the decomposition $\Omega^k(M)_{\mathbb{C}} = \oplus_{p+q=k}\Omega^{p,q}(M)$ yields a decomposition

$$\ker(\Delta_{\mathbb{C}} \colon \Omega^k(M)_{\mathbb{C}} \to \Omega^k(M)_{\mathbb{C}}) = \bigoplus_{p+q=k} \mathscr{H}^k(M)_{\mathbb{C}} \cap \Omega^{p,q}(M),$$

and so by (58), Proposition 2.1.3, and Theorem 2.5.2 we obtain:

THEOREM 2.5.4. *If (M,J,g) is a compact Kähler manifold there is an isomorphism*

$$H^k(M;\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(M).$$

Thus the Betti numbers b_k of M are related to the Hodge numbers by

$$b_k(M) = \sum_{p+q=k} h^{p,q}(M) \quad \text{if } (M,J,g) \text{ is Kähler.}$$

COROLLARY 2.5.5. *If (M,J,g) is a compact Kähler manifold then, the odd-degree Betti numbers $b_{2j+1}(M)$ of M are even.*

PROOF. Recalling the conjugation symmetry $h^{p,q} = h^{q,p}$, we see that

$$b_{2j+1}(M) = \sum_{p=0}^{2j+1} h^{p,2j+1-p}(M) = \sum_{p=0}^{j}(h^{p,2j+1-p}(M) + h^{2j+1-p,p}(M)) = 2\sum_{p=0}^{j} h^{p,2j+1-p}(M).$$

□

Let us also mention another, simpler, topological consequence of the Kähler condition. If (M,J,g) is a compact Kähler manifold, then its fundamental 2-form ω is closed, and is given locally by $\omega|_U = i\sum_{j=1}^n e^k \wedge \bar e^k$ where $\{e^k\}$ is the dual basis to an orthonormal frame $\{e_1,\ldots,e_n\}$ for $T_{1,0}M|_U$. Consequently $\omega^n|_U = i^n n! e^1 \wedge \bar e^1 \wedge \cdots \wedge \bar e^n \wedge \bar e^n$; in particular ω^n is nowhere-vanishing (and is also real, since $\omega \in \Omega^2(M) \subset \Omega^2(M)_{\mathbb{C}}$). Thus ω^n induces an orientation on M, and (using this orientation to define integration of $2n$-forms) $\int_M \omega^n > 0$. Stokes' theorem then shows that ω^n is not exact. But since ω is closed, we have a cohomology class $[\omega] \in H^2(M)$, and the cup product in de Rham cohomology is given by wedge product, so that $[\omega]^n = [\omega^n]$. Thus:

[2]More generally, in the non-Kähler case one can use the notion of the spectral sequence of a filtered complex to obtain a spectral sequence from the Dolbeault to the de Rham cohomology, but this is not sufficient to calculate one in terms of the other unless one has additional information

COROLLARY 2.5.6. *If (M,J,g) is a compact n-complex-dimensional Kähler manifold then there is $a \in H^2(M)$ such that $a^n \neq 0$. In particular $H^2(M) \neq \{0\}$*

For instance this shows that smooth manifolds such as S^{2n} or $S^2 \times S^{2n}$ can never be made into Kähler manifolds for $n \geq 2$.

There are a variety of other interesting topological consequences of the Kähler condition that are beyond the scope of these notes; for instance the "Hard Lefschetz Theorem" (see [**GH**, p. 122]) asserts that the class $a = [\omega]$ from Corollary 2.5.6 has the property that, for each $i \in \{0, \dots, n\}$, the map

$$\begin{aligned} H^{n-i}(M) &\to H^{n+i}(M) \\ c &\mapsto a^i c \end{aligned}$$

is an isomorphism. The paper [**DGMS**] contains some additional topological results on Kähler manifolds from the viewpoint of rational homotopy theory.

We close this chapter on a more concrete note by considering some examples of complex manifolds which do or do not admit Kähler structures.

EXAMPLE 2.5.7. *If $n = 1$ and (M,J) is a compact complex 1-manifold (so its real dimension is 2), then for any choice of J-compatible metric g it will hold that (M,J,g) is Kähler, for the trivial reason that the fundamental 2-form ω, like any 2-form on a 2-manifold, is closed. Now a compact smooth 2-manifold M can be made complex if and only if it is orientable (the forward implication is trivial; for the backward implication, choose a Riemannian metric and define an almost complex structure J by the property that for each unit vector $v \in T_m M$ the pair $\{v, Jv\}$ is an oriented orthonormal basis for $T_m M$, then use the Newlander-Nirenberg theorem—which is a bit easier to prove in two real dimensions, see* [**MS1**, Theorem 4.16]*—to see that J is induced by a complex manifold structure on M). So the fact that complex 1-manifolds can always be made Kähler is, via Corollaries 2.5.5 and 2.5.6, consistent with the fact that every compact orientable surface has $H^2 \neq \{0\}$ and $\dim H^1$ even, as is familiar from the classification of surfaces. Indeed one could even say that we have given a proof of this latter fact that is independent of the classification of surfaces, though admittedly significantly simpler proofs than this are possible.*

EXAMPLE 2.5.8. *Moving up one dimension, consider the "Hopf surface," defined topologically as a quotient space*

$$H = \frac{\mathbb{C}^2 \setminus \{\vec{0}\}}{\vec{z} \sim 2\vec{z}}.$$

It is easy to give a complex-2-manifold atlas for H: note that the quotient projection $p\colon \mathbb{C}^2 \setminus \{\vec{0}\} \to H$ restricts to either $\{\vec{z} \in \mathbb{C}^2 | 1 < |\vec{z}| < 2\}$ or to $\{\vec{z} \in \mathbb{C}^2 | \frac{2}{3} < |\vec{z}| < \frac{4}{3}\}$ as an embedding of an open subset; let $U_0, U_1 \subset H$ be the respective images of these embeddings and let $\phi_0\colon U_0 \to \{1 < |\vec{z}| < 2\}$ and $\phi_1\colon U_1 \to \{\frac{2}{3} < |\vec{z}| < \frac{4}{3}\}$ be the inverses of the embeddings. Then $H = U_0 \cup U_1$, while $\phi_0(U_0 \cap U_1) = \{1 < |\vec{z}| < \frac{4}{3}\} \cup \{\frac{4}{3} < |\vec{z}| < 2\}$ and $\phi_1(U_0 \cap U_1) = \{1 < \vec{z} < \frac{4}{3}\} \cup \{\frac{2}{3} < |\vec{z}| < 1\}$. The transition function $\phi_0 \circ \phi_1^{-1}$ restricts to $\{1 < \vec{z} < \frac{4}{3}\}$ as the identity and to $\{\frac{2}{3} < |\vec{z}| < 1\}$ as multiplication by 2, so $\phi_0 \circ \phi_1^{-1}$ is holomorphic and we have indeed presented an atlas making H a complex-two-manifold.

Now there is an obvious diffeomorphism $F\colon S^3 \times S^1 \to H$: we can view S^1 as $\frac{[1,2]}{1 \sim 2}$ and S^3 as the unit sphere in C^2, and define $F(\vec{z}, r) = p(r\vec{z})$ where again $p\colon \mathbb{C}^2 \setminus \{\vec{0}\} \to H$ is the projection. So H is an example of a complex manifold that cannot *be given the structure of a Kähler manifold: indeed we have $b_1(H) = b_1(S^1 \times S^3) = 1$ and $b_2(H) = b_2(S^1 \times S^3) = 0$, so H violates both Corollary 2.5.5 and Corollary 2.5.6.*

EXAMPLE 2.5.9. *Consider the complex projective space*

$$\mathbb{C}P^n = \frac{\mathbb{C}^{n+1}\setminus\{\vec{0}\}}{\vec{z}\sim\lambda\vec{z} \text{ for } \lambda\in\mathbb{C}^*} = \frac{S^{2n+1}}{\vec{z}\sim e^{i\theta}\vec{z} \text{ for } e^{i\theta}\in S^1}.$$

Denote by $p\colon \mathbb{C}^{n+1}\setminus\{\vec{0}\}\to\mathbb{C}P^n$ and $\pi\colon S^{2n+1}\to\mathbb{C}P^n$ the quotient projections. One can define a complex manifold atlas on $\mathbb{C}P^n$ by covering it with open sets $U_k = p(\{\vec{z}\in\mathbb{C}^{n+1}|z_k\neq 0\})$, and defining charts $\phi_k\colon \mathbb{C}P^n\to\mathbb{C}^n$ by

$$\phi_k([z_0:\ldots:z_n]) = \left(\frac{z_0}{z_k},\ldots,\frac{z_{k-1}}{z_k},\frac{z_{k+1}}{z_k},\ldots,\frac{z_n}{z_k}\right).$$

It's easy to see that the transition functions $\phi_j\circ\phi_k^{-1}$ have components given by rational functions of the z_l, and so are holomorphic.

*The **Fubini-Study** metric is, roughly speaking, the Riemannian metric induced on $\mathbb{C}P^n$ using the quotient map $\pi\colon S^{2n+1}\to\mathbb{C}P^n$ and the standard metric on S^{2n+1} (this makes sense because the maps $\vec{z}\mapsto e^{i\theta}\vec{z}$ are isometries). More precisely, let $\langle\cdot,\cdot\rangle$ be the standard inner product on $\mathbb{C}^{n+1}$ (which we freely identify with $T_{\vec{z}}\mathbb{C}^{n+1}$ for any $\vec{z}\in\mathbb{C}^{n+1}$), so $Re\langle\cdot,\cdot\rangle$ is just the (real) dot product. Then, for $\vec{z}\in S^{2n+1}$ we have $T_{\vec{z}}S^{2n+1} = \{\vec{w}\in\mathbb{C}^{n+1}|Re\langle\vec{z},\vec{w}\rangle = 0\}$.*

Now the linearization $(\pi_*)_{\vec{z}}\colon T_{\vec{z}}S^{2n+1}\to T_{[\vec{z}]}\mathbb{C}P^n$ is surjective and has image spanned by $J\vec{z}$ where J is the standard complex structure on $\mathbb{C}^n$. This leads to an identification of $T_{[\vec{z}]}\mathbb{C}P^n$ with the (real) orthogonal complement of $\{J\vec{z}\}$ inside $T_{\vec{z}}S^{2n+1}$, i.e. *with the real orthogonal complement of $\{\vec{z},J\vec{z}\}$ in $\mathbb{C}^{n+1}$,* i.e. *with the* complex *orthogonal complement of $\vec{z}$ in $\mathbb{C}^{n+1}$. Thus we have an identification*

$$T_{[\vec{z}]}\mathbb{C}P^n \cong \{\vec{w}\in\mathbb{C}^{n+1}|\langle\vec{w},\vec{z}\rangle = 0\}. \tag{59}$$

Notice that the set on the right-hand side above is unchanged if we replace $\vec{z}$ by $e^{i\theta}\vec{z}$, i.e. *it only depends on the element $[\vec{z}]$ of $\mathbb{C}P^n$ and not on the choice of preimage $\vec{z}\in S^{2n+1}$.*

Under this identification, it is not hard to see that the (almost) complex structure J on $\mathbb{C}P^n$ given by simply acting on the complex subspace $T_{\vec{z}}\mathbb{C}P^n = \{\vec{w}\in\mathbb{C}^{n+1}|\langle\vec{w},\vec{z}\rangle = 0\}$ of $\mathbb{C}^{n+1}$ by multiplication by i is the same as the complex structure induced by our coordinate atlas for $\mathbb{C}P^n$. For instance, one can see this by noting first that the projection $p\colon \mathbb{C}^{n+1}\setminus\{\vec{0}\}\to\mathbb{C}P^n$ is holomorphic and then that the linearization of this projection at any $\vec{z}\in S^{2n+1}$ maps any element of $\{\vec{w}\in\mathbb{C}^{n+1}|\langle\vec{w},\vec{z}\rangle = 0\}$ to itself.

We can now define the Fubini-Study metric g by simply putting $g(\vec{v},\vec{w}) = Re\langle\vec{v},\vec{w}\rangle$ for $\vec{v},\vec{w}\in T_{\vec{z}}\mathbb{C}P^n = \{\vec{w}\in\mathbb{C}^{n+1}|\langle\vec{w},\vec{z}\rangle = 0\}\leq\mathbb{C}^{n+1}$. Thus the corresponding fundamental 2-form on $\mathbb{C}P^n$ is given by $\omega(\vec{v},\vec{w}) = Re\langle J\vec{v},\vec{w}\rangle$. Part (b) of the exercise after this example will confirm that this form is closed.

Assuming this result, we obtain a wealth of examples of Kähler manifolds by taking an arbitrary complex submanifold of $\mathbb{C}P^n$ (for instance, one could use the vanishing locus of a collection of homogeneous polynomials, provided that this locus is smooth), and endowing it with the restriction of the Fubini-Study metric; the Kähler condition is satisfied because the restriction of a closed form is closed.

EXERCISE 2.5.10. *Throughout this exercise p,π,ω are as in Example 2.5.9*

(a) Show that $\pi^*\omega$ is the restriction to S^{2n+1} of the two-form $\omega_0(\vec{v},\vec{w}) = Re\langle J\vec{v},\vec{w}\rangle$ on $\mathbb{C}^{n+1}$. (This is not completely trivial, because you have to account for the direction in $T_{\vec{z}}S^{2n+1}$ that is annihilated by π_*.)

(b) Show that $\pi^*\omega$ is exact, and deduce from this that ω is closed.

(c) Show that $p^*\omega = \frac{i}{2}\partial\bar{\partial}\log\left(\sum_{j=0}^n |z_j|^2\right)$. (It will help think of p as the composition $p = \pi\circ r$ where $r\colon \mathbb{C}^{n+1}\setminus\{\vec{0}\}\to S^{2n+1}$ is the radial projection, so that $p^*\omega = r^*(\pi^*\omega)$, and also to find a convenient α such that $d\alpha = \pi^*\omega$ and work with α instead of ω.) Deduce that, where $\phi_0\colon U_0\to\mathbb{C}^n$ is the local coordinate chart described at the start of Example 2.5.9, $\omega|_{U_0}$ may be expressed in this coordinate chart as $\frac{i}{2}\partial\bar{\partial}\log\left(1+\sum_{j=1}^n |z_j|^2\right)$.

CHAPTER 3

Pseudoholomorphic curves

3.1. Introducing the nonlinear Cauchy-Riemann equation

We now turn to a different geometrically-interesting elliptic PDE:

DEFINITION 3.1.1. Let (Σ, j) and (M, J) be almost complex manifolds. A differentiable map $u\colon \Sigma \to M$ is said to be (J, j)-holomorphic (or just J-holomorphic if j is understood) provided that, for all $p \in \Sigma$, we have the following equality of linear maps $T_p\Sigma \to T_{u(p)}M$:

$$u_* \circ j = J \circ u_*. \tag{60}$$

In other words, when we use the almost complex structures j and J to view $T_p\Sigma$ and $T_{u(p)}M$ as complex vector spaces, we are asking for the derivative of u at each $p \in \Sigma$ to be complex-linear and not just real-linear. The equation (60) will be called the Cauchy-Riemann equation. (As should soon be clear, if Σ and M are both open subsets of $\mathbb{C}$ it is equivalent to the usual Cauchy-Riemann equation for a holomorphic map).

We will focus entirely on the case that the (real) dimension of Σ is 2, and in applications a standard example of (Σ, j) (such as $\mathbb{C}P^1$ with its standard complex structure) will be specified. The Newlander-Nirenberg Theorem guarantees quite generally that j will then be an integrable complex structure, in the sense that Σ has the structure of a complex 1-manifold (a "curve" as in the title of the chapter) with j the almost complex structure induced by the complex manifold structure. As for M, Proposition 2.2.2 shows that the real dimension of M is an even number, say $2n$, but if $n \geq 2$ we generally cannot expect that J will arise from a complex manifold structure on M, as the following exercise shows. A "holomorphic curve" would be a map from a complex one-manifold to another complex manifold (or perhaps the image of that map); the prefix "pseudo" in the term "pseudoholomorphic curve" alludes to the fact that the codomains of our maps are typically not complex manifolds.

EXERCISE 3.1.2. *Construct an almost complex structure J on $\mathbb{R}^4$ such that, in terms of the decompositions $\Omega^k(\mathbb{R}^4)_{\mathbb{C}} = \oplus_{p+q=k}\Omega^{p,q}(\mathbb{R}^4)$ induced by J, the restriction of d to $\Omega^{1,0}(\mathbb{R}^4)$ does* not *have image contained in $\Omega^{2,0}(\mathbb{R}^4) \oplus \Omega^{1,1}(\mathbb{R}^4)$. Hence by Corollary 2.3.4 there does not exist a complex manifold structure on $\mathbb{R}^4$ with J as its induced almost complex structure. (*Suggestion*: Since it's not obvious how to write down a large family of matrices whose square is $-I$, a good way of constructing an almost complex structure is to choose a suitable complex basis $\{e_1, e_2\}$ for the space that you intend to be $T_{1,0}\mathbb{R}^4$ at each point (and then $T_{0,1}\mathbb{R}^4$ is obliged to be the complex conjugate of $T_{1,0}\mathbb{R}^4$ and J is determined by these data). Your basis will probably need to vary from point to point; do this in such a way that that the corresponding dual basis element e^1 obeys $de^1 \notin \Omega^{2,0}(\mathbb{R}^4) \oplus \Omega^{1,1}(\mathbb{R}^4)$.)*

Accordingly if $p \in \Sigma$ we can choose a complex coordinate chart $z = s + it\colon U \to \mathbb{C}$, mapping z to 0 (where s, t are real-valued). That j is the almost complex structure induced by these coordinates means that $j\partial_s = \partial_t$ and $j\partial_t = -\partial_s$ (where we use notation such as ∂_s as an abbreviation for the coordinate vector field $\frac{\partial}{\partial s}$). So applying both sides of (60) to ∂_s yields the equation $\frac{\partial u}{\partial t} = J\frac{\partial u}{\partial s}$, while applying both sides to ∂_t yields $-\frac{\partial u}{\partial s} = J\frac{\partial u}{\partial t}$. In fact these latter two equations are equivalent, as can be seen by applying J to both sides of either of them and using that $J^2 = -I$. So in terms of a local complex coordinate $s + it$ on Σ, the Cauchy-Riemann equation is equivalent to the equation

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0.$$

Let us now introduce local coordinates on M near $u(p)$ for $p \in \Sigma$. Note first of all that there exist bases for $T_{u(p)}\Sigma$ having the form $\{v_1, Jv_1, \dots, v_n, Jv_n\}$; to prove this quickly from prior results, based on Proposition 2.2.2 we may take a basis for $(T_{0,1}M)_p$ having the form $\{v_1 + iw_1, \dots, v_n + iw_n\}$ where each $v_k, w_k \in T_{u(p)}M \subset T_{u(p)}M_{\mathbb{C}}$, and then taking real and imaginary parts of the equation $J(v_k + iw_k) = -i(v_k + iw_k)$ shows that $Jv_k = w_k$ and $Jw_k = -v_k$; moreover it's not hard to see from the various parts of Proposition 2.2.2 that $\{v_1, w_1, \dots, v_n, w_n\}$ spans V. We can then take a neighborhood V of $u(p)$ and a local coordinate chart $\vec{z} = (\vec{x}, \vec{y})\colon V \to \mathbb{R}^{2n}$ mapping $u(p)$ to $\vec{0}$ and whose derivative at $u(p)$ maps $\{v_1, Jv_1, \dots, v_n, Jv_n\}$ to the standard basis $\{\partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_n}, \partial_{y_n}\}$ for (the tangent space to) $\mathbb{R}^{2n} = \{(\vec{x}, \vec{y}) | \vec{x}, \vec{y} \in \mathbb{R}^n\}$. Now identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via $(\vec{x}, \vec{y}) \mapsto \vec{x} + i\vec{y}$, the **standard complex structure** on $\mathbb{C}^n$ (given by componentwise multiplication by i) is the linear map $J_0\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by $J_0\partial_{x_k} = \partial_{y_k}$ and $J_0\partial_{y_k} = -\partial_{x_k}$. Thus our coordinate chart identifies $J|_{T_{u(p)}M}$ with the standard complex structure J_0 on $\mathbb{R}^{2n}$. However, at other points $x \in V$, we must expect that the coordinate chart $\vec{z}\colon V \to M$ will identify $J|_{T_xM}$ with some other complex structure from $\mathbb{R}^{2n}$; the most we can say is that this complex structure will vary smoothly with x.

So using both the complex coordinate chart $z = s + it\colon U \to \mathbb{C}$ to identify a neighborhood of p with a neighborhood of $0 \in C$, and the above coordinate chart $\vec{z}\colon V \to \mathbb{C}^n$ to identify a neighborhood of $u(p)$ with a neighborhood $W := \vec{z}(V)$ of $\vec{0} \in \mathbb{C}^n$, the Cauchy-Riemann equation becomes

$$\frac{\partial u}{\partial s} + J(u(z))\frac{\partial u}{\partial t} = 0 \tag{61}$$

where $J\colon W \to End_{\mathbb{R}}(\mathbb{C}^n)$ is a smooth function with $J(\vec{0}) = J_0$ and $J(\vec{z})^2 = -I$ for all $\vec{z} \in W$. We emphasize that the appearance of the term $J(u(z))$ makes this local version of the equation nonlinear. (If J were constant the local version would be linear, though from a global standpoint our equation (60) cannot considered to be linear for the simple reason that there is no vector space structure on the set of maps from a surface Σ into a manifold M.)

It is useful to rephrase this equation in term of the operators $\partial_z = \frac{\partial}{\partial z}$ and $\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}}$ from (53). Adapting these to functions with image in $\mathbb{C}^n$ rather than $\mathbb{C}$ by simply working component by component, and recalling that J_0 just denotes componentwise multiplication by i, for $u\colon \mathbb{C} \to \mathbb{C}^n$ we have

$$\partial_z u = \frac{1}{2}\left(\frac{\partial u}{\partial s} - J_0\frac{\partial u}{\partial t}\right) \qquad \partial_{\bar{z}} u = \frac{1}{2}\left(\frac{\partial u}{\partial s} + J_0\frac{\partial u}{\partial t}\right).$$

Hence

$$\frac{\partial u}{\partial s} = \partial_z u + \partial_{\bar{z}} u, \quad \frac{\partial u}{\partial t} = J_0^{-1}(-\partial_z u + \partial_{\bar{z}} u) = J_0(\partial_z u - \partial_{\bar{z}} u),$$

and

$$\frac{\partial u}{\partial s} + J(u(z))\frac{\partial u}{\partial t} = (I - J(u(z))J_0)\partial_{\bar{z}} u + (I + J(u(z))J_0)\partial_z u.$$

Now we have chosen coordinates so that $u(0) = \vec{0}$ and $J(\vec{0}) = J_0$, so when the right-hand side above is evaluated at $z = 0$ the term $(I - J(u(z))J_0)$ simplifies to $I - J_0^2 = 2I$ and the term $(I + J(u(z))J_0)$ becomes zero. Since J and u are continuous, it follows that, at least after shrinking to a smaller coordinate chart, $I - J(u(z))J_0$ will be invertible throughout our coordinate chart. So (after possibly shrinking our open sets V and W) we may define

$$q(\vec{z}) = (I - J(\vec{z})J_0)^{-1}(I + J(\vec{z})J_0),$$

and then $q\colon W \to End_{\mathbb{R}}(\mathbb{C}^n)$ is a smooth function with $q(\vec{0}) = 0$, and (61) is equivalent to the equation

$$\partial_{\bar{z}} u + q(u(z))\partial_z u = 0. \tag{62}$$

Since q is small near our origin of coordinates (and is identically zero in the case that $J(\vec{z})$ is identically equal to J_0), we might think of this locally as a nonlinear perturbation of the usual Cauchy-Riemann equation $\partial_{\bar{z}} u = 0$.

3.2. Some properties of the linear Cauchy-Riemann operator

Nonlinear equations such as (62) are generally more difficult to analyze than linear ones; it will help us to have some background about the linear Cauchy-Riemann operator $\partial_{\bar{z}}$ acting on functions from $\mathbb{C}$ to $\mathbb{C}$. So consider the equation $\partial_{\bar{z}} u = f$ where we regard f as known. The following shows that, if u is compactly supported, we can reconstruct u from f.

PROPOSITION 3.2.1. *Let $u \in C^1_0(\mathbb{C};\mathbb{C})$. Then for all $z_0 \in \mathbb{C}$, we have*

$$u(z_0) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial_{\bar{z}} u}{z - z_0} dz \wedge d\bar{z}.$$

(In particular the integral on the right exists.)

PROOF. Since u is continuous, taking the average value of u on small circles around z_0 gives

$$u(z_0) = \lim_{\epsilon \to 0} \frac{1}{2\pi}\int_{\partial B_\epsilon(z_0)} u(z_0 + \epsilon e^{i\phi}) d\phi. \tag{63}$$

If ρ, ϕ are polar coordinates relative to z_0, so that $\partial B_\epsilon(z_0) = \{\rho = \epsilon\}$ and the usual coordinate on $\mathbb{C}$ is given by $z = z_0 + \rho e^{i\phi}$, we have an equation of complexified one-forms $dz = e^{i\phi} d\rho + i\rho e^{i\phi} d\phi$, so since $d\rho|_{\partial B_\epsilon(z_0)} = 0$ and $\rho e^{i\phi} = z - z_0$ we have $d\phi|_{\partial B_\epsilon(z_0)} = \left.\frac{dz}{i(z-z_0)}\right|_{\partial B_\epsilon(z_0)}$. So (63) gives

$$\begin{aligned} u(z_0) &= \frac{1}{2\pi i}\lim_{\epsilon \to 0}\int_{\partial B_\epsilon(z_0)} \frac{u(z)}{z - z_0} dz \\ &= -\frac{1}{2\pi i}\lim_{\epsilon \to 0}\int_{\mathbb{C}\setminus B_\epsilon(z_0)} d\left(\frac{u(z)}{z - z_0} dz\right) \\ &= -\frac{1}{2\pi i}\lim_{\epsilon \to 0}\int_{\mathbb{C}\setminus B_\epsilon(z_0)} \frac{\partial_{\bar{z}} u}{z - z_0} d\bar{z} \wedge dz. \end{aligned}$$

Here in the second equation we have used Stokes' theorem (which applies despite the noncompactness of $\mathbb{C}$ since u is compactly supported; the sign arises because the boundary orientation for $\partial(\mathbb{C} \setminus B_\epsilon(z_0))$ is opposite to the boundary orientation for $B_\epsilon(z_0)$) and the third equation uses the quotient rule and the fact that $d(z - z_0) \wedge dz = 0$. So the Proposition follows provided that $\lim_{\epsilon \to 0}\int_{\mathbb{C}\setminus B_\epsilon(z_0)} \frac{\partial_{\bar{z}} u}{z - z_0} dz \wedge d\bar{z}$ exists (for then $\int_{\mathbb{C}} \frac{\partial_{\bar{z}} u}{z - z_0} dz \wedge d\bar{z}$ would, essentially by definition, be equal to this limit). We have assumed that u is C^1 and compactly supported, so $\partial_{\bar{z}} u$ is bounded and compactly supported. So it suffices to check that $\int_{B_\epsilon(z_0)} \left|\frac{1}{z - z_0}\right| dA \to 0$ as $\epsilon \to 0$. But converting to the polar coordinates ρ, ϕ centered at z_0, this integral is equal to $\int_0^{2\pi}\int_0^\epsilon \frac{1}{\rho}\rho\, d\phi\, d\rho = 2\pi\epsilon$, so it indeed converges to zero. □

Let us define $\phi\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by

$$\phi(w) = \frac{1}{\pi w}.$$

Since $dz \wedge d\bar{z} = -2i ds \wedge dt$ we find from Proposition 3.2.1 that a compactly supported C^1 function $u\colon \mathbb{C} \to \mathbb{C}$ satisfies

$$u(z_0) = \int_{\mathbb{C}} \phi(z_0 - z)\partial_{\bar{z}} u(z) dA = (\phi * \partial_{\bar{z}} u)(z_0)$$

where $*$ denotes convolution. As in the proof of Proposition 3.2.1, we note that $\int_{B_R(0)} |\phi| dA = \int_0^{2\pi} \int_0^R \frac{1}{\pi r} r\, dr\, d\theta = 2R$, so ϕ is locally integrable; thus $\phi * f$ is a well-defined function for any $f \in C_0(\mathbb{C};\mathbb{C})$. On the other hand the same computation (sending $R \to \infty$) shows that $\phi \notin L^1(\mathbb{C};\mathbb{C})$.

For $f \in C_0(\mathbb{C};\mathbb{C})$ let us define

$$Pf = \phi * f,$$

so $Pf \in C(\mathbb{C};\mathbb{C})$ (by Proposition 1.2.4) but generally we cannot expect Pf to be compactly supported even though f is. It follows from Proposition 1.2.5 that if $f \in C_0^1(\mathbb{C};\mathbb{C})$ then also $Pf \in C^1(\mathbb{C};\mathbb{C})$, and that

$$\partial_z Pf = P\partial_z f, \quad \partial_{\bar{z}} Pf = P\partial_{\bar{z}} f. \tag{64}$$

But by Proposition 3.2.1, in this case $P\partial_{\bar{z}} f = f$. Thus:

PROPOSITION 3.2.2. *For $f \in C_0^1(\mathbb{C};\mathbb{C})$, the function $Pf = \phi * f$ is a solution to the equation $\partial_{\bar{z}} u = f$.*

We would now like to generalize this to the case that $f \in L^p(\Omega;\mathbb{C})$ for some bounded domain $\Omega \subset \mathbb{C}$ (where $p < \infty$); note that $C_0^1(\Omega;\mathbb{C})$ is dense in $L^p(\Omega;\mathbb{C})$. For any $f \in C_0^1(\Omega;\mathbb{C})$ we may extend f by zero to be defined on all of $\mathbb{C}$, then apply the operator P to obtain $Pf \in C^1(\mathbb{C};\mathbb{C})$, and finally restrict Pf to Ω. We will mildly abuse notation and still write the resulting operator from $C_0^1(\Omega;\mathbb{C})$ to $C^1(\Omega;\mathbb{C})$ as P.

LEMMA 3.2.3. *For any bounded domain Ω and any $p < \infty$, the above operator $P\colon C_0^1(\Omega;\mathbb{C}) \to C^1(\Omega;\mathbb{C})$ extends to a bounded linear operator $P\colon L^p(\Omega;\mathbb{C}) \to L^p(\Omega;\mathbb{C})$.*

PROOF. Since $C_0^1(\Omega;\mathbb{C})$ is dense in $L^p(\Omega;\mathbb{C})$, it suffices to show that there is a constant C_Ω such that we have a bound $\|Pf\|_{L^p(\Omega;\mathbb{C})} \le C_\Omega \|f\|_{L^p(\Omega;\mathbb{C})}$ for all $f \in C_0^1(\Omega)$. Since $Pf = \phi * f$ this almost follows from Young's inequality $\|g * f\|_{L^p(\mathbb{C};\mathbb{C})} \le \|g\|_{L^1(\mathbb{C};\mathbb{C})} \|f\|_{L^p(\mathbb{C};\mathbb{C})}$, except for the issue that our function ϕ is only locally integrable and not in L^1. But since our domain is bounded we can work around this as follows. Choose $R > 0$ so that $\Omega \subset B_R(0)$, and notice that, if $z_0 \in \Omega$, then we have (via the change of variables $w = z_0 - z$)

$$Pf(z_0) = \int_{\mathbb{C}} \phi(z_0 - z) f(z) dA_z = \int_{\mathbb{C}} \phi(w) f(z_0 - w) dA_w,$$

and the term $f(z_0 - w)$ vanishes for all w outside the doubled ball $B_{2R}(0)$ (since if $w \notin B_{2R}(0)$ and $z_0 \in \Omega \subset B_R(0)$ then $z_0 - w \notin B_R(0)$). Hence the integral on the right above is unchanged if we replace ϕ by $\chi\phi$ where χ is any compactly supported continuous function that equals 1 everywhere in $B_{2R}(0)$. (Note that we can choose χ to depend only on Ω (or really only on R) and not on f.). Since ϕ is locally integrable, we will then have $\chi\phi \in L^1(\mathbb{C};\mathbb{C})$ with $Pf|_\Omega = ((\chi\phi) * f)|_\Omega$. So Young's inequality gives $\|Pf\|_{L^p(\Omega;\mathbb{C})} \le \|(\chi\phi) * f\|_{L^p(\mathbb{C};\mathbb{C})} \le \|\chi\phi\|_{L^1(\mathbb{C};\mathbb{C})} \|f\|_{L^p(\mathbb{C};\mathbb{C})}$ and the desired bound holds with $C_\Omega = \|\chi\phi\|_{L^1(\mathbb{C};\mathbb{C})}$. □

We can thus attempt to generalize Proposition 3.2.2 to this extended operator P, thus considering the equation $\partial_{\bar{z}} u = f$ where $f \in L^p(\Omega;\mathbb{C})$. As usual when we work with partial differential equations and L^p spaces, it works best to consider weak solutions to this equation, according to the following definition.

DEFINITION 3.2.4. If u, f are locally integrable functions on a bounded domain $\Omega \subset \mathbb{C}$, we say that u is a weak solution to $\partial_{\bar{z}} u = f$ and write $\partial_{\bar{z}} u \overset{\mathrm{W}}{=} f$ provided that, for all $g \in C_0^\infty(\mathbb{C};\mathbb{C})$, we have

$$\int_\Omega u \partial_{\bar{z}} g \, dA = -\int_\Omega f g \, dA.$$

(As usual, the motivation for this definition is that integration by parts shows that, in the case that u is continuously differentiable, the left-hand side above would be equal to $-\int_\Omega (\partial_{\bar{z}} u) g \, dA$, and so in this case $\partial_{\bar{z}} u \overset{\mathrm{W}}{=} f$ if and only if $\partial_{\bar{z}} u = f$.)

PROPOSITION 3.2.5. *Let $\Omega \subset \mathbb{C}$ be a bounded domain and let $p < \infty$. For any $f \in L^p(\Omega;\mathbb{C})$ we have $\partial_{\bar{z}}(Pf) \overset{\mathrm{W}}{=} f$.*

PROOF. Choose a sequence $\{f_n\}$ of functions in $C_0^1(\Omega;\mathbb{C})$ such that $f_n \to f$ in L^p. By Proposition 3.2.2, for each n we have $\partial_{\bar{z}}(Pf_n) = f_n$, and so $\partial_{\bar{z}}(Pf_n) \overset{\mathrm{W}}{=} f_n$.

Now $Pf_n \to Pf$ in L^p and $f_n \to f$ in L^p, so (using Hölder's inequality, for instance), one therefore finds for every $g \in C_0^\infty(\Omega;\mathbb{C})$,

$$\int_\Omega (Pf) \partial_{\bar{z}} g \, dA = \lim_{n\to\infty} \int_\Omega (Pf_n) \partial_{\bar{z}} g \, dA = -\lim_{n\to\infty} \int_\Omega f_n g \, dA = -\int_\Omega f g \, dA,$$

as desired. □

If $f \in W^{m,2}(\Omega;\mathbb{C})$ where $m \geq 1$ and if $\partial_{\bar{z}} u \overset{\mathrm{W}}{=} f$ (for instance we could take $u = Pf$), then applying ∂_z to both sides of the equation $\partial_{\bar{z}} u = f$ gives $\partial_z \partial_{\bar{z}} u = \partial_z f$; note moreover that

$$\partial_z \partial_{\bar{z}} = \frac{1}{4}\left(\frac{\partial}{\partial s} - i\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}\right).$$

So since $\partial_z f \in W^{m-1,2}(\Omega;\mathbb{C})$ with $\|\partial_z f\|_{W^{m-1,2}(\Omega;\mathbb{C})} \leq \|f\|_{W^{m,2}(\Omega;\mathbb{C})}$ our Regularity Theorem 1.5.7 applies to show that, for any $\Omega' \Subset \Omega$, there is a constant C such that $u \in W^{m+1,2}(\Omega';\mathbb{C})$ with $\|u\|_{W^{m+1,2}(\Omega';\mathbb{C})} \leq C(\|u\|_{W^{1,2}(\Omega;\mathbb{C})} + \|f\|_{W^{m,2}(\Omega;\mathbb{C})})$. The above required the right-hand side f of the equation $\partial_{\bar{z}} u \overset{\mathrm{W}}{=} f$ to be at least class $W^{1,2}$ (since we needed to apply $\partial_{\bar{z}} u$ to f before applying the main regularity theorem), but by a separate argument we can (at least if we choose the specific solution $u = Pf$) require f to be only L^2. The following (perhaps somewhat surprising) lemma will be helpful here:

LEMMA 3.2.6. *If $h \in C_0^2(\mathbb{C};\mathbb{C})$ then $\|\partial_z h\|_{L^2} = \|\partial_{\bar{z}} h\|_{L^2} = \frac{1}{2}\|\nabla h\|_{L^2}$.*

PROOF. We have

$$\|\partial_{\bar{z}} h\|_{L^2}^2 = \frac{1}{4}\int_{\mathbb{C}} \left(\frac{\partial h}{\partial s} + i\frac{\partial h}{\partial t}\right)\left(\overline{\frac{\partial h}{\partial s}} - i\overline{\frac{\partial h}{\partial t}}\right) dA \quad = \frac{1}{4}\left(\int_{\mathbb{C}}\left(\left|\frac{\partial h}{\partial s}\right|^2 + \left|\frac{\partial h}{\partial t}\right|^2\right) dA + i\int_{\mathbb{C}}\left(\frac{\partial h}{\partial t}\overline{\frac{\partial h}{\partial s}} - \frac{\partial h}{\partial s}\overline{\frac{\partial h}{\partial t}}\right)\right) dA.$$

Since h is compactly supported, the last integral may be converted via integration by parts to $\int_{\mathbb{C}}\left(-h\overline{\frac{\partial^2 h}{\partial t \partial s}} + h\overline{\frac{\partial^2 h}{\partial s \partial t}}\right) dA = 0$, and so indeed $\|\partial_{\bar{z}} h\|_{L^2}^2 = \frac{1}{4}\|\nabla h\|_{L^2}^2$.

The statement that $\|\partial_z h\|_{L^2} = \frac{1}{2}\|\nabla h\|_{L^2}$ can be proven by an essentially identical argument by changing some signs in the above calculation; alternatively one can just note that $\overline{\partial_z h} = \partial_{\bar{z}} \bar{h}$ and so applying what we have already proven to $\bar{h}$ shows that $\|\partial_z h\|_{L^2} = \frac{1}{2}\|\nabla \bar{h}\|_{L^2} = \frac{1}{2}\|\nabla h\|_{L^2}$. □

Motivated in part by this, let us define an operator $T\colon C_0^\infty(\mathbb{C};\mathbb{C}) \to C^\infty(\mathbb{C};\mathbb{C})$ by $T = \partial_z \circ P$. Thus, for any $u \in C_0^\infty(\mathbb{C};\mathbb{C})$ we have:

$$T\partial_{\bar{z}} u = \partial_z P \partial_{\bar{z}} u = \partial_z u.$$

So Proposition 3.2.6 shows that, if $f \in C_0^\infty(\mathbb{C};\mathbb{C})$ has the property that $f = \partial_{\bar{z}}u$ for some $u \in C_0^\infty(\mathbb{C};\mathbb{C})$, then $\|Tf\|_{L^2} = \|\partial_z u\|_{L^2} = \|\partial_{\bar{z}}u\|_{L^2} = \|f\|_{L^2}$. In fact with only slightly more work one can show that the same identity holds without the assumption that $f = \partial_{\bar{z}}u$ for a compactly supported u. (Actually this generalization isn't strictly necessary for our purposes since we will ultimately only apply T to functions of the form $\partial_{\bar{z}}u$ where u is compactly supported; I am including the following mainly as a not-too-hard-to-prove version of Theorem 3.2.8.)

PROPOSITION 3.2.7. *For any $f \in C_0^\infty(\mathbb{C};\mathbb{C})$ we have $\|Tf\|_{L^2} = \|f\|_{L^2}$. Hence T extends to an isometry of $L^2(\mathbb{C};\mathbb{C})$.*

PROOF. The trick is to evaluate the following exterior derivative:

$$\begin{aligned} d\left(Pf\overline{Tf}\,d\bar{z} + \bar{f}Pf\,dz\right) &= \left(\partial_z Pf\overline{Tf} + Pf\partial_z\overline{Tf} - \partial_{\bar{z}}\bar{f}Pf - \bar{f}\partial_{\bar{z}}Pf\right)dz\wedge d\bar{z} \\ &= \left(|Tf|^2 + Pf\overline{\partial_{\bar{z}}Tf} - \overline{\partial_z f}Pf - |f|^2\right)dz\wedge d\bar{z}. \end{aligned}$$

Now $\partial_{\bar{z}}Tf = \partial_{\bar{z}}\partial_z Pf = \partial_z f$ since $\partial_{\bar{z}}Pf = f$, so the middle two terms above cancel and we obtain

$$|Tf|^2 dz\wedge d\bar{z} = |f|^2 dz\wedge d\bar{z} + d\left(Pf\overline{Tf}\,d\bar{z} + \bar{f}Pf\,dz\right). \tag{65}$$

Now the definition of P makes clear that there is a constant C (depending on f) such that $|Pf(z)| < \frac{C}{|z|}$, and then since $Tf = P\partial_z f$ a similar estimate holds for Tf. So if R is so large that the support of f is contained in the interior of $B_R(0)$, we will have by Stokes' theorem

$$\int_{B_R(0)} d\left(Pf\overline{Tf}\,d\bar{z} + \bar{f}Pf\right) = \int_{\partial B_R(0)} Pf\overline{Tf}\,d\bar{z} \leq \frac{2\pi CR}{R^2}.$$

So integrating 65 over $B_R(0)$ and then taking the limit as $R \to \infty$ proves the result. □

In fact, the following theorem whose proof is beyond the scope of these notes (see [**A**, Section V.D] for one proof) shows that the above result extends to other values of p. We will later find it useful to take $p > 2$, since (in the context of pseudoholomorphic curves) in this case all $W^{1,p}$ functions are continuous.

THEOREM 3.2.8 (Calderón-Zygmund Theorem). *Let $1 < p < \infty$. Then there is $C_p > 0$ such that, for all $f \in C_0^\infty(\mathbb{C};\mathbb{C})$ we have $\|Tf\|_{L^p} \leq C_p\|f\|_{L^p}$. Thus T extends to a bounded linear operator on $L^p(\mathbb{C};\mathbb{C})$.*

This leads to the following regularity theorem, analogous to Theorem 1.5.7.

THEOREM 3.2.9. *Let $1 < p < \infty$, let $f \in W^{m,p}(\Omega;\mathbb{C})$ for a bounded domain Ω (where $m \geq 0$), and let $u \in L^p(\Omega)$ obey $\partial_{\bar{z}}u \overset{\mathrm{W}}{=} f$. Then for all $\Omega' \Subset \Omega$ we have $u \in W^{m+1,p}(\Omega';\mathbb{C})$, with a bound*

$$\|u\|_{W^{m+1,p}(\Omega';\mathbb{C})} \leq C(\|u\|_{L^p(\Omega;\mathbb{C})} + \|f\|_{W^{m,p}(\Omega;\mathbb{C})}) \tag{66}$$

where C depends only on m, p and not on u.

PROOF. Note first that if $u \in W^{k,p}(\Omega;\mathbb{C})$ where $k \leq m$ then $\partial_{\bar{z}}(D^\alpha u) \overset{\mathrm{W}}{=} D^\alpha f$. So by induction on m (and by replacing u with appropriate $D^\alpha u$) it suffices to prove the result in the case that $m = 0$, so that $f, u \in L^p(\Omega;\mathbb{C})$ with $\partial_{\bar{z}}u \overset{\mathrm{W}}{=} f$ and we wish to show that $u \in W^{1,p}(\Omega';\mathbb{C})$.

Choose a cutoff function $\chi \in C_0^\infty(\Omega;\mathbb{C})$ with $\chi|_{\Omega'} = 1$. Then χu is an L^p function supported in Ω with $\partial_{\bar{z}}(\chi u) \overset{\mathrm{W}}{=} g$ where we define $g = \chi f + (\partial_{\bar{z}}\chi)u$. Note that $g \in L^p(\Omega;\mathbb{C})$ and there is a bound $\|g\|_{L^p(\Omega;\mathbb{C})} \leq A(\|u\|_{L^p(\Omega;\mathbb{C})} + \|f\|_{L^p(\Omega;\mathbb{C})})$ where A depends only on the cutoff function χ.

Since χ has support compactly contained within Ω, for large n the mollifications $v_n = \eta_{1/n}*(\chi u)$ will be smooth and supported within Ω and, by Theorem 1.2.10, will converge in L^p to χu. Moreover by Proposition 1.3.5 we will have $\partial_{\bar z} v_n = \eta_{1/n} * g$; thus $\partial_{\bar z} v_n \to g$ in L^p. Note that, since the v_n are compactly supported, we have $P\partial_{\bar z} v_n = v_n$, and hence Lemma 3.2.3 shows that $Pg = \chi u$.

By Theorem 3.2.8, we find that

$$\begin{aligned}\|v_n - v_m\|_{W^{1,p}(\Omega;\mathbb{C})} &\le \|v_n - v_m\|_{L^p(\Omega;\mathbb{C})} + 2(\|\partial_{\bar z}(v_n - v_m)\|_{L^p(\Omega;\mathbb{C})} + \|\partial_z(v_n - v_m)\|_{L^p(\Omega;\mathbb{C})}) \\ &\le \|v_n - v_m\|_{L^p(\Omega;\mathbb{C})} + 2(C_p + 1)\|\partial_{\bar z}(v_n - v_m)\|_{L^p(\Omega;\mathbb{C})}\end{aligned}$$

and so since $\{v_n\}_{n=1}^\infty$ and $\{\partial_{\bar z} v_n\}_{n=1}^\infty$ both converge in L^p it follows that $\{v_n\}_{n=1}^\infty$ is Cauchy in $W^{1,p}(\Omega;\mathbb{C})$ and hence that its L^p-limit χu in fact lies in $W^{1,p}(\Omega;\mathbb{C})$. So $u \in W^{1,p}(\Omega';\mathbb{C})$, and

$$\begin{aligned}\|u\|_{W^{1,p}(\Omega';\mathbb{C})} &\le \|\chi u\|_{W^{1,p}(\Omega;\mathbb{C})} \le \|u\|_{L^p(\Omega;\mathbb{C})} + 2(\|\partial_{\bar z}(\chi u)\|_{L^p(\Omega;\mathbb{C})} + \|\partial_z(\chi u)\|_{L^p(\Omega;\mathbb{C})}) \\ &\le \|u\|_{L^p(\Omega;\mathbb{C})} + 2(C_p + 1)\|g\|_{W^{1,p}(\Omega;\mathbb{C})} \le \|u\|_{L^p(\Omega;\mathbb{C})} + 2(C_p + 1)A(\|u\|_{L^p(\Omega;\mathbb{C})} + \|f\|_{L^p(\Omega;\mathbb{C})})\end{aligned}$$

where we have used that $\partial_{\bar z}(\chi u) \overset{W}{=} g$ and that $Tg = \partial_z Pg = \partial_z(\chi u)$. □

3.3. Regularity for the nonlinear Cauchy-Riemann equation

Having learned more about the linear operator $\partial_{\bar z}$, we now return to the local version of the nonlinear Cauchy-Riemann equation, which we rewrote in (62) as

$$\partial_{\bar z} u + q(u(z))\partial_z u = 0$$

where we have chosen coordinates so that $u\colon D \to \mathbb{C}^n$ obeys $u(0) = \vec{0}$ and that $q \in C^\infty(\mathbb{C}^n, End_{\mathbb{R}}(\mathbb{C}^n))$ obeys $q(\vec{0}) = 0$. (Since for now we will just be concerned with local behavior, we will assume the domain of u to be a disk D centered at the origin.) The goal of this section is to show that any class-$W^{1,p}$ solution to this equation is in fact smooth; we will assume that $p > 2$, since by Morrey's inequality this forces u to be continuous. To be a bit more specific about the strategy, we will show that if $u \in W^{k,p}(D;\mathbb{C}^n)$ solves (62) then there is a smaller disk D' centered at the origin such that $u \in W^{k+1,p}(D';\mathbb{C}^n)$. By induction (and Theorem 1.3.14) this will show that u is smooth at the origin, and then since we could have chosen an arbitrary point as the origin by a suitable coordinate change it will show that u is globally smooth.

We will use the properties of the operator T from the previous section (extended to $\mathbb{C}^n$-valued functions by working component by component). First of all we modify the problem to one involving a compactly supported function defined on all of $\mathbb{C}$, by our usual device of choosing a cutoff function $\chi \in C_0^\infty(D,[0,1])$ which is identically equal to 1 on some subdisk centered at the origin, and then considering the function χu. Assuming that u satisfies (62), χu evidently satisfies the equation

$$\partial_{\bar z}(\chi u) + (q \circ u)\partial_z(\chi u) = (\partial_{\bar z}\chi)u + (\partial_z \chi)(q \circ u)u. \tag{67}$$

This equation holds at all points of D; moreover since $\partial_{\bar z}(\chi u), \partial_z(\chi u), \partial_{\bar z}\chi, \partial_z\chi$ are all supported in the interior of D, each term on either side of the equation naturally extends by zero to all of $\mathbb{C}$. To work in a context where this is a little more explicit, choose another cutoff function $\eta \in C_0^\infty(D;[0,1])$ such that η is identically equal to 1 on the support of χ. If we let $v = \chi u, w = \eta u$, both v, w lie in $W_0^{k,p}(D;\mathbb{C}^n)$ (and hence extend by zero to functions in $W^{k,p}(\mathbb{C};\mathbb{C}^n)$) and (67) gives

$$\partial_{\bar z} v + (q \circ w)\partial_z v = (\partial_{\bar z}\chi)w + (\partial_z \chi)(q \circ w)w. \tag{68}$$

(Indeed, we have $u = w$ on the support of χ, and each term in (67) in which u appears vanishes outside the support of χ, so such terms are unaffected by replacing u by w.) In view of this, our desired regularity statement for u will follow if we show that there is a subdisk centered at 0 on which a function v obeying (68) is of class $W^{k+1,p}$, where we assume that $v, w \in W^{k+1,p}(\mathbb{C};\mathbb{C}^n)$ with $v(0) = w(0) = 0$ and q is as before.

The left hand side of (68) can be rewritten as $(I+(q\circ u)T)\,\partial_{\bar z}v$, and the key will be to show that the right-hand side of (68) lies in $W^{k,p}$, and to reduce to a case where the operator $I+(q\circ w)T$ is (or can be arranged to be) invertible on $W^{k,p}$.

We will need results about the behavior of Sobolev norms under multiplication and composition; here is where the assumption that $p>2$ makes things relatively simple.

PROPOSITION 3.3.1. *Let $p>n$ and $k\geq 1$. There is a constant C such that if $f,g\in C_0^\infty(\mathbb{R}^n)$ then $\|fg\|_{k,p}\leq C\|f\|_{k,p}\|g\|_{k,p}$. Consequently if $f,g\in W^{k,p}(\mathbb{R}^n)$ then $fg\in W^{k,p}(\mathbb{R}^n)$ with $\|fg\|_{k,p}\leq C\|f\|_{k,p}\|g\|_{k,p}$.*

PROOF. Let $f,g\in C_0^\infty(\mathbb{R}^n)$. Morrey's inequality (Theorem 1.3.11) shows that, for any multi-index α with $|\alpha|\leq k-1$, we have (for a constant A independent of f) $\max|D^\alpha f|\leq A\|D^\alpha f\|_{1,p}\leq A\|f\|_{k,p}$ and likewise $\max|D^\alpha g|\leq A\|g\|_{k,p}$. Now if $|\alpha|\leq k$, then

$$D^\alpha(fg)=\sum_{\beta+\gamma=\alpha}(D^\beta f)(D^\gamma g)$$

where (since $k\geq 1$, so $|\alpha|<2k$) each term in the sum has either (or both) $|\beta|\leq k-1$ or $|\gamma|\leq k-1$. Terms with $|\beta|\leq k-1$ obey $\|(D^\beta f)(D^\gamma g)\|_p\leq \max|D^\beta f|\|g\|_{k,p}\leq A\|f\|_{k,p}\|g\|_{k,p}$, and terms with $|\gamma|\leq k-1$ likewise obey $\|(D^\beta f)(D^\gamma g)\|_p\leq A\|f\|_{k,p}\|g\|_{k,p}$. So for each $|\alpha|\leq k$ we have $\|D^\alpha(fg)\|_p\leq A\sum_{\beta+\gamma=\alpha}\|f\|_{k,p}\|g\|_{k,p}$, and so indeed $\|fg\|_{k,p}\leq C\|f\|_{k,p}\|g\|_{k,p}$ for appropriate C.

To prove the last sentence we use the usual approach of approximating f,g in $W^{k,p}$ norm by sequences $\{f_m\},\{g_m\}$. What we have already done, together with the triangle inequality, gives an estimate

$$\|f_lg_l-f_mg_m\|_{k,p}\leq C\|f_l\|_{k,p}\|g_l-g_m\|_{k,p}+C\|g_m\|_{k,p}\|f_l-f_m\|_{k,p}$$

and so proves that $\{f_mg_m\}_{m=1}^\infty$ is Cauchy in $W^{k,p}$, and so its limit, namely fg, belongs to $W^{k,p}(\mathbb{R}^n)$, with $\|fg\|_{k,p}\leq \lim_{m\to\infty}C\|f_m\|_{k,p}\|g_m\|_{k,p}=C\|f\|_{k,p}\|g\|_{k,p}$. □

PROPOSITION 3.3.2. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain, let $k\geq 0$ and $p>n$, and let $g\colon\mathbb{R}^m\to\mathbb{R}$ be a compactly supported C^{k+1} function. Then for each $u\in W_0^{k,p}(\Omega;\mathbb{R}^m)$ it holds that $g\circ u\in W^{k,p}(\Omega)$. Moreover if $g(\vec{0})=0$ and if $\{u_l\}_{l=1}^\infty$ is a sequence with $u_l\to 0$ in $W^{k,p}(\Omega;\mathbb{R}^m)$ then $g\circ u_l\to 0$ in $W^{k,p}(\Omega)$.*

PROOF. To see that the result holds for $k=0$, note that the fact that g is compactly supported and C^1 implies that it is Lipschitz (with Lipschitz constant given by the maximum value of $\|\nabla g\|$), so we have a bound of the form $|g(\vec{x})-g(\vec{0})|\leq A\|\vec{x}\|$ and hence $|(g\circ u)(z)|\leq A|u(z)|+|g(\vec{0})|$ for all $z\in\mathbb{R}^n$. So since Ω is bounded we obtain that $g\circ u\in L^p(\Omega)$ with a bound $\|g\circ u\|_{L^p(\Omega)}\leq A\|u\|_{L^p(\Omega)}+|g(\vec{0})|vol(\Omega)^{1/p}$. This suffices to establish the $k=0$ case of the proposition.

Let us now consider the case that $k=1$. First of all we will show that the chain rule formula $\frac{\partial(g\circ u)}{\partial x_i}=\sum_r\left(\frac{\partial g}{\partial x_r}\circ u\right)\frac{\partial u_r}{\partial x_i}$ continues to be valid (in the sense of weak derivatives) in the present case where u is of class $W_0^{1,p}$ and g is compactly supported and C^2. Indeed if we take a sequence of compactly supported smooth functions $\{u^{(\ell)}\}$ with $u^{(\ell)}\to u$ in $W^{1,p}$, the fact that $\frac{\partial g}{\partial x_r}$ is Lipschitz yields, as in the previous paragraph, a uniform bound $\left\|\frac{\partial g}{\partial x_r}\circ u^{(\ell)}-\frac{\partial g}{\partial x_r}\circ u\right\|_\infty\leq A\|u^{(\ell)}-u\|_\infty\leq B\|u^{(\ell)}-u\|_{W^{1,p}(\Omega)}$ for constants A,B, where the second inequality uses Theorem 1.3.11. So since also $\frac{\partial u^{(\ell)}}{\partial x_i}\to\frac{\partial u}{\partial x_i}$ in L^p, it follows that

$$\frac{\partial(g\circ u^{(\ell)})}{\partial x_i}=\sum_r\left(\frac{\partial g}{\partial x_r}\circ u_r^{(\ell)}\right)\frac{\partial u_r^{(\ell)}}{\partial x_i}\to\sum_r\left(\frac{\partial g}{\partial x_r}\circ u\right)\frac{\partial u_r}{\partial x_i}\quad\text{in }L^p,$$

by the general (easily-verified) fact that the product of a uniformly convergent sequence and an L^p-convergent sequence is L^p-convergent. So the usual argument with Hölder's inequality (used *e.g.* in the proof of Proposition 1.3.9) shows that the weak derivative of $g \circ u$ with respect to x_i exists and is equal to $\sum_r \left(\frac{\partial g}{\partial x_r} \circ u\right) \frac{\partial u_r}{\partial x_i}$.

With this chain rule in hand, we see that for any $u \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, $\frac{\partial(g \circ u)}{\partial x_i} = \sum_r \left(\frac{\partial g}{\partial x_r} \circ u\right) \frac{\partial u_r}{\partial x_i}$ is a sum of finitely many terms each of which is a product of a bounded function $\frac{\partial g}{\partial x_r} \circ u$ with an L^p function $\frac{\partial u_r}{\partial x_i}$ and hence is of class L^p, with L^p norm converging to zero for any sequence of u whose $W^{1,p}$ norms converge to zero (since in this case the $\frac{\partial g}{\partial x_r} \circ u$ terms remain uniformly bounded by the compact support of g, while the $\frac{\partial u_r}{\partial x_i}$ terms L^p-converge to zero). Combined with what we have already done in the $k = 0$ case this proves the $k = 1$ case of the proposition.

Now let $K \geq 2$ and assume that we have proven the result for all $k < K$. Given any $u \in W^{K,p}(\Omega; \mathbb{R}^m)$ we have $g \circ u \in L^p(\Omega)$, with $g \circ u \to 0$ in L^p as $u \to 0$ in L^p provided that $g(\vec{0}) = 0$, by the $k = 0$ case of the proposition. So it suffices to show that $\frac{\partial(g \circ u)}{\partial x_i} \in W^{K-1,p}(\Omega)$, with norm converging to zero as $u \to 0$ in $W^{K,p}$. But as we have already shown (in the proof of the $k = 1$ case), $\frac{\partial(g \circ u)}{\partial x_i} = \sum_r \left(\frac{\partial g}{\partial x_r} \circ u\right) \frac{\partial u_r}{\partial x_i}$. By the inductive hypothesis we have $\frac{\partial g}{\partial x_r} \circ u \in W^{K-1,p}(\Omega)$ while by assumption we have $\frac{\partial u_r}{\partial x_i} \in W^{K-1,p}$. Since $K - 1 \geq 1$ and $p > n$, we can now appeal to Proposition 3.3.1 to see that the product of these terms belongs to $W^{K-1,p}(\Omega)$, and converges to zero as $u \to 0$ in $W^{K,p}$ (as this causes $\frac{\partial u_r}{\partial x_i} \to 0$ in $W^{K-1,p}$). $\square$

We can now make the key argument leading to the regularity of solutions to the nonlinear Cauchy-Riemann equation.

PROPOSITION 3.3.3. *Fix a disk $D \subset \mathbb{C}$ centered at 0 and a C^{k+1} function $q \colon \mathbb{C}^n \to End_{\mathbb{R}}(\mathbb{C}^n)$ with $q(\vec{0}) = 0$, and let $k \geq 1$ and $p > 2$. Then there is $\epsilon > 0$ such that for any functions $v, w \in W_0^{k,p}(D; \mathbb{C}^n)$ with $\|w\|_{k,p} < \epsilon$ such that $\partial_{\bar{z}} v + (q \circ w)\partial_z v \in W_0^{k,p}(D; \mathbb{C}^n)$ we have $v \in W^{k+1,p}(D; \mathbb{C}^n)$.*

PROOF. The Calderón-Zygmund operator T appearing in Theorem 3.2.8 obeys $TD^\alpha = D^\alpha T$ for every multi-index α, so by Theorem 3.2.8 T maps $W^{k,p}(\mathbb{C}; \mathbb{C}^n)$ to itself, with an estimate $\|Tf\|_{k,p} \leq C_p \|f\|_{k,p}$. Note that since $k \geq 1$ an estimate $\|w\|_{k,p} < 1$ leads to a uniform bound $|w(z)| \leq R$, so as long as we choose $\epsilon < 1$ the function $q \circ w$ is unchanged if we modify q to some function $\tilde{q} = \beta q$ where β is a compactly supported smooth function equal to 1 on the ball of radius R. So Proposition 3.3.2 implies that (all matrix elements of) $q \circ w$ will be of class $W^{k,p}$, and moreover that for any $\delta > 0$ we can choose $\epsilon > 0$ sufficiently small that if $\|w\|_{k,p} < \epsilon$ then each of these matrix elements has $W^{k,p}$ norm less than δ. So using Proposition 3.3.1, for sufficiently small ϵ, it will hold that for all $f \in W^{k,p}(\mathbb{C}; \mathbb{C}^n)$ we have $\|(q \circ w)Tf\|_{k,p} \leq \frac{1}{2}\|f\|_{k,p}$. If $A \colon W^{k,p}(\mathbb{C}; \mathbb{C}^n) \to W^{k,p}(\mathbb{C}; \mathbb{C}^n)$ is defined by $Af = (q \circ w)Tf$, for any $v \in W_0^{k,p}(D; \mathbb{C}^n) \subset W^{k,p}(C^n)$ we have $A\partial_{\bar{z}} v = (q \circ w)T\partial_{\bar{z}} v = (q \circ w)\partial_z v$. But, for $\|w\|_{k,p} < \epsilon$, we have shown that the operator A on $W^{k,p}(\mathbb{C}; \mathbb{C}^n)$ has operator norm at most $\frac{1}{2}$; hence the operator $I + A$ is invertible, with inverse $(I + A)^{-1} = \sum_{j=0}^{\infty} (-A)^j$. (That the right hand side converges to a well-defined operator on $W^{k,p}(\mathbb{C}; \mathbb{C}^n)$ is a corollary of the completeness of $W^{k,p}(\mathbb{C}; \mathbb{C}^n)$.)

So if $\partial_{\bar{z}} v + (q \circ w)\partial_z v = h \in W_0^{k,p}(D; \mathbb{C}^n) \subset W^{k,p}(\mathbb{C}; \mathbb{C}^n)$, then (provided that $\|w\|_{k,p} < \epsilon$) $\partial_{\bar{z}} v = (I + A)^{-1} h \in W^{k,p}(\mathbb{C}; \mathbb{C}^n)$. But then Theorem 3.2.9 (applied to a set Ω such that $D \Subset \Omega$) proves that $v \in W^{k+1,p}(D; \mathbb{C}^n)$. $\square$

By implementing the procedure described at the start of the section, we find:

COROLLARY 3.3.4. *Fix disks $D' \Subset D \subset \mathbb{C}$ centered at* 0 *and a C^{k+1} function $q\colon \mathbb{C}^n \to End_{\mathbb{R}}(\mathbb{C}^n)$ with $q(\vec{0}) = 0$, and let $k \geq 1$ and $p > 2$. Then there is $\delta > 0$ such that if $u \in W^{k,p}(D;\mathbb{C}^n)$ satisfies $\|u\|_{k,p} < \delta$ and $\partial_{\bar{z}}u + (q \circ u)\partial_z u = 0$ then $u \in W^{k+1,p}(D';\mathbb{C}^n)$.*

PROOF. As suggested at the start of the section, if we let χ be a smooth cutoff function supported in the interior of D and equal to 1 on D', and if we let η be a smooth cutoff function supported in the interior of D and equal to 1 on the support of χ, then the equation $\partial_{\bar{z}}u + (q \circ u)\partial_z u = 0$ implies that $\partial_{\bar{z}}v + (q \circ w)\partial_z w = (\partial_z \chi)w + (\partial_z \chi)(q \circ w)w$ where we put $v = \chi u, w = \eta u$. Now Propoisition 3.3.1 implies that $v, w \in W_0^{k,p}(D;\mathbb{C})$, and we in particular have $\|w\|_{k,p} \leq \|\eta\|_{k,p}\|u\|_{k,p}$, so taking δ to be sufficiently small and requiring that $\|u\|_{k,p} < \delta$ implies that $\|w\|_{k,p} < \epsilon$ where ϵ is the constant from Proposition 3.3.3. Since Propositions 3.3.1 and 3.3.2 (together with the same argument in the proof of Proposition 3.3.3 involving replacing q by a compactly supported $\tilde{q}$) show that $(\partial_z \chi)w + (\partial_z \chi)(q \circ w)w \in W_0^{k,p}(D;\mathbb{C}^n)$, the previous proposition shows that $v \in W^{k+1,p}(D;\mathbb{C}^n)$. This implies the corollary since u coincides with v on D'. □

All that remains to do now is remove the hypothesis that $\|u\|_{k,p} < \delta$ in Corollary 3.3.4. This involves using a renormalization trick which requires passage to a significantly smaller disk D_0 and also (again) requires $p > 2$.

PROPOSITION 3.3.5. *Let $D \subset \mathbb{C}$ be a disk centered at the origin, let $k \geq 1$ and $p > 2$, let $q\colon \mathbb{C}^n \to End_{\mathbb{R}}(\mathbb{C}^n)$ be a C^{k+1} function with $q(\vec{0}) = 0$, and let $u \in W^{k,p}(D;\mathbb{C}^n)$ satisfy $u(0) = 0$ and $\partial_{\bar{z}}u + (q \circ u)\partial_z u = 0$. Then there is a disk $D_0 \Subset D$ centered at the origin such that $u \in W^{k+1,p}(D_0;\mathbb{C}^n)$.*

PROOF. For each positive integer m define $u_m\colon D \to \mathbb{C}^n$ by $u_m(z) = u(z/m)$. So for all $z \in \mathbb{C}$ we have
$$\partial_{\bar{z}}u_m(z) + (q \circ u_m)(z)\partial_z u_m(z) = \frac{1}{m}\partial_{\bar{z}}u(z/m) + \frac{1}{m}q(u(z/m))\partial_z u(z/m) = 0,$$
i.e. u_m satisfies the same equation that we have assumed to be satisfied by u. I claim that $u_m \to 0$ (in $W^{k,p}(D;\mathbb{C}^n)$-norm) as $m \to \infty$. Indeed, we find that
$$\int_D |D^\alpha u_m(z)|^p dV_z = \int_D \left|m^{-|\alpha|}D^\alpha u(z/m)\right|^p dV_z = \int_{\frac{1}{m}D} m^{n-p|\alpha|}|D^\alpha u(w)|dV_w \leq m^{n-p|\alpha|}\|u\|_{W^{k,p}(D;\mathbb{C}^n)}$$
where $\frac{1}{m}D$ is the disk centered at the origin with radius $\frac{1}{m}$ times the radius of D. Since we assume that $p > n$, this proves that $D^\alpha u_m \to 0$ in L^p for $|\alpha| \geq 1$. This argument does not work for $\alpha = 0$, but in this case Theorem 1.3.14 (together with our hypothesis that $u(0) = 0$) shows that, for $z \in D$, $|u_m(z)| = |u(z/m)| \leq \left|\frac{z}{m}\right|^{1-2/p}$, so since D has finite radius $u_m \to 0$ uniformly, and hence also in $L^p(D;\mathbb{C}^n)$. Thus indeed $u_m \to 0$ in $W^{k,p}(D;\mathbb{C}^n)$.

Thus we can find m_0 such that $\|u_{m_0}\|_{W^{k,p}(D;\mathbb{C})} \leq \delta$ where δ is the constant from Corollary 3.3.4. So $u_{m_0} \in W^{k+1,p}(D;\mathbb{C}^n)$. But from the definition of u_{m_0} it is clear that the statement that $u_{m_0} \in W^{k+1,p}(D;\mathbb{C}^n)$ is equivalent to the statement that $u \in W^{k+1,p}(\frac{1}{m_0}D;\mathbb{C}^n)$. So the result holds with $D_0 = \frac{1}{m_0}D$. □

We now rephrase this in more global language, along the lines of the start of the chapter. To formulate this we should introduce language for Sobolev spaces of functions between manifolds. If Σ, M are smooth manifolds, a natural definition for a Sobolev space $W^{k,p}_{loc}(\Sigma;M)$ would be as the space of functions $f\colon \Sigma \to M$ such that for each $p \in \Sigma$ there exist local coordinate charts around p and $f(p)$ in terms of which f is given by a class $W^{k,p}$ function; if Σ is compact we would just call this $W^{k,p}(\Sigma;M)$, deleting the loc (for noncompact manifolds there would be a more subtle question of convergence of various integrals to consider in defining $W^{k,p}$). Provided that $p > \dim\Sigma$, it is easy to see using Proposition 3.3.2 (and an analogous but easier version of it for compositions $u \circ g$ rather than $g \circ u$) that the question of whether a function is locally of class $W^{k,p}$ in this sense is independent of which coordinate charts are used, provided at least that they are taken from a bounded atlas as in Definition 1.6.1.

For the following we will slightly generalize the class of almost complex structures we consider, allowing an almost complex structure J on M to be just C^k for some k (as a map of smooth manifolds $TM \to TM$) instead of being smooth.

THEOREM 3.3.6. *Let (Σ, j) be a compact complex 2-manifold, and let M be a smooth manifold equipped with a C^k almost complex structure J where $k \geq 2$. If $u\colon \Sigma \to M$ is a class-$W^{1,p}$ map obeying the Cauchy-Riemann equation $u_* \circ j = J \circ u_*$, then $u \in W^{k,p}(\Sigma; M)$. In particular, if J is a smooth almost complex structure then $u \in C^\infty(\Sigma, M)$.*

PROOF. As discussed at the start of the chapter, given any $x \in \Sigma$ we may choose a holomorphic coordinate chart U around x (mapping x to 0) and a smooth coordinate chart V around $u(x)$ (mapping $u(x)$ to $\vec{0}$) in terms of which $J|_{T_{u(x)}M}$ is given by the standard almost complex structure J_0. Moreover defining (on possibly a smaller coordinate chart than V) $q(\vec{z}) = (I - J(\vec{z})J_0)^{-1}(I + J(\vec{z})J_0)$, q will be a C^k function with $q(\vec{0}) = 0$, and u will be represented in a subchart[1] of U by a function satisfying $\partial_{\bar{z}} u + q(u(z))\partial_z u = 0$. Applying Proposition 3.3.5 inductively gives a still-smaller coordinate chart around x on which u is of class $W^{k,p}$. Since this can be done around every point $x \in \Sigma$ we have indeed proven that $u \in W^{k,p}(\Sigma; M)$. □

3.4. Differential topology of $\bar{\partial}_J$

Given a compact complex curve (Σ, j) and an almost complex $2n$-manifold (M, J) we now intend to study (perhaps under additional conditions on J to be indicated later) geometric properties of the "moduli space" of solutions $u\colon \Sigma \to M$ to the Cauchy-Riemann equation $u_* \circ j = J \circ u_*$. Using that $J^2 = -I$, this equation is equivalent to

$$\bar{\partial}_J u := \frac{1}{2}(u_* + J \circ u_* \circ j) = 0.$$

We thus propose to think of the moduli space of J-holomorphic curves as something like the zero-level set of a function, namely $\bar{\partial}_J$, defined on the space $W^{k,p}(\Sigma, M)$ of class-$W^{k,p}$ maps $u\colon \Sigma \to M$. Ideally, this zero-level set would be a manifold.

We will understand this (somewhat sketchily; see [**MS2**] for many more details) from the viewpoint of differential topology on Banach manifolds. By definition, a Banach manifold is a second-countable Hausdorff space X with an atlas of coordinate charts $\phi_\alpha\colon U_\alpha \to E_\alpha$ where the U_α form an open cover of X and the E_α are Banach spaces, with transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ being smooth maps between open subsets of the Banach spaces E_α, E_β. Here one says a map F between open subsets of Banach spaces E_α, E_β is differentiable at a point x in its domain provided that there is a bounded linear operator $DF(x)$ with $\lim_{h\to 0} \frac{F(x+h)-F(x)-DF(x)h}{\|h\|} = 0$, that F is twice-differentiable if the map $x \mapsto DF(x)$ (with codomain the Banach space of bounded linear operators from E_α to E_β) is differentiable, and so on to define smoothness.

Our first contention is that (at least for $p > 2$) the space $W^{k,p}(\Sigma, M)$ (as defined in the previous section) is in fact a Banach manifold. To show this it suffices to, for all $u \in W^{k,p}(\Sigma, M)$, find a Banach space T_u and a homeomorphism Φ_u from a neighborhood B_u of the origin in T_u to a neighborhood $\mathcal{U}_u$ of u, in such a way that the transition functions $\Phi_v \circ \Phi_u^{-1}$ are smooth. The Banach space T_u will then be identified with the tangent space to $W^{k,p}(\Sigma, M)$ at u, and should be thought of as parametrizing ways to perturb u within $W^{k,p}(\Sigma, M)$.

[1]Note that since $u \in W^{1,p}(\Sigma, M)$ where $p > 2$, u is automatically continuous, so it maps a sufficiently small neighborhood of x into the region where $I - J(\vec{z})J_0$ is invertible.

Such a perturbation should involve, for each $z \in \Sigma$, a choice of perturbation of the point $u(z) \in M$, which is to say a tangent vector $\xi(z) \in T_{u(z)}M$. This is conveniently expressed in terms of the *pullback bundle* $u^*TM = \{(z,v) | v \in T_{u(z)}M\}$, which fits into the commutative diagram

$$\begin{tikzcd} u^*TM \arrow[r] \arrow[d] & TM \arrow[d] \\ \Sigma \arrow[r, "u"] & M \end{tikzcd}$$

(where the maps out of u^*TM are the projections to either factor and the map $TM \to M$ is the bundle projection). So $u^*TM \to \Sigma$ is a vector bundle, with fiber over z naturally identified with $T_{u(z)}M$, and our perturbation ξ of u may be viewed as a *section* of u^*TM (*i.e.* $\xi \colon \Sigma \to u^*TM$ with $\pi \circ \xi = I_\Sigma$ where $\pi \colon u^*TM \to \Sigma$ is the bundle projection).

To actually define the Banach manifold chart Φ_u around u we need to have a way of converting an "infinitesimal perturbation" ξ as in the previous paragraph into a new map $\Phi_u(\xi) \colon \Sigma \to M$. For this it suffices to have a way, for each $z \in \Sigma$, of converting the tangent vector $\xi(z) \in T_{u(z)}M$ into a new point in M (which would be the value of the map $\Phi_u(\xi)$ at z). If M were $\mathbb{R}^{2n}$ we would simply use vector addition—our new point would be $u(z) + \xi(z)$. On a general manifold M there is of course no notion of vector addition, but Riemannian geometry provides a substitute appropriate for our purposes. So let us choose a J-compatible Riemannian metric g on M. As is explained in any Riemannian geometry book (*e.g.* [**dC**]), for each $x \in M$ and each sufficiently short (as measured by g) $v \in T_xM$, there is a unique geodesic $\gamma_{x,v} \colon [0,1] \to M$ with $\gamma_{x,v}(0) = x$ and $\gamma'_{x,v}(0) = v$. So denoting by $\mathcal{N} \subset TM$ the set of pairs (x,v) which are "sufficiently short" in this sense, we obtain an "exponential map" $\exp \colon \mathcal{N} \to M$ defined by $\exp(x,v) = \gamma_{x,v}(1)$. So as long as our section ξ of u^*TM is pointwise-small enough that $(u(z), \xi(z)) \in \mathcal{N}$ for each z, we can define a map $\Phi_u(\xi) \colon \Sigma \to M$ by $(\Phi_u(\xi))(z) = \exp(u(z), \xi(z))$.

Of course the section ξ of u^*TM should be chosen in such a way that the resulting map $\Phi_u(\xi)$ still lies in $W^{k,p}(\Sigma, M)$. The appropriate condition, probably not surprisingly (in view of the case $M = \mathbb{R}^{2n}$ in which case $\Phi_u(\xi) = u + \xi$), is that ξ should itself be of class $W^{k,p}$, viewed as a section of u^*TM (*i.e.* a particular kind of map between the manifolds Σ and u^*TM, so the discussion above Theorem 3.3.6 suffices to define what it means for ξ to be of class $W^{k,p}$). The space of class $W^{k,p}$ sections of u^*TM (which from now on we will just denote by $W^{k,p}(u^*TM)$) is indeed a Banach space: it is a vector space since sections of a vector bundle can be added and scalar multiplied in the obvious way, and the most natural way to put a norm on it is to, for $j \le k$, use the Levi-Civita connection associated to the Riemannian metric g to make sense of the jth-order derivative $\nabla^j \xi$ as a j-linear bundle bundle map $TM^{\oplus j} \to TM$ and then use g to measure the norm of this bundle map. Since we always assume $k \ge 1$ and $p > 2$, Morrey's inequality implies that if $\xi \in W^{k,p}(u^*TM)$ has sufficiently small $W^{k,p}$ norm then we will have $(u(z), \xi(z)) \in \mathcal{N}$ for all $z \in \Sigma$. So for all ξ in a sufficiently small neighborhood B_u of zero in $W^{k,p}(u^*TM)$, the map $\Phi_u(\xi) \colon \Sigma \to M$ is well-defined and belongs to $W^{k,p}(\Sigma, M)$. We omit the proofs (based, somewhat tediously, on standard properties of the exponential map) that $\Phi_u \colon B_u \to W^{k,p}(\Sigma, M)$ is an embedding of an open subset, and that the transition functions $\Phi_v \circ \Phi_u^{-1}$ are smooth.

The upshot of the above discussion is that $W^{k,p}(\Sigma, M)$ is indeed a Banach manifold, with tangent space at u given by $T_u W^{k,p}(\Sigma, M) = W^{k,p}(u^*TM)$ (the space of class $W^{k,p}$ sections of the bundle $u^*TM \to \Sigma$). We wish to think of $\bar{\partial}_J$ as a function on this Banach manifold; if this function has zero as a regular value we would then conclude (by a Banach-manifold version of the implicit function theorem, which is proven in just the same way as the usual implicit function theorem) that our moduli space of J-holomorphic curves (solutions to $\bar{\partial}_J u = 0$) is a smooth manifold.

If we are to think of $\bar{\partial}_J$ as a function defined on the space $W^{k,p}(\Sigma, M)$ and ask whether it has zero as a regular value, we should first understand what the codomain of $\bar{\partial}_J$ is. For any $u \in W^{k,p}(\Sigma, M)$ we have defined $\bar{\partial}_J u = \frac{1}{2}(u_* + J \circ u_* \circ j)$. Thus $\bar{\partial}_J u$ gives, for each $z \in \Sigma$, a linear map $(\bar{\partial}_J u)_z \colon T_z\Sigma \to T_{u(z)}M$. Now as before we can think of $T_{u(z)}M$ as the fiber $(u^*TM)_z$ of the vector bundle $u^*TM \to \Sigma$. So $\bar{\partial}_J u$ is a section of the bundle over Σ whose fiber over z consists of linear maps $T_z\Sigma \to (u^*TM)_z$; this latter bundle is denoted $\mathrm{Hom}(T\Sigma, u^*TM)$. More specifically, it's easy to see from the formula for $\bar{\partial}_J u$ that $(\bar{\partial}_J u) \circ j = \frac{1}{2}(u_* \circ j - J \circ u^*) = -J \circ (\bar{\partial}_J u)$, so if we view $T\Sigma$ and u^*TM as complex vector bundles using the structures j and J, respectively, then $\bar{\partial}_J u$ is a section of the bundle $\overline{\mathrm{Hom}}(T\Sigma, u^*TM)$ whose fiber over z consists of *conjugate-linear* maps $T_z\Sigma \to (u^*TM)_z$. Moreover the assumption that $u \in W^{k,p}(\Sigma, M)$ readily implies that $\bar{\partial}_J u$ is a class $W^{k-1,p}$ section of this latter bundle. Thus,

$$\text{For } u \in W^{k,p}(\Sigma, M), \text{ we have } \bar{\partial}_J u \in W^{k-1,p}\left(\overline{\mathrm{Hom}}(T\Sigma, u^*TM)\right).$$

A mild difficulty is that the set in which we have located $\bar{\partial}_J u$ depends on u. This is because, from a global perspective, $\bar{\partial}_J$ is better seen not as a function whose zero-level set consists of J-holomorphic curves, but rather as a section of a vector bundle, whose intersection with the zero section consists of J-holomorphic curves.

We thus introduce a (Banach) vector bundle $\mathscr{E}^{k-1,p} \to W^{k,p}(\Sigma, M)$, such that the fiber $\mathscr{E}_u^{k-1,p}$ over u is equal to $W^{k-1,p}\left(\overline{\mathrm{Hom}}(T\Sigma, u^*TM)\right)$. So set-theoretically

$$\mathscr{E}^{k-1,p} = \bigcup_{u \in W^{k,p}(\Sigma, M)} \{u\} \times W^{k-1,p}\left(\overline{\mathrm{Hom}}(T\Sigma, u^*TM)\right).$$

For this to actually be a vector bundle it should admit local trivializations, *i.e.* around each $u \in W^{k,p}(\Sigma, M)$ there should be a neighborhood $\mathscr{U}$ such that the restriction $\mathscr{E}^{k-1,p}|_{\mathscr{U}}$ is diffeomorphic in fiberwise-linear fashion to the trivial bundle $\mathscr{U} \times \mathscr{E}_u^{k-1,p}$. This entails identifying, for each $v \in \mathscr{U}$, the fiber $\mathscr{E}_v^{k-1,p} = W^{k-1,p}\left(\overline{\mathrm{Hom}}(T\Sigma, v^*TM)\right)$ with $\mathscr{E}_u^{k-1,p} = W^{k-1,p}\left(\overline{\mathrm{Hom}}(T\Sigma, u^*TM)\right)$. Provided that we choose $\mathscr{U}$ sufficiently small, this can again be done by a construction with geodesics. Pointwise, the identification in question is a matter of setting up a correspondence between conjugate-linear maps $T_z\Sigma \to T_{u(z)}M$ and conjugate-linear maps $T_z\Sigma \to T_{v(z)}M$. If the neighborhood $\mathscr{U}$ is taken small enough, then for all $z \in \Sigma$ the points $u(z)$ and $v(z)$ will be sufficiently small enough that (with respect to our fixed J-compatible metric) there is a unique geodesic γ with $\gamma(0) = u(z)$ and $\gamma(1) = v(z)$. Given any affine connection (see [**dC**, Chapter 2]) $\tilde{\nabla}$ on M, the path γ determines an parallel-transport isomorphism $\mathscr{P}_\gamma^{\tilde{\nabla}} \colon T_{v(z)}M \to T_{u(z)}M$. If we choose $\tilde{\nabla}$ so that $\tilde{\nabla}J = 0$ (specifically, if ∇ is the Levi-Civita connection let us set $\tilde{\nabla}_v X = \frac{1}{2}(\nabla_v X - J\nabla_v(JX))$) then $\mathscr{P}_\gamma^{\tilde{\nabla}}$ will be a complex-linear map, and so the correspondence $\alpha \mapsto \mathscr{P}_\gamma^{\tilde{\nabla}} \circ \alpha$ sets up an isomorphism between conjugate-linear maps $T_z\Sigma \to T_{v(z)}M$ and conjugate-linear maps $T_z\Sigma \to T_{u(z)}M$. Allowing z to vary through Σ in this construction gives an isomorphism $\mathscr{E}_v^{k-1,p} \cong \mathscr{E}_u^{k-1,p}$ for any $v \in \mathscr{U}$, and then allowing v to vary through $\mathscr{U}$ gives the desired local trivialization for the Banach vector bundle $\mathscr{E}^{k-1,p}$.

Accordingly, we obtain a section $\underline{\bar{\partial}}_J \to W^{k,p}(\Sigma, M) \to \mathscr{E}^{k-1,p}$ given by $u \mapsto (u, \bar{\partial}_J u)$. The moduli space of J-holomorphic curves is then give as the intersection of the image of $\underline{\bar{\partial}}_J$ with the zero section of $\mathscr{E}^{k-1,p}$, and by the implicit function theorem this moduli space will be a manifold provided that $\underline{\bar{\partial}}_J$ is transverse to the zero section Z. Here transversality means that, for each u with $\bar{\partial}_J u = 0$, and some Banach-space complement F_u to $T_{(u,0)}Z$ in $T_{(u,0)}\mathscr{E}^{k-1,p}$, with projection $\pi_u \colon T_{(u,0)}\mathscr{E}^{k-1,p} = T_{(u,0)}Z \oplus F_u \to F_u$, the composition of π_u with the derivative of $\underline{\bar{\partial}}_J$ at u is surjective (with a bounded right inverse). In our context the obvious choice of a Banach-space complement F_u to the zero section at $(u, 0)$ is the fiber $\mathscr{E}_u^{k-1,p}$. Moreover, our local trivializations for the bundle

$\mathscr{E}^{k-1,p}$ give, for a neighborhood $\mathscr{U}$ of u, an identification of $\mathscr{E}^{k-1,p}|_{\mathscr{U}}$ with the product $\mathscr{U}\times\mathscr{E}_u^{k-1,p}$, under which the section $\underline{\bar\partial}_J$ is identified with the map $v\mapsto(v,\mathscr{P}^{\tilde\nabla}\bar\partial_J v)$. Here $\mathscr{P}^{\tilde\nabla}$ is the identification of $\overline{\mathrm{Hom}}(T\Sigma,v^*TM)$ with $\overline{\mathrm{Hom}}(T\Sigma,u^*TM)$ given by parallel translation along geodesics with respect to the connection $\tilde\nabla$ of the previous paragraph. Thus the transversality condition (at a given u satisfying $\bar\partial_J u=0$) is equivalent to the condition that the map $v\mapsto\mathscr{P}^{\tilde\nabla}\bar\partial_J v$ from $\mathscr{U}$ to $\mathscr{E}_u^{k-1,p}=W^{k-1,p}\left(\overline{\mathrm{Hom}}(T\Sigma,u^*TM)\right)$ has linearization at u which is surjective, with a bounded right inverse.

PROPOSITION 3.4.1. *For $u\in W^{k,p}(\Sigma,M)$, the derivative of the map $\mathscr{P}^{\tilde\nabla}\bar\partial_J\colon\ \mathscr{U}\to W^{k-1,p}\left(\overline{\mathrm{Hom}}(T\Sigma,u^*TM)\right)$ is the map $D_u\colon W^{k,p}(u^*TM)\to W^{k-1,p}\left(\overline{\mathrm{Hom}}(T\Sigma,u^*TM)\right)$ defined by, for $\xi\in W^{k,p}(u^*TM)$ and $v\in T_z\Sigma$,*

$$(D_u\xi)_z(v)=\frac{1}{2}\left(\nabla_v\xi+J(u(z))\nabla_{jv}\xi\right)+\frac{1}{2}(\nabla_{\xi(z)}J)(\partial_J u)_z(jv). \tag{69}$$

(Here ∇ is the pullback of the Levi-Civita connection on M to u^*TM, and $(\partial_J u)_z(v)=\frac{1}{2}(u_*v-Ju_*jv)$, so that if $\bar\partial_J u=0$ then $\partial_J u=du$.)

PROOF. Let $\xi\in W^{k,p}(u^*TM)$ and $v\in T_z\Sigma$ be given. For small t define γ_t to be the geodesic with $\gamma_t(0)=z$ and $\gamma_t'(0)=t\xi(z)$, so that we have a parallel transport map $\mathscr{P}_{\gamma_t}^{\tilde\nabla}\colon T_{\exp_{u(z)}(t\xi(z))}M\to T_{u(z)}M$ given by the Hermitian connection $\tilde\nabla$. As before define the map $\Phi_u(t\xi)\colon\Sigma\to M$ by $\Phi_u(t\xi)(z)=\exp_{u(z)}(t\xi(z))$; thus $t\mapsto\Phi_u(t\xi)$ defines an arc in $W^{k,p}(\Sigma,M)$ whose velocity at $t=0$ is precisely $\xi\in W^{k,p}(u^*TM)=T_uW^{k,p}(\Sigma,M)$, and so the derivative of $\mathscr{P}^{\tilde\nabla}\bar\partial_J$ at u in the direction ξ is $\frac{d}{dt}\big|_{t=0}\mathscr{P}^{\tilde\nabla}\bar\partial_J(\Phi_u(t\xi))$. So writing $D_u\xi$ for this derivative, we have

$$\begin{aligned}(D_u\xi)_z(v)&=\frac{1}{2}\left(\frac{d}{dt}\bigg|_{t=0}\mathscr{P}_{\gamma_t}^{\tilde\nabla}\left(\Phi_u(t\xi)_*v+J(\exp_{u(z)}(t\xi(z)))\Phi_u(t\xi)_*jv\right)\right)\\ &=\frac{1}{2}\left(\frac{d}{dt}\bigg|_{t=0}\mathscr{P}_{\gamma_t}^{\tilde\nabla}(\Phi_u(t\xi)_*v)+J(u(z))\,\frac{d}{dt}\bigg|_{t=0}\mathscr{P}_{\gamma_t}^{\tilde\nabla}(\Phi_u(t\xi)_*jv)\right).\end{aligned} \tag{70}$$

Here we use the fact that $\tilde\nabla J=0$ so that $\tilde\nabla$-parallel transport preserves J; thus $\mathscr{P}_{\gamma_t}^{\tilde\nabla}J(\exp_{u(z)}(t\xi(z)))=J(u(z))\mathscr{P}_{\gamma_t}^{\tilde\nabla}$.

Choose an arc $\eta\colon(-\delta,\delta)\to\Sigma$ with $\eta(0)=z$ and $\eta'(0)=v$, and define $\Gamma(s,t)=\exp_{u(\eta(s))}(t\xi(\eta(s)))$. Thus $\Phi_u(t\xi)_*v=\frac{d}{ds}\big|_{s=0}\Gamma(s,t)$, which gives a vector field along the curve $t\mapsto\Phi_u(t\xi(z))$, and $\frac{d}{dt}\big|_{t=0}\mathscr{P}_{\gamma_t}^{\tilde\nabla}\Phi_u(t\xi)_*v$ is the derivative $\frac{D^{\tilde\nabla}}{dt}\left[\frac{d}{ds}\big|_{s=0}\Gamma(s,t)\right]_{t=0}$ of this vector field along the curve with respect to our Hermitian connection $\tilde\nabla$. If we instead used the Levi-Civita connection ∇ to define parallel transport $\mathscr{P}_{\gamma_t}^{\nabla}$, we would likewise have $\frac{d}{dt}\big|_{t=0}\mathscr{P}_{\gamma_t}^{\nabla}\Phi_u(t\xi)_*v=\frac{D^{\nabla}}{dt}\left[\frac{d}{ds}\big|_{s=0}\Gamma(s,t))\right]_{t=0}$. Now by [**dC**, Lemma 3.3.4], the Levi-Civita connection has the symmetry property that

$$\frac{D^{\nabla}}{dt}\left[\frac{d}{ds}\bigg|_{s=0}\Gamma(s,t))\right]_{t=0}=\frac{D^{\nabla}}{ds}\left[\frac{d}{dt}\bigg|_{t=0}\Gamma(s,t))\right]_{s=0}.$$

But the right-hand side above is just the time-zero derivative (with respect to ∇) of the vector field ξ along the curve $u\circ\eta$; since $\eta'(0)=v$ this is precisely (by definition) $\nabla_v\xi$.

We have thus shown that $\frac{d}{dt}\big|_{t=0}\mathscr{P}_{\gamma_t}^{\nabla}\Phi_u(t\xi)_*v=\nabla_v\xi$. The object that we are actually interested in uses the Hermitian connection $\tilde\nabla$ to define parallel transport instead of the Levi-Civita connection ∇, but these are related in a simple way. Namely, for any vector field X and tangent vector w on M

we have by definition

$$\nabla_w X - \tilde\nabla_w X = \frac{1}{2}(\nabla_w X + J\nabla_w(JX)) = \frac{1}{2}\left(\nabla_w X + J^2\nabla_w X + J(\nabla_w J)X\right) = \frac{1}{2}J(\nabla_w J)X,$$

which implies that, for any vector field X along the curve $t\mapsto \exp_{u(z)}(t\xi(z))$, one has

$$\frac{d}{dt}\bigg|_{t=0}\left(\mathscr{P}^{\tilde\nabla}_{\gamma_t} - \mathscr{P}^{\nabla}_{\gamma_t}\right)X(t) = -\frac{1}{2}J(\nabla_{\xi(z)}J)X(0).$$

In particular we obtain

$$\frac{d}{dt}\bigg|_{t=0}\mathscr{P}^{\nabla}_{\gamma_t}\Phi_u(t\xi)_* v = \nabla_v\xi - \frac{1}{2}(J\nabla_{\xi(z)}J)(u_*v) \tag{71}$$

for $v\in T_z\Sigma$. This applies equally well with v replaced by jv, so together with (70) we get

$$\begin{aligned}(D_u\xi)_z(v) &= \frac{1}{2}\left((\nabla_v\xi - \frac{1}{2}J(u(z))(\nabla_{\xi(z)}J)(u_*v)) + J(u(z))(\nabla_{jv}\xi - \frac{1}{2}J(u(z))(\nabla_{\xi(z)}J)(u_*jv))\right)\\ &= \frac{1}{2}\left(\nabla_v\xi + J(u(z))\nabla_{jv}\xi\right) + \frac{1}{4}((\nabla_{\xi(z)}J)(u_*jv + Ju_*v))\end{aligned}$$

(where we have used that $\nabla_{\xi(z)}J$ anticommutes with J since $\nabla_{\xi(z)}(J^2)=0$. Since $u_*j + Ju_* = (u_* - Ju_*j)\circ j = 2(\partial_J u)\circ j$ this is precisely the desired formula. □

We will see that the operator D_u from Proposition 3.4.1 is an example of the following class of operators between Banach spaces. (Recall in general that the *cokernel* of a linear map $L\colon E\to F$ is by definition the quotient vector space $\frac{F}{\mathrm{Im}\,L}$.)

DEFINITION 3.4.2. Let E,F be two Banach spaces. A bounded linear map $L\colon E\to F$ is said to be a **Fredholm operator** provided that $\mathrm{Im}\,L$ is a closed subspace of F, and $\ker L$ and $\mathrm{coker}\,L$ are both finite-dimensional.

THEOREM 3.4.3. *Let $k\ge 1$ and $p>2$. If $u\in W^{k,p}(\Sigma,M)\cap C^1(\Sigma,M)$, the operator $D_u\colon W^{k,p}(u^*TM)\to W^{k-1,p}\left(\overline{\mathrm{Hom}}(T\Sigma,u^*TM)\right)$ given by (69) is a Fredholm operator.*

PROOF. We begin by expressing D_u in local coordinates. Let $U\subset\Sigma$ be an open set on which we have a holomorphic coordinate $z=s+it$ on Σ, and a complex orthonormal frame $\{e_1,\dots,e_n\}$ for u^*TM (so $\{e_1,Je_1,\dots,e_n,Je_n\}$ evaluates at each point $p\in U$ as an orthonormal basis for $(u^*TM)_p = T_{u(p)}M$ with respect to our fixed J-compatible Riemannian metric). Since u is of class $W^{k,p}\cap C^1$, the e_m can likewise be taken be of class $W^{k,p}\cap C^1$. Using this frame, we identify each $(u^*TM)_p$ with $\mathbb{C}^n$, so that multiplication by i in $\mathbb{C}^n$ corresponds to the action of $J(u(p))$.

A general element $\xi\in W^{k,p}(u^*TM)$ has restriction to U given by $\xi|_U = \sum_m f_m e_m$ where $f_i\in W^{k,p}(U;\mathbb{C})$. At the same time, since for each $p\in U$ $(D_u\xi)_p$ is a conjugate-linear map $T_p\Sigma\to (u^*TM)_p$ and since $\partial_t = i\partial_s$, we have $(D_u\xi)_p(\partial_t) = -i(D_u\xi)_p(\partial_s)$, and hence

$$(D_u\xi)|_U = ((D_u\xi)(\partial_s))\,d\bar z|_U \tag{72}$$

where as usual $d\bar z = ds - i dt$. (In other words, for each $v\in T_p\Sigma$ with $p\in U$, $(D_u\xi)_p(v) = ((D_u\xi)(\partial_s))\,d\bar z(v)$.)

If $\xi|_U = \sum_m f_m e_m$ then one finds (by the Leibniz rule for the connection ∇) that

$$(\nabla_{\partial_s}\xi)|_U = \frac{\partial f}{\partial s}e_m + f_m\nabla_{\partial_s}e_m + \Im f_m(\nabla_{\partial_s}J)e_m$$

and similarly for $(\nabla_{\partial_t}\xi)|_U$, where $\Im$ denotes imaginary part. This (69) yields, if $\xi|_U = \sum_m f_m e_m$ for $f_m \to U \to \mathbb{C}$,

(73)

$$(D_u\xi)(\partial_s) = \frac{1}{2}\sum_m\left[\left(\frac{\partial f_m}{\partial s} + i\frac{\partial f_m}{\partial t}\right)e_m + f_m(\nabla_{\partial_s}e_m + i\nabla_{\partial_t}e_m) + \Im f_m(\nabla_{\partial_s}J + i\nabla_{\partial_t}J)e_m + f_m(\nabla_{e_m}J)(\partial_J u)(\partial_t)\right].$$

The key point about the above formula is that the only terms that involve differentiation of the functions f_m yield precisely $\sum_m(\partial_{\bar z}f_m)e_m$. A little more specifically, based on (72) and (73), for $u \in W^{k,p}(\Sigma, M)$ there is a class-$W^{k-1,p}$ real endomorphism of A of $u^*TM|_U$ such that

$$D_u\left(\sum_m f_m e_m\right) = \left(\sum_m(\partial_{\bar z}f_m)e_m + A(z)(\sum_m f_m e_m)\right)d\bar z.$$

In the case that $k = 1$, since we have assumed that $u \in C^1(\Sigma, M)$, A will still be continuous (for $k > 1$ this follows automatically by Morrey's inequality). It then follows from Theorem 3.2.9 that whenever $V \Subset U' \Subset U$ we have a bound

$$\|\xi\|_{W^{k,p}(u^*TM|_V)} \le C\left(\|D_u\xi\|_{W^{k-1,p}(\overline{\mathrm{Hom}}(T\Sigma,u^*TM)|_{U'})} + \|A\xi\|_{W^{k-1,p}(u^*TM|_{U'})} + \|\xi\|_{W^{k-1,p}(u^*TM|_U)}\right)$$

for some constant C. If $k \ge 2$ then Proposition 3.3.1 bounds $\|A\xi\|_{W^{k-1,p}(u^*TM|_{U'})}$ in terms of $\|\xi\|_{W^{k-1,p}(u^*TM|_{U'})}$. If $k = 1$ then such a bound simply follows from the continuity of A on a set which contains the (compact) closure of the set U'. So in fact we have, whenever $V \Subset U$ (in which case we can always find an intermediate set U' as above),

$$\|\xi\|_{W^{k,p}(u^*TM|_V)} \le C\left(\|D_u\xi\|_{W^{k-1,p}(\overline{\mathrm{Hom}}(T\Sigma,u^*TM)|_U)} + \|\xi\|_{W^{k-1,p}(u^*TM|_U)}\right)$$

for a different constant C.

By applying this to a finite collection of local frames over open subsets $U_1, \ldots, U_r$ with subsets $V_1 \Subset U_1, \ldots, V_r \Subset U_r$ with the V_m still covering Σ (as is possible since Σ is compact), we obtain a global bound (with still a new constant C)

$$\|\xi\|_{k,p} \le C(\|D_u\xi\|_{k-1,p} + \|\xi\|_{k-1,p}).$$

Now it is a straightforward consequence of Theorem 1.3.16 that the inclusion $W^{k,p}(u^*TM) \hookrightarrow W^{k-1,p}(u^*TM)$ is a compact operator. Hence the above estimate puts us precisely in the context of Lemma 1.6.14 (as generalized to Banach spaces in Remark 1.6.15), and so proves that D_u has **closed range and finite-dimensional kernel**.

It now remains only to show that the cokernel of D_u is finite-dimensional. Because we know that $\mathrm{Im}(D_u)$ is closed, the dimension of $\mathrm{coker}(D_u)$ is equal to the dimension of $\ker(D_u^*)$ where D_u^* is the adjoint to D_u, provided that $\ker(D_u^*)$ is finite-dimensional.[2] Let us first assume that $k = 1$, so the codomain of D_u is $L^p(\overline{\mathrm{Hom}}(T\Sigma, u^*TM))$, which has dual space given by $L^q(\overline{\mathrm{Hom}}(T\Sigma, u^*TM)^*)$ where $\frac{1}{p} + \frac{1}{q} = 1$. If $\eta \in \ker(D_u^*)$, then in terms of a local frame $\{e_1, \ldots, e_n\}$ for u^*TM and local holomorphic coordinate z for Σ as at he start of the proof, η is locally represented as $\eta|_U = h\,d\bar z^*$ for some $h \in L^q(U; \mathbb{C}^n)$, and the local description of D_u from earlier shows that we must have $\int_U \langle \partial_{\bar z} f + A(z)f(z), h(z)\rangle\, ds\,dt = 0$ for every $f \in C_0^\infty(U; \mathbb{C}^n)$, where $\langle\cdot,\cdot\rangle$ is the usual Hermitian inner product. But then this means that $\bar h$ is a weak, class L^q solution to the equation $-\partial_{\bar z}\bar h + A(z)^T\bar h = 0$, which since A is continuous forces h to be of class $W^{1,q}$ by Theorem 3.2.9. So (when $k = 1$), $\ker(D_u^*)$ in fact consists of class $W^{1,q}$ solutions to an equation $\tilde D_u^*\eta = 0$ of the same basic form as the equation

[2]Sketch of proof: If $y \in W^{k-1,p}\left(\overline{\mathrm{Hom}}(T\Sigma, u^*TM)\right)$ does not lie in $\mathrm{Im}(D_u)$, the Hahn-Banach theorem allows one to construct an element $\eta \in W^{k-1,p}\left(\overline{\mathrm{Hom}}(T\Sigma, u^*TM)^*\right)$ which vanishes on $\mathrm{Im}(D_u)$ but not on y. Since $\eta|_{\mathrm{Im}(D_u)} = 0$ we have $\eta \in \ker(D_u^*)$. Consequently the natural pairing between $\ker(D_u^*)$ and $\mathrm{coker}(D_u)$ is nondegenerate, and so these spaces either have the same finite dimension or are both infinite-dimensional.

$D_u\xi = 0$, where $\tilde{D}_u^*$ maps class $W^{1,q}$ sections to class L^q sections. In particular just as with D_u one gets an estimate $\|\eta\|_{1,q} \leq C(\|\tilde{D}_u^*\eta\|_q + \|\eta\|_q)$, so by Theorem 1.3.23 and Lemma 1.6.14 $\tilde{D}_u^*$ has finite-dimensional kernel. So indeed the cokernel of D_u is finite-dimensional when $k = 1$.

The case that $k > 1$ can quickly be inferred from this, together with Theorem 3.2.9. Let $u \in W^{k,p}(\Sigma, M)$, so that in particular $u \in W^{1,p}(\Sigma, M)$ and we can view D_u either as acting on $W^{k,p}(u^*TM)$ or as acting on $W^{1,p}(u^*TM)$. If $V \leq W^{k-1,p}\left(\overline{\mathrm{Hom}}(T\Sigma, u^*TM)\right)$ has trivial intersection with $\mathrm{Im}(D_u|_{W^{k,p}(u^*TM)})$ then by Theorem 3.2.9 V would also have trivial intersection with the image of D_u acting on class-$W^{1,p}$ sections, since if $D_u\xi$ is of class $W^{k-1,p}$ then ξ is of class $W^{k,p}$. But then V can have dimension no larger than the dimension of the cokernel of D_u on $W^{1,p}$ sections. Thus the cokernel is no larger for $D_u|_{W^{k,p}(u^*TM)}$ than it is for $D_u|_{W^{1,p}(u^*TM)}$. □

REMARK 3.4.4. If $u \in W^{1,p}(\Sigma, M)$ is a *J-holomorphic* map and if J is a smooth almost complex structure, then Theorem 3.3.5 shows that u is in fact smooth, and so we could apply Theorem 3.4.3 for any value of k. The description of D_u in the proof, together with Theorem 3.2.9, then shows inductively that (for $k \geq 1$) any class $W^{k,p}$ element ξ of D_u is in fact of class $W^{k+1,p}$, and hence is actually smooth. Thus $\ker(D_u)$ is independent of our choice of k. Similarly since $\mathrm{coker}(D_u)$ can be identified with the dual of the kernel of the adjoint D_u^*, the discussion of the kernel of D_u^* in the proof shows, again via Theorem 3.2.9, that $\mathrm{coker}(D_u)$ is independent of our choice of k.

3.4.1. Generalities about Fredholm operators. Having shown that D_u is a Fredholm operator, we now discuss some general properties of such operators.

DEFINITION 3.4.5. Let E and F be Banach spaces and let $L\colon E \to F$ be a Fredholm operator. The **index** of L is the integer

$$ind(L) = \dim\ker L - \dim\mathrm{coker}\, L.$$

EXAMPLE 3.4.6. *Let $E = \mathbb{R}^n$ and $F = \mathbb{R}^m$. Then obviously any linear map $L\colon E \to F$ is a Fredholm operator since all subspaces of E and F are closed and finite-dimensional. We find*

$$ind(L) = \dim\ker L - \dim\mathrm{coker}\, L = \dim\ker L - (m - \dim\mathrm{Im}\, L) = (\dim\ker L + \dim\mathrm{Im}\, L) - m = n - m$$

where the last equality is the rank-plus-nullity theorem. So for linear maps between two given finite-dimensional Banach spaces, the index depends only on the spaces and not on the specific map, even though different maps will certainly have different-sized kernels or cokernels.

We will see that in the infinite-dimensional context the index is likewise a fairly robust invariant of the operator, though it is not completely independent of the operator—for instance among maps $\ell^2 \to \ell^2$ the identity has index zero while the "backwards shift" operator $\{x_n\}_{n=1}^\infty \mapsto \{x_{n+1}\}_{n=1}^\infty$ has index one.

Here we give another characterization of the Fredholm property.

LEMMA 3.4.7. *Let $L\colon E \to F$ be a bounded linear operator. The following are equivalent:*

(i) *L is Fredholm.*
(ii) *There is a finite-dimensional Banach space C and a surjective bounded linear operator $L'\colon E \times C \to F$ such that $L'|_{E\times\{0\}} = L$ and $\ker L'$ is finite-dimensional.*
(iii) *There are finite-dimensional Banach spaces C and K and a bounded linear isomorphism $\hat{L}\colon E \times C \to F \times K$ having the form $\hat{L}(e, c) = (Le + \alpha c, \beta e + \gamma c)$ for some bounded linear operators $\alpha\colon C \to F$, $\beta\colon E \to K$, $\gamma\colon C \to K$.*

Moreover in case (ii) we have $ind(L) = \dim\ker L' - \dim C$, and in case (iii) we have $ind(L) = \dim K - \dim C$.

PROOF. (i)⇒(ii): Since $\mathrm{Im}(L)$ is closed with $\frac{F}{\mathrm{Im}\,L}$ finite-dimensional, we can find a finite-dimensional (and hence closed) subspace $C \le F$ such that $\mathrm{Im}(L) \oplus C = F$ (just take the span of an arbitrary set of lifts of the basis elements of $\mathrm{coker}\, L$ via the projection $F \to \mathrm{coker}\, L$). Then $L' \colon E \times C \to F$ defined by $L'(e,c) = Le + c$ will satsify the required property. (This choice of L' in fact gives $\ker L' = (\ker L) \times \{0\}$.)

(ii)⇒(iii): Let $K = \ker L' \le E \times C$. Since this is finite-dimensional, it is a standard consequence of the Hahn-Banach theorem that there is a bounded linear map $\Pi\colon E\times C \to K$ such that $\Pi k = k$ for all $k \in K$. So define $\hat{L}\colon E\times C \to F\times K$ by $\hat{L}(e,c) = (L'(e,c), \Pi(e,c))$. This is bounded since L' and Π are bounded, and it is clearly injective since $\ker L' = K$ and Π restricts injectively to K. Surjectivity is only slightly harder: if $(f,k) \in F \times K$ we can find $x \in E \times C$ such that $L'x = f$. Now since L' vanishes on K we have $L'\Pi = 0$ and so $\hat{L}(x - \Pi x + k) = (L'x, \Pi x - \Pi x + \Pi k) = (f,k)$. So $\hat{L}$ is a bounded linear isomorphism, which is clearly of the desired form (with α given by $c \mapsto L'(0,c)$, β by $e \mapsto \Pi(e,0)$, and γ by $c \mapsto \Pi(0,c)$).

(iii)⇒(ii): By the open mapping theorem, the inverse $\hat{L}^{-1}\colon F\times K \to E\times C$ is also bounded, so $\|\hat{L}(e,c)\| \ge \delta\|(e,c)\|$ for all e,c and some number $\delta > 0$ independent of e,c. We thus have

$$\|e\| = \|(e,0)\| \le \delta^{-1}\|\hat{L}(e,0)\| = \delta^{-1}\|(Le,\beta e)\| \le \delta^{-1}(\|Le\| + \|\beta e\|).$$

But β is a bounded operator to the *finite-dimensional* Banach space K, so β is a compact operator (as any bounded set in K is sequentially precompact). Thus Lemma 1.6.14 (and Remark 1.6.15) proves that L has finite-dimensional kernel and closed range. Moreover the surjectivity of $\hat{L}$ shows that $F = \mathrm{Im}\, L + \mathrm{Im}\,\alpha$ where $\alpha\colon C \to F$ has finite rank since C is finite-dimensional. Thus $\mathrm{coker}\, L$ is finite-dimensional, completing the proof that L is Fredholm.

Now let us obtain the conclusions about the index, first for case (ii). Define $\alpha\colon C \to F$ by $\alpha(c) = L'(0,c)$, so that L' is given by $L'(e,c) = Le + \alpha c$. Since L' is surjective we have

$$rank(\alpha) = \dim \mathrm{coker}\, L + \dim(\mathrm{Im}\, L \cap \mathrm{Im}\,\alpha).$$

Meanwhile there is a surjective linear map $\ker L' \to \mathrm{Im}\, L \cap \mathrm{Im}\,\alpha$ given by $(e,c) \mapsto Le$ (or equivalently $(e,c) \mapsto -\alpha c$), and the kernel of this map is $\ker L \times \ker\alpha$, so we have

$$\dim\ker L' = \dim\ker L + \dim\ker\alpha + \dim(\mathrm{Im}\, L \cap \mathrm{Im}\,\alpha).$$

Subtracting the last two displayed equations from each other gives $\dim\ker L' - rank(\alpha) = ind(L) + \dim\ker\alpha$, which by the rank-plus-nullity theorem proves that $\dim\ker L' = ind(L) + \dim C$.

Finally in case (iii), let π denote the projection $F \times K \to F$. Since $\hat{L}\colon E\times C \to F\times K$ is a linear isomorphism, it follows that $\pi\circ\hat{L}\colon E\times C \to F$ is surjective, with $\dim\ker(\pi\circ\hat{L}) = \dim K$. On the other hand, the assumption on $\hat{L}$ shows that $\pi\circ\hat{L}(e,0) = L(e)$ for $e \in E$, so $\pi\circ\hat{L}$ satisfies the requirements of case (ii), in view of which, by what we have already shown, $\dim\ker(\pi\circ\hat{L}) = ind(L) + \dim C$. Thus $\dim K = ind(L) + \dim C$, as claimed. □

As the proof shows, given a Fredholm operator $L\colon E \to F$, we can choose the spaces C and K in part (iii) to be equal to the cokernel and the kernel of L, respectively; however the freedom to make other choices is helpful in the following corollary.

COROLLARY 3.4.8. *Let E and F be Banach spaces, denote by $B(E,F)$ the space of bounded linear operators from E to F, and let $Fred(E,F) \subset B(E,F)$ be the subset consisting of Fredholm operators. Then $Fred(E,F)$ is open with respect to the operator norm topology on $B(E,F)$, and the map $Fred(E,F) \to \mathbb{Z}$ defined by $L \mapsto ind(L)$ is locally constant.*

PROOF. If $L \in Fred(E,F)$, construct a bounded linear isomorphism $\hat{L}\colon E\times C \to F\times K$ as in (iii) of Proposition 3.4.7, so $\hat{L}(e,c) = (Le + \alpha c, \beta e + \gamma c)$ for certain bounded operators α,β,γ. Recall that the set of bounded linear isomorphisms between two Banach spaces is open with respect to the operator norm topology.[3] So if L_1 is sufficiently close to L, the operator $\hat{L}_1\colon E \times C \to F \times K$ given by $\hat{L}_1(e,c) = (L_1 e + \alpha c, \beta e + \gamma c)$ will still be a bounded linear isomorphism and hence will be Fredholm by the implication (iii)⇒(i) of Proposition 3.4.7. Moreover we will have $ind(L_1) = ind(L) = \dim K - \dim C$. Thus all bounded linear operators in a sufficiently small neighborhood of a Fredholm operator L are Fredholm and have the same index as does L, which suffices to prove the corollary. □

COROLLARY 3.4.9. *If E and F are two Banach spaces and $\gamma\colon [0,1] \to Fred(E,F)$ is a path of Fredholm operators which is continuous with respect to the operator norm topology, then $ind(L_0) = ind(L_1)$.*

PROOF. The preceding corollary shows that the map $t \mapsto ind(L_t)$ is a locally constant function on $[0,1]$, hence is constant since $[0,1]$ is connected. □

3.4.2. Local structure of moduli spaces of J-holomorphic curves. An immediate consequence of Corollary 3.4.9 is that, given a homotopy $\{u_t\}_{t\in[0,1]}$ of $W^{k,p}$ maps $u_t\colon \Sigma \to M$ (assumed also to be C^1 if $k=1$), the Fredholm index of the linearization D_u^t from Theorem 3.4.3 is independent of t. In fact, by applying Corollary 3.4.9 to a path of Fredholm operators beginning at D_u and ending at the $\bar{\partial}$ operator on sections of u^*TM with respect to a holomorphic structure on the complex vector bundle u^*TM, it can be shown using the Hirzebruch Riemann-Roch theorem (see [**MS2**, Appendix C]) that

$$ind(D_u) = 2\left(n(1-g(\Sigma)) + \langle c_1(TM), u_*[\Sigma]\rangle\right) \tag{74}$$

where $g(\Sigma)$ is the genus of Σ, $c_1(TM) \in H^2(M;\mathbb{Z})$ is the first Chern class of the complex vector bundle (TM,J), and $[\Sigma] \in H_2(\Sigma;\mathbb{Z})$ is the fundamental class determined by the orientation that is induced by the complex structure j on Σ. So the index of D_u in fact only depends on the class $u_*[\Sigma] \in H_2(M;\mathbb{Z})$.

For $A \in H_2(M;\mathbb{Z})$ let us define

$$\mathcal{M}(A,J) = \{u\colon \Sigma \to M \,|\, \bar{\partial}_J u = 0,\, u_*[\Sigma] = A\}.$$

(We do not specify the regularity of u because Theorem 3.3.5 shows that all elements will automatically be smooth if J is smooth, or $W^{k,p}$ if J is C^k.) We would like to describe the structure of $\mathcal{M}(A,J)$ in a neighborhood of a general element $u \in \mathcal{M}(A,J)$.

First of all, *if* D_u is surjective, then the implicit function theorem for Banach manifolds asserts[4] that a neighborhood of u in $\mathcal{M}(A,J)$ will be a smooth manifold of dimension equal to $\dim\ker D_u$. The surjectivity of D_u is equivalent to the statement that $\operatorname{coker} D_u = \{0\}$, so in this case $\dim\ker D_u = ind(D_u) = 2\left(n(1-g(\Sigma)) + \langle c_1(TM), A\rangle\right)$. So if D_u is surjective *for all* $u \in \mathcal{M}(A,J)$ (typically J is said to be *regular* if this is the case) then all of $\mathcal{M}(A,J)$ is a smooth manifold of dimension $2\left(n(1-g(\Sigma)) + \langle (TM), A\rangle\right)$ (and in particular is empty if this number is negative).

If D_u is not surjective we can make a weaker statement. Since D_u is Fredholm, Lemma 3.4.7 shows that we can find a finite-dimensional Banach space C with a bounded linear map $\alpha\colon C \to \mathcal{E}_u^{k-1,p}$ such that the map $L'\colon W^{k,p}(u^*TM) \times C \to \mathcal{E}^{k-1,p}$ defined by $L'(\xi,c) = D_u\xi + \alpha c$ is surjective and has kernel of dimension equal to $ind(D_u) + \dim C$. So the implicit function theorem shows that a neighborhood $\mathcal{N}$ of the point $(u,0)$ in the space $\{(v,c) \in \mathcal{U} \times C \,|\, \mathcal{P}^{\tilde{\nabla}}\bar{\partial}_J v + \alpha c = 0\}$ is a smooth manifold of dimension $ind(D_u) + \dim C$; here $\mathcal{U}$ is the neighborhood of u in $W^{k,p}(u^*TM)$

[3] Sketch proof: If A is a bounded linear isomorphism (which implies that A^{-1} is bounded by the open mapping theorem) then $A+\epsilon$ will have inverse given by $\left(\sum_{k=0}^{\infty}(-A^{-1}\epsilon)^k\right)A^{-1}$ provided that ϵ is small enough in operator norm $\|\cdot\|_{op}$ that $\|A^{-1}\epsilon\|_{op} < 1$.

[4] In general the implicit function theorem would require that D_u have a bounded right inverse, but this holds automatically in the present context because $\ker(D_u)$ is finite-dimensional and hence admits a closed complement.

from Proposition 3.4.1. A neighborhood of u in $\mathcal{M}(A,J)$ can then be identified with the preimage of 0 under the projection $\mathcal{N} \to C$. Thus denoting $m = \dim C$, it is quite generally the case that a neighborhood of u in $\mathcal{M}(A,J)$ is homeomorphic to the preimage of 0 under a smooth map $\psi\colon \mathbb{R}^{m+ind(D_u)} \to \mathbb{R}^m$. Of course such a set in general need not be a manifold (and it also need not be empty if $ind(D_u) < 0$), but it is at least a finite-dimensional object; this map ψ is an example of what is called a *Kuranishi neighborhood* for u. As described in the proof of Lemma 3.4.7, the space C can be taken equal to $\operatorname{coker} D_u$; if we do this then our Kuranishi neighborhood will be given by a map $\mathbb{R}^{\dim\ker D_u} \to \mathbb{R}^{\dim \operatorname{coker} D_u}$, or, just as well, as a map $\ker D_u \to \operatorname{coker} D_u$.

One would naturally prefer to work with manifolds instead of Kuranishi neighborhoods. Sadly this is not always possible, but it often is, at least if one is willing to modify the almost complex structure J. The idea is to consider a whole (possibly infinite-dimensional Banach) manifold $\mathcal{J}$ of almost complex structures, and then look at the map which sends a pair (u,J) to $\bar{\partial}_J u$. As with $u \mapsto \bar{\partial}_J u$ this is map is properly seen as a section of a Banach vector bundle; if we trivialize the bundle around some pair (u,J) with $\bar{\partial}_J u = 0$, the linearization will be a map $\mathcal{D}_{u,J}\colon W^{k,p}(u^*TM) \times T_J\mathcal{J} \to \mathcal{E}_u^{k-1,p}$ of the form $(\xi, Y) \mapsto D_u^J\xi + AY$ where D_u^J is the operator from Proposition 3.4.1 (and we include J in the notation since it is now variable). Since D_u^J is already "close to" surjective (it has finite-dimensional cokernel), it is perhaps plausible that if we take $\mathcal{J}$ to be large enough then $\mathcal{D}_{u,J}$ will always be surjective. This runs into trouble when u is a multiple cover, *i.e.* $u = v \circ \pi$ for some branched covering map $\pi\colon \Sigma \to \Sigma'$, but can generally be arranged otherwise, see [**MS2**, Chapter 3]. If this is the case, we obtain a "universal moduli space" $\tilde{M}(A) = \{(u,J)|\bar{\partial}_J u = 0\}$ which is a smooth (maybe infinite-dimensional) manifold. There is then a projection $p\colon \tilde{M}(A) \to \mathcal{J}$ given by $(u,J) \mapsto J$. It is a good exercise for the reader to show that, if $J \in \mathcal{J}$ is a *regular value* for this projection p, then it will hold that D_u^J is surjective for every $u \in \mathcal{M}(A,J)$, and hence that $\mathcal{M}(A,J)$ is a manifold of the "expected dimension" $ind(D_u)$. While a particular $J \in \mathcal{J}$ may not be a regular value for p, Sard's theorem (and Smale's infinite-dimensional generalization of it) implies that many regular values do exist, allowing this sketch to be carried out at least if one removes multiple covers from consideration. See [**MS2**] for a much more complete discussion.

3.5. Pseudoholomorphic curves in the presence of a symplectic structure

While the notion of a pseudoholomorphic curve just requires the target manifold M to carry an almost complex structure J, Gromov observed [**G**] that if M carries a symplectic structure with which J is compatible then J-holomorphic curves satisfy compactness properties which allow them to be used in a powerful way to study the properties of M; this has been a major tool in symplectic topology ever since. First we define the relevant terms.

DEFINITION 3.5.1. A **symplectic manifold** is a pair (M,ω) where M is a smooth manifold and $\omega \in \Omega^2(M)$ is a two-form which is *closed* and *nondegenerate* (*i.e.*, if $x \in M$ and $0 \neq v \in T_xM$ then there is $w \in T_xM$ with $\omega(v,w) \neq 0$).

DEFINITION 3.5.2. An almost complex structure J on a symplectic manifold (M,ω) is said to be **ω-compatible** if for all $x \in M$ and all $v,w \in T_xM$ with $v \neq 0$ we have $\omega(Jv,Jw) = \omega(v,w)$ and $\omega(v,Jv) > 0$. We write $\mathcal{J}(M,\omega)$ for the space of ω-compatible almost complex structures (equipped with the compact-open topology on maps $TM \to TM$).

We have already seen a large family of examples of symplectic manifolds. If (M,J,g) is a Hermitian manifold (so (M,J) is a complex manifold and g is a Riemannian metric obeying $g(Jv,Jw) = g(v,w)$) then in Section 2.4.1 we defined the fundamental 2-form $\omega \in \Omega^2(M)$ by $\omega(v,w) = g(Jv,w)$, and we said that (M,J,g) is a Kähler manifold if $d\omega = 0$. Note that in this situation ω is certainly nondegenerate, since for any nonzero $v \in TM$ we have $\omega(v, Jv) = g(Jv, Jv) > 0$. So Kähler manifolds (M, J, g) can be seen as symplectic manifolds, using the fundamental 2-form ω as the symplectic form. Since (as just noted) $\omega(v, Jv) > 0$ for nonzero v, and since $\omega(Jv, Jw) = g(J^2v, Jw) = g(Jw, -v) = \omega(w, -v) = \omega(v, w)$, in this situation we have $J \in \mathcal{J}(M, \omega)$. It's worth mentioning that none of the above depended on J being a genuine complex structure as opposed to an almost complex structure; if (M, J, g) is only almost Hermitian then we can still form the fundamental 2-form ω, and we say that (M, J, g) is almost Kähler if ω is closed. In this situation (M, ω) will again be a symplectic manifold with $J \in \mathcal{J}(M, \omega)$.

Taking a different perspective, we can start with the triple (M, ω, J) with (M, ω) symplectic and $J \in \mathcal{J}(M, \omega)$ and then obtain a Riemannian metric g_J by the formula $g_J(v, w) = \omega(v, Jw)$. It's easy to check that (M, J, g_J) is then an almost Hermitian manifold with ω as its fundamental 2-form. Accordingly the term "almost Kähler manifold" (or "Kähler manifold" if J is a genuine complex structure) is sometimes assigned to the data (M, ω, J) rather than the (equivalent) data (M, J, g_J).

Before trying to discuss J-holomorphic curves in a given symplectic manifold (M, ω) for ω-compatible almost complex structures J, we should make sure that such almost complex structures exist. Indeed they do:

PROPOSITION 3.5.3. *If (M, ω) is a symplectic manifold then the space $\mathcal{J}(M, \omega)$ of ω-compatible almost complex structures is nonempty and contractible.*

PROOF. Let $\mathcal{G}$ denote the space of Riemannian metrics on M (equipped with the compact open topology on maps $TM \oplus TM \to \mathbb{R}$). We will show that $\mathcal{J}(M, \omega)$ is homotopy equivalent to $\mathcal{G}$, from which the result immediately follows since $\mathcal{G}$ is nonempty (as can be seen by a construction with partitions of unity by working in local coordinates) and convex.

More specifically, we will show that the map $\iota\colon \mathcal{J}(M, \omega) \to \mathcal{G}$ given by $\iota(J) = g_J$ is a homotopy equivalence; in particular we must construct a homotopy inverse $g \mapsto J_g$. Let $g \in \mathcal{G}$. For each $x \in M$, the maps $w \mapsto \omega(\cdot, w)$ and $w \mapsto g(\cdot, w)$ each define isomorphisms $T_xM \to T_x^*M$ (they are injective by nondegeneracy, and hence surjective by dimensional considerations). Composing one of these isomorphisms with the inverse of the other gives a smooth invertible map $A_g\colon TM \to TM$ such that $g(v, w) = \omega(v, A_g w)$. This latter property uniquely determines A_g, for if A'_g is another such map then the map $v \mapsto \omega(v, A_g w - A'_g w)$ would be trivial for all w, forcing $A_g w = A'_g w$ for all w by the nondegeneracy of ω. In particular it follows that if $g = g_J$ for some $J \in \mathcal{J}(M, \omega)$ then $A_g = J$.

For general g we cannot expect A_g to be an almost complex structure. However we find that, for $x \in M$ and $v, w \in T_xM$,

$$g(A_g^2 v, w) = \omega(A_g^2 v, A_g w) = -\omega(A_g w, A_g^2 v) = -g(A_g w, A_g v) \tag{75}$$

and likewise $g(v, A_g^2 w) = -g(A_g v, A_g w)$. Thus A_g^2 is a symmetric operator on each T_xM (with respect to the inner product given by g). Moreover we see from (75) that $g(A_g^2 v, v) = -g(A_g v, A_g v) < 0$ for all nonzero v. So A_g^2 is a symmetric negative definite operator on each T_xM, from which it follows that $-A_g^2$ is diagonalizable with all eigenvalues positive. This allows us to define, for any $s \in \mathbb{R}$, the matrix power $(-A_g^2)^s$: one simply has $(-A_g^2)^s$ act by multiplication by λ^s on each λ-eigenspace of $-A_g^2$. Now A_g preserves each eigenspace of $-A_g^2$, so since $(-A_g^2)^s$ acts by scalar multiplication on each such eigenspace (and since the eigenspaces together span T_xM) it follows that

$$A_g(-A_g^2)^s = (-A_g^2)^s A_g \text{ for all } s \in \mathbb{R}. \tag{76}$$

Let us define $J_g = A_g(-A_g^2)^{-1/2}$ for each $g \in \mathcal{G}$. Then by (76),

$$J_g^2 = A_g(-A_g^2)^{-1/2} A_g(-A_g^2)^{-1/2} = A_g^2(-A_g^2)^{-1} = -I,$$

so J_g is an almost complex structure. Moreover for $x \in M$ and $v, w \in T_xM$ we have (using (75), (76), and the fact that each $(-A_g^2)^s$, like $-A_g^2$, is symmetric and positive definite):

$$\begin{aligned}\omega(J_g v, J_g w) &= g\left(A_g(-A_g^2)^{-1/2}v, A_g(-A_g^2)^{-1/2}A_g^{-1}w\right) = -g\left((-A_g^2)^{-1/2}v, A_g^2(-A_g^2)^{-1/2}A_g^{-1}w\right)\\ &= g\left((-A_g^2)^{-1/2}v, (-A_g^2)^{1/2}A_g^{-1}w\right) = g(v, A_g^{-1}w) = \omega(v,w);\end{aligned}$$

and

$$\omega(v, J_g v) = \omega(v, A_g(-A_g^2)^{-1/2}v) = g(v, (-A_g^2)^{-1/2}v) > 0 \text{ for } v \neq 0.$$

Thus for all $g \in \mathcal{G}$ we have $J_g \in \mathcal{J}(M,\omega)$ and we can define a map $\phi\colon \mathcal{G} \to \mathcal{J}(M,\omega)$ by $\phi(g) = J_g$.

As mentioned earlier, if $g = g_J$ then $A_g = J$ (and so $-A_g^2 = I$); thus where again $\iota(J) = g_J$ we have $\phi \circ \iota = I$. On the other hand $\iota\circ\phi$ sends a general metric $g(\cdot,\cdot) = \omega(\cdot, A_g\cdot)$ to the metric given by $\omega(\cdot, A_g(-A_g^2)^{-1/2}\cdot) = g(\cdot, (-A_g^2)^{-1/2}\cdot)$. If we define $H_t\colon \mathcal{G} \to \mathcal{G}$ by setting $H_t(g) = g(\cdot, (-A_g^2)^{-t/2}\cdot)$ (which is indeed a Riemannian metric since $-A_g^2$ and hence also each $(-A_g^2)^{-t/2}$ is positive definite) we thus see that $\{H_t\}_{t\in[0,1]}$ defines a homotopy from the identity to $\iota\circ\phi$. Thus ι and ϕ are indeed homotopy inverses. □

Thus any symplectic manifold (M,ω) can be viewed as an almost Kähler manifold by choosing some $J \in \mathcal{J}(M,\omega)$. Such a choice is very non-unique; in fact (by considering the linearization of the equations on J imposed by requiring that $J \in \mathcal{J}(M,\omega)$ and appealing to the implicit function theorem) it is not hard to see that $\mathcal{J}(M,\omega)$ is an infinite-dimensional manifold. One large family of elements of $\mathcal{J}(M,\omega)$ can be obtained by starting with one element $J_0 \in \mathcal{J}(M,\omega)$ and considering $\phi_* J_0 \phi_*^{-1}$ for arbitrary diffeomorphisms $\phi\colon M \to M$ such that $\phi^*\omega = \omega$. (There are many such ϕ, obtained for instance as the flows of vector fields X such that $\omega(X,\cdot)$ is closed—the nondegeneracy of ω shows that any closed 1-form determines such an X.)

While symplectic manifolds can always be made almost Kähler, they typically cannot be made Kähler; the earliest proof of this (in [**T**]) involves constructing compact symplectic manifolds whose first Betti number are odd (Thurston's first example was given by $S^1 \times Y$ for the three-manifold Y given by taking the mapping torus of a Dehn twist on the two-torus); we have seen in Corollary 2.5.5 that such a manifold is not Kähler. By now it is well-understood that the class of symplectic manifolds is vastly larger than the class of Kähler manifolds.

We will now do a computation that indicates the significance of J-holomorphic curves in a symplectic manifold (M,ω) with $J \in \mathcal{J}(M,\omega)$. Let (Σ, j) be a compact (almost) complex 2-manifold, and choose a j-compatible Riemannian metric h on Σ. For any $p \in \Sigma$, if $e_1 \in T_p\Sigma$ is any unit vector (as measured by h), then the j-compatibility of h shows that je_1 is also a unit vector and that $h(e_1, je_1) = h(je_1, -e_1)$, so $h(e_1, je_1) = 0$; thus $\{e_1, je_1\}$ is an orthonormal basis for $T_p\Sigma$. Accordingly, for a C^1 map $u\colon \Sigma \to M$ we define

$$|du|^2_{g_J}(p) = g_J(u_*e_1, u_*e_1) + g_J(u_*je_1, u_*je_1).$$

Since any other unit vector in $T_p\Sigma$ is given by $ae_1 + bje_1$ with $a^2 + b^2 = 1$, it's easy to see that $|du(p)|^2_{g_J}$ is independent of the choice of unit vector e_1. We now compute, for any C^1 map $u\colon \Sigma \to M$ and any $p \in \Sigma$ and unit vector $e_1 \in T_p\Sigma$,

$$\begin{aligned}|du|^2_{g_J}(p) &= g_J(u_*e_1, u_*e_1) + g_J(u_*je_1, u_*je_1) = \omega(u_*e_1, Ju_*e_1) + \omega(u_*je_1, Ju_*je_1)\\ &= \omega(u_*e_1, u_*je_1) + \omega(u_*e_1, Ju_*e_1 - u_*je_1) + \omega(u_*je_1, -u_*e_1) + \omega(u_*je_1, u_*e_1 + Ju_*je_1)\\ &= 2u^*\omega(e_1, je_1) + \omega\left(u_*e_1, J(u_*e_1 + Ju_*je_1)\right) + \omega\left(Ju_*je_1, J(u_*e_1 + Ju_*je_1)\right)\\ &= 2u^*\omega(e_1, je_1) + \omega\left(2(\bar\partial_J u)_p(e_1), 2J(\bar\partial_J u)_p(e_1)\right)\\ &= 2u^*\omega(e_1, je_1) + 4g_J((\bar\partial_J u)_p(e_1), (\bar\partial_J u)_p(e_1))\\ &= 2\left(u^*\omega(e_1, je_1) + g_J((\bar\partial_J u)_p(e_1), (\bar\partial_J u)_p(e_1)) + g_J((\bar\partial_J u)_p(je_1), (\bar\partial_J u)_p(je_1))\right)\end{aligned}$$

where the last equation uses that $(\bar\partial_J u)_p(je_1) = -J(\bar\partial_J u)_p(e_1)$, so that $(\bar\partial_J u)_p(je_1)$ and $(\bar\partial_J u)_p(e_1)$ have the same magnitude with respect to g_J.

So if, consistently with our definition of $|du|^2_{g_J}(p)$, we define $|\bar\partial_J u|^2_{g_J}(p) = g_J((\bar\partial_J u)_p(e_1), (\bar\partial_J u)_p(e_1)) + g_J((\bar\partial_J u)_p(je_1), (\bar\partial_J u)_p(je_1))$, we have thus shown that, for any $p \in M$ and any unit vector $e_1 \in T_p\Sigma$, we have

$$\frac{1}{2}|du|^2_{g_J}(p) = u^*\omega(e_1, je_1) + |\bar\partial_J u|^2_{g_J}(p).$$

Now we can integrate this equation with respect to the volume form vol_h determined by the Riemannian metric h on Σ together with the orientation given by j; this volume form evaluates at p as $e^1 \wedge e^2$ where $\{e^1, e^2\}$ is the dual basis to $\{e_1, je_1\}$. In particular $u^*\omega(e_1, je_1)\mathrm{vol}_h = u^*\omega(e_1, je_1)e^1 \wedge e^2 = u^*\omega$. So we obtain

$$\frac{1}{2}\int_\Sigma |du|^2_{g_J}\mathrm{vol}_h = \int_\Sigma u^*\omega + \int_\Sigma |\bar\partial_J u|^2_{g_J}\mathrm{vol}_h. \tag{77}$$

Now the fact that ω is closed means that the first term on the right-hand side is a *topological* quantity: it is just $\langle[\omega], u_*[\Sigma]\rangle$ where $[\omega] \in H^2(M)$ is the de Rham cohomology class of ω. Meanwhile the second term on the right-hand side is nonnegative, and is zero if and only if u is J-holomorphic. So we obtain:

COROLLARY 3.5.4. *Let (Σ, j, h) be a compact Hermitian manifold of real dimension 2, let (M, ω) be a symplectic manifold with $J \in \mathcal{J}(M, \omega)$, and for any C^1 map $v\colon \Sigma \to M$ define the* energy *of v as the quantity $E(v) = \frac{1}{2}\int_\Sigma |dv|^2_{g_J}\mathrm{vol}_h$. Suppose that $u\colon \Sigma \to M$ has $\bar\partial_J u = 0$. Then:*

(i) $E(u) = \langle[\omega], u_*[\Sigma]\rangle$.
(ii) *If $v\colon \Sigma \to M$ represents the same class in $H_2(M;\mathbb{Z})$ as does u, then $E(v) \geq E(u)$ with equality if and only if v is J-holomorphic.*

Thus the energy of a J-holomorphic map—seemingly a geometric quantity—is determined by its topology, and J-holomorphic maps minimize the energy in their homology classes. One obvious consequence of Corollary 3.5.4 and the definition of $E(u)$ is that if $A \in H_2(M;\mathbb{Z})$ has $\langle[\omega], A\rangle \leq 0$ then A can never be represented by a J-holomorphic curve, except in the case that $A = 0$ in which case the only J-holomorphic representatives of A are the constant maps.

3.5.1. Compactness. If we fix a class $A \in H_2(M;\mathbb{Z})$ and a complex curve (Σ, j) and let $\mathcal{M}(\Sigma, A, J) = \{u\colon \Sigma \to M | \bar\partial_J u = 0, u_*[\Sigma] = A\}$, then Corollary 3.5.4 gives something like a $W^{1,2}$ bound on all elements of $\mathcal{M}(A, J)$. A bit more precisely, we could embed M in some Euclidean space $\mathbb{R}^N$ and use the usual norm on $W^{1,2}(\Sigma;\mathbb{R}^N)$ to make sense of the $W^{1,2}$ norm of a map $u\colon \Sigma \to M$, and then as long as M is compact (implying a bound on the L^2 norm of any map $\Sigma \to M$) we would have an upper bound on all elements of $\mathcal{M}(\Sigma, A, J)$ in this $W^{1,2}$ norm. If we instead were to have a bound on the $W^{1,p}$ norms of elements of $\mathcal{M}(\Sigma, A, J)$ for some $p > 2$, then Theorem 1.3.16 would imply that every sequence in $\mathcal{M}(A, J)$ has a uniformly convergent subsequence (and with a bit more work based on the proof of Theorem 3.3.5 one could even show that the limit is a genuine J-holomorphic curve). Unfortunately since we are in the "Sobolev borderline" case $p = \dim \Sigma = 2$, this reasoning does not quite apply, and indeed $\mathcal{M}(\Sigma, A, J)$ is generally not sequentially compact. However, as we will discuss (briefly) next, a somewhat weaker form of compactness does hold, in the sense that $\mathcal{M}(A, J)$ can be compactified by adding other objects that are built out of J-holomorphic curves and adjusting for reparametrization issues.

Before making positive statements, let us give a couple examples which should disabuse the reader of any hope that $\mathcal{M}(\Sigma, A, J)$ should generally be compact, while suggesting the limited senses in which compactness can fail.

EXAMPLE 3.5.5. *Let $\Sigma = M = \mathbb{C}P^1$, and let $A = [\mathbb{C}P^1]$ be the fundamental class. Then $\mathcal{M}(\Sigma, A, J)$ is just the set of degree-one holomorphic maps from $\mathbb{C}P^1$ to itself; these are precisely the Möbius transformations $\phi([w : z]) = [cz+d : az+b]$ for $a, b, c, d \in \mathbb{C}$ with $ad \neq bc$. (Identifying $\mathbb{C}$ with a dense subset of $\mathbb{C}P^1$ via $z \mapsto [1 : z]$, these restrict to $\mathbb{C}$ as $z \mapsto \frac{az+b}{cz+d}$.) But the group $PGL(2; \mathbb{C})$ of Möbius transformations is not compact under any reasonable topology. If we let $\phi_n([w : z]) = [nw : z] = [w : \frac{z}{n}]$, then for all $[w : z] \in \mathbb{C}P^1 \setminus \{[0 : 1]\}$ we have $\phi_n([w : z]) \to [1 : 0]$, while $\phi_n([0 : 1]) = [0 : 1]$ for all n. So $\{\phi_n\}$ converges pointwise to a discontinuous function, and so no subsequence of $\{\phi_n\}$ can converge uniformly to anything.*

This issue affects all nonempty spaces $\mathcal{M}(\mathbb{C}P^1, A, J)$ for all nonzero classes $A \in H_2(M; \mathbb{Z})$, the point being that if $u \in \mathcal{M}(\mathbb{C}P^1, A, J)$ then $u \circ \phi \in \mathcal{M}(\mathbb{C}P^1, A, J)$ for all $\phi \in PGL(2; \mathbb{C})$. For $A \neq 0$, any element u of $\mathcal{M}(\mathbb{C}P^1, A, J)$ is nonconstant, so after precomposition by some Möbius transformation can be arranged to have the property that $u([1 : 0]) \neq u([0 : 1])$. But then if ϕ_n is the sequence of the previous paragraph, $u \circ \phi_n$ will converge pointwise to a function taking exactly the two values $u([1 : 0]), u([0 : 1])$, so $u \circ \phi_n$ has no subsequence converging to any element of $\mathcal{M}([\Sigma], A, J)$ (or indeed to any continuous map, much less a pseudoholomorphic curve).

This suggests that we may have been asking the wrong question, at least when $\Sigma = \mathbb{C}P^1$: instead of asking for compactness of $\mathcal{M}([\Sigma], A, J)$ we should ask for compactness of the quotient of this space by the equivalence relation given by identifying a general element u with any of its reparametrizations $u \circ \phi$ for $\phi \in PGL(2; \mathbb{C})$. Another way of dealing with this reparametrization issue is to choose distinct codimension-two submanifolds C_1, C_2, C_3 each having nontrivial intersection number with the class A and then restrict attention to those elements u of $\mathcal{M}(\Sigma, A, J)$ such that $u(p_i) \in C_i$ for $i = 1, 2, 3$, where p_1, p_2, p_3 are three fixed points on $\mathbb{C}P^1$. Since reparametrizations of u will typically no longer satisfy the latter property, this has the effect of eliminating reparametrizations as a source of noncompactness (at least in generic situations).

EXAMPLE 3.5.6. *For this example let $M = \mathbb{C}P^2$, equipped with its standard complex structure J and the Fubini-Study symplectic form ω from Example 2.5.9. Let H denote the positive (*i.e.* $\langle [\omega], H \rangle > 0$) generator for $H_2(\mathbb{C}P^2; \mathbb{Z})$.*

The idea in this example is to consider the limit of the of the complex curves $\{(x, y) \in \mathbb{C}^2 | xy = 1/n\}$; after including them into $\mathbb{C}P^2$ and taking the closure to obtain a curve $C_n \subset \mathbb{C}P^2$, these curves can each be seen as copies of $\mathbb{C}P^1$ representing the homology class $2H$, while their limit, which would be the closure in $\mathbb{C}P^2$ of $\{x = 0\} \cup \{y = 0\} \subset \mathbb{C}^2$, is a union of two copies of $\mathbb{C}P^1$ each representing the class H.

To phrase this in terms of holomorphic maps, the C_n are the images of maps $u_n : \mathbb{C}P^1 \to \mathbb{C}P^2$ obtained by extending the map $z \mapsto \left(\frac{1}{nz}, z\right)$ from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C}^2$ to a map between projective spaces. So for $w \neq 0$ we should have $u_n([1 : \frac{z}{w}]) = \left[1 : \frac{1}{nz/w} : \frac{z}{w}\right]$; clearing denominators yields a map on all of $\mathbb{C}P^1$, namely

$$u_n([w : z]) = \left([nzw : w^2 : nz^2]\right) = \left[zw : \frac{w^2}{n} : z^2\right].$$

From this formula it should be clear that

$$\text{As } n \to \infty,\ u_n([w:z]) \to \begin{cases} [w:0:z] & \text{if } [w:z] \neq [1:0] \\ [0:1:0] & \text{if } [w:z] = [1:0] \end{cases}.$$

So somewhat similarly to Example 3.5.5, the sequence $\{u_n\}_{n=1}^{\infty}$ converges pointwise to a discontinuous function, so does not converge uniformly to any function. For any given compact subset K of $\mathbb{C}P^1 \setminus \{[1:0]\}$, $u_n|_K$ does converge uniformly to the function $u|_K$, where $u\colon \mathbb{C}P^1 \to \mathbb{C}P^2$ is defined by $u([w:z]) = [w:0:z]$. In other words u has image given by the closure in $\mathbb{C}P^2$ of the complex line $\{(x,y) : x = 0\} \subset \mathbb{C}^2$. This is one half of the union of two complex projectivized lines that we expected to be the limit of C_n. So our maps u_n converge away from the point $[1:0]$, but the image of the limit u is "missing" the copy of $\mathbb{C}P^1$ corresponding to the line $\{(x,y) : y = 0\}$.

It turns out that we can recover this missing line by reparametrizing the u_n; this essentially allows for a better understanding of the behavior of u_n near the "bubble point" $[1:0]$. As in Example 3.5.5 let $\phi_n([w:z]) = [nw:z] = [w:z/n]$. Note that for large n and any fixed neighborhood U of $[1:0]$ with closure contained in $\mathbb{C}P^1 \setminus \{[0:1]\}$, ϕ_n maps U to a very small neighborhood of $[1:0]$. Hence if we consider the behavior of the sequence $\{u_n \circ \phi_n\}$ then we are essentially "zooming in" on $[1:0]$, which is the point at which our original sequence $\{u_n\}$ behaved poorly. We find that

$$u_n \circ \phi_n([w:z]) = [nzw : nw^2 : z^2] = \left[zw : w^2 : \frac{z^2}{n}\right]$$

and so

$$\text{As } n \to \infty,\ u_n \circ \phi_n([w:z]) \to \begin{cases} [z:w:0] & \text{if } [w:z] \neq [0:1] \\ [0:0:1] & \text{if } [w:z] = [0:1]. \end{cases}.$$

So the sequence $\{u_n \circ \phi_n\}$ converges uniformly on compact subsets of $\mathbb{C}P^1 \setminus \{[0:1]\}$ to the map $v([w:z]) = [z:w:0]$; this latter map has image equal precisely to the "other half" $\{y = 0\}$ of the expected limit of the curves C_n.

Thus the sequence of maps $u_n \in \mathcal{M}(\mathbb{C}P^1, 2A, J)$ has a uniform-on-compact-subsets limit u on the complement of a single bubble point (and u extends to a map over the bubble point, though it is no longer the limit of $\{u_n\}$ there); by reparametrizing so as to zoom in on this bubble point we obtain a sequence having a limit v which likewise extends to a map of $\mathbb{C}P^1$. This map v is called a "bubble," and the sequence $\{u_n\}$ is said to weakly converge to the "bubble tree" (u,v). The formal definition of weak convergence is rather involved and I will not attempt to give it, but obviously it is not convergence in any Sobolev space or C^k sense, since (u,v) is not a map but rather a pair of maps. In particular this reflects that $\mathcal{M}(\mathbb{C}P^1, 2H, J)$ (and likewise its quotient by reparametrizations) is not compact. However it also suggests that the one can obtain from $\mathcal{M}(\mathbb{C}P^1, 2H, J)$ a space that is at least closer to being compact by adding objects which are made out of J-holomorphic curves whose homology classes have sum equal to $2H$.

With very little time left in the semester, I don't intend to give a serious proof of the compactness theorem, but will indicate now some of the basic ingredients. I gave a more complete account in [**U1**, Section 8], and [**MS2**, Chapter 5] contains a still-more-detailed treatment. Fix a compact symplectic manifold (M,ω) with an almost complex structure $J \in \mathcal{J}(M,\omega)$, let (Σ, j, h) be a compact Hermitian manifold of real dimension two, and let $A \in H_2(M;\mathbb{Z})$. Also let $D(r)$ denote the closed disk in $\mathbb{C}$ of radius r around the origin. Suppose that $\{u_n\}_{n=1}^{\infty}$ is a sequence in the moduli space $\mathcal{M}(\Sigma, A, J) = \{u\colon \Sigma \to M | \bar{\partial}_J u = 0,\ u_*[\Sigma] = A\}$. We can use h and g_J to measure quantities such as $|du(p)|$ for any point $p \in \Sigma$. The energy of a map $u\colon C \to M$ (where C is now a complex curve possibly with boundary—usually C will be a subset of Σ or its image under a holomorphic coordinate chart) is again defined as $E(u) = \frac{1}{2}\int_C |du|^2 \mathrm{vol}$. By Corollary 3.5.4 we have $E(u) = \int_C u^*\omega$ when u is J-holomorphic.

(i) Suppose that $K \subset \Sigma$ is any compact subset on which there is a uniform bound $\|du_n|_K\|_{L^\infty} < C$. Then the Arzelà-Ascoli theorem gives a subsequence of u_n which converges uniformly on K. In fact with a little more work using the methods of Section 3.3, one can obtain $W^{2,p}$ bounds on the u_n for $p > 2$, which implies that (after passing to an appropriate subsequence) their derivatives also converge uniformly, and so the limit will in fact be a J-holomorphic curve.

(ii) A result sometimes called "Gromov's Schwarz Lemma" ([**U1**, Corollary 8.14]) asserts that there are constants $C, \hbar > 0$ depending only on (M, J) such that any J-holomorphic map $u\colon D(1) \to M$ such that $E(u) < \hbar$ obeys $|du(0)| \le C$. By reparametrizing we see that any J-holomorphic map $u\colon D(r) \to M$ such that $E(u) < \hbar$ obeys $|du(0)| \le \frac{C}{r}$. It follows from this that if $p \in \Sigma$ and if $|du_n(p)| \to \infty$ then p must be a "bubble point" in the following sense:

$$\text{For every neighborhood } U \text{ of } p,\ \liminf_{n\to\infty} E(u_n|_U) \ge \hbar. \tag{78}$$

A little more strongly, if U is an neighborhood of p with each $E(u_n|_U) < \hbar$ and $V \Subset U$ then we can apply Gromov's Schwarz Lemma to coordinate neighborhoods centered at points in V to obtain a bound $\|du_n|_V| \le C_V$ for a constant C_V. Thus for any $p \in \Sigma$, either p has a neighborhood V_p on which $\|du_n|_{V_p}\|_{L^\infty}$ is bounded or else p is a bubble point in the sense of (78).

(iii) Note that since $E(u_n) = \langle [\omega], A\rangle$ is independent of n, we can pass to a subsequence of u_n for which there are only finitely many (more specifically, at most $\frac{\langle[\omega],A\rangle}{\hbar}$) bubble points $p_1, \dots, p_N$. (Choose one bubble point p_1 and pass to a subsequence—still called u_n—for which $E(u_n|_U) \ge \hbar$ for all neighborhoods U of p_1 and *all* sufficiently large n, *i.e.* so that every subsubsequence of our subsequence still has p_1 as a bubble point, then repeat for a bubble point p_2 of this subsequence, and so on until no bubble points are left. This process stops after at most $\frac{\langle[\omega],A\rangle}{\hbar}$ steps since arbitrarily small neighborhoods of each of the bubble points contribute energy at least $\hbar$ to every subsequence.) Moreover, the last sentence of (ii) combined with (i) shows that a further subsequence of $\{u_n\}$ converges uniformly to a pseudoholomorphic curve $u\colon \Sigma \setminus \{p_1, \dots, p_N\} \to M$ on compact subsets of $\Sigma \setminus \{p_1, \dots, p_N\}$. Passing to a further subsequence if necessary (and perhaps deleting some of the p_i), we may as well assume that each p_i is a bubble point for every subsequence of $\{u_n\}$.

(iv) A pseudoholomorphic version of the removal of singularities theorem from complex analysis ([**U1**, Theorem 8.11]) asserts that a finite energy J-holomorphic map $v\colon D(1)\setminus\{0\} \to M$ extends over the origin to a J-holomorphic map on all of $D(1)$. Applying this in coordinate neighborhoods of $p_1, \dots, p_N$ to the map u from the end of (iii) shows that u extends to a J-holomorphic map $u\colon \Sigma \to M$. Thus, much like in Example 3.5.6, we have found a map J-holomorphic map $u\colon \Sigma \to M$ and a finite set $\{p_1, \dots, p_N\}$ such that a subsequence of $\{u_n\}$ converges uniformly on compact subsets of $\Sigma \setminus \{p_1, \dots, p_N\}$ to u, and such that the derivatives likewise converge uniformly on compact subsets of $\Sigma \setminus \{p_1, \dots, p_N\}$. By the definition of a bubble point (and the condition in the last sentence of (iii)), for every $K \Subset \Sigma\setminus\{p_1, \dots, p_N\}$ we will have $E(u_n|_K) \le \langle[\omega], A\rangle - N\hbar$, and hence $E(u) \le \langle[\omega], A\rangle - N\hbar$. In particular if our subsequence has any bubble points, then $u\colon \Sigma \to M$ will represent a class having strictly smaller pairing with $[\omega]$ than does A.

(v) We now consider the behavior near the bubble points. Let p_i be one of these points, identify a neighborhood of p_i by a holomorphic coordinate chart with $D(1) \subset \mathbb{C}$ (with p_i corresponding to 0), and identify the maps u_n with their representations $u_n\colon D(1) \to M$ in these coordinate charts. A fairly elementary argument (see [**U1**, Lemmas 8.19 and 8.20]) allows one to find a sequence $\zeta_n \to 0$ and numbers $\epsilon_n \to 0$ such that $r_n := \epsilon_n |du_n(\zeta_n)| \to \infty$ and $|du_n(\zeta_n + w)| \leq 2|du_n(\zeta_n)|$ whenever $|w| \leq \epsilon_n$. One then defines maps $v_n : D(r_n) \to M$ by

$$v_n(z) = u_n\left(\zeta_n + \frac{z}{|du_n(\zeta_n)|}\right).$$

In other words we are zooming in on a sequence of points ζ_n converging to 0 at which $|du_n|$ is large. By construction $|dv_n(0)| = 1$ for all n, and the condition that $|du_n(\zeta_n + w)| \leq 2|du_n(\zeta_n)|$ whenever $|w| \leq \epsilon_n$ implies that $|dv_n(z)| \leq 2$ for all z in the domain $D(r_n)$ of v_n. But $r_n \to \infty$, so it follows from (i) that a subsequence of $\{v_n\}$ converges uniformly on compact subsets to a J-holomorphic map $v : \mathbb{C} \to M$. Certainly the energy of v is no greater than $\langle [\omega], A\rangle$. Even better, viewing $\mathbb{C}$ as the complement of a point in $\mathbb{C}P^1$ in the usual way, by applying the removal of singularities theorem from (iv) to a coordinate neighborhood of the point at infinity we find that v extends to a J-holomorphic map $v : \mathbb{C}P^1 \to M$. Thus associated to each bubble point p we have produced a holomorphic sphere v in M, by rescaling the domains of restrictions of the v_n to certain open subsets contained in small neighborhoods of v. This sphere is nonconstant, since $|dv(0)| = \lim_{n\to\infty} |dv_n(0)| = 1$. Also for any neighborhod U of p, the quantity $\liminf_{n\to\infty} E(u_n|_U)$ is at least equal to the energy of v.

The above outline shows how a sequence $\{u_n\}$ in $\mathcal{M}(\Sigma, A, J)$ has a subsequence which converges on the complement of finitely many bubble points to an element $u \in \mathcal{M}(\Sigma, B, J)$ (with $B = A$ if and only if there are no bubble points, and $\langle [\omega], B\rangle < \langle [\omega], A\rangle$ otherwise), and moreover produces a nonconstant J-holomorphic sphere $v^{(i)}$ from the behavior of the u_n near each bubble point p_i. A more refined argument (which in particular involves a different approach to rescaling in (v) above) builds a "bubble tree" consisting of u together with collections of J-holomorphic spheres sprouting off of each of the bubble points (and possibly bubbling off of each other), such that the sum of all of the homology classes of all the resulting curves is equal to A. Note that while our spheres $v^{(i)}$ are nonconstant, it is possible for u to be constant—this is what happens with the sequence $\{\phi_n\}$ in Example 3.5.5.

By the way, the same conclusions about convergence to a J-holomorphic bubble tree hold if, instead of assuming that $u_n \in \mathcal{M}(\Sigma, A, J)$ for some fixed J, we assume that $u_n \in \mathcal{M}(\Sigma, A, J_n)$ for some sequence J_n of almost complex structures that C^0-converges to J, in which case the limit will be J-holomorphic.

3.6. Sketch of a proof of the non-squeezing theorem

In this final section we indicate how the general theory that we have discussed can be used to prove Gromov's non-squeezing theorem [**G**]; this is one of the earliest applications of pseudoholomorphic curves, and remains one of the most famous.

We now set up notation for the theorem, using contemporary notation for the sets involved. For $n \in \mathbb{N}$ we write points in $\mathbb{R}^{2n}$ as $(\vec{x}, \vec{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$, and use the symplectic form $\omega_0 = \sum_i dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$. For $a \geq 0$ we define the ball

$$B^{2n}(a) = \{(\vec{x}, \vec{y} \in \mathbb{R}^{2n} | \pi \sum_i (x_i^2 + y_i^2) \leq a\},$$

so $B^{2n}(a)$ intersects the $x_1 y_1$ plane in a disk of area a, and $B^{2n}(a)$ is a ball of radius $\sqrt{\frac{a}{\pi}}$. Also let

$$Z^{2n}(a) = \{(\vec{x}, \vec{y}) \in \mathbb{R}^{2n} | \pi(x_1^2 + y_1^2) \leq a\},$$

so $Z^{2n}(a) = B^2(a) \times \mathbb{R}^{2n-2}$, with the natural product symplectic structure.

A basic feature of symplectic geometry is that a symplectic manifold (M,ω) admits many symplectomorphisms (*i.e.* diffeomorphisms obeying $\phi^*\omega=\omega$), obtained for instance as flows of vector fields X such that $\omega(X,\cdot)$ is closed. For some time it was unclear what sorts of qualitative properties a symplectomorphism ϕ necessarily satisfies, other than the easy fact that ϕ must preserve the volume form given as the top exterior power of ω. The following non-squeezing theorem was the first significant constraint going beyond volume preservation.

THEOREM 3.6.1 (Gromov's non-squeezing theorem). *Suppose that $\phi\colon B^{2n}(a)\to Z^{2n}(A)$ is an embedding with $\phi^*\omega_0=\omega_0$. Then $a\le A$.*

So in particular a symplectomorphism of $\mathbb{R}^{2n}$ can never map $B^{2n}(a)$ into $Z^{2n}(A)$ if $A<a$; on the other hand certainly there are (for $n\ge 2$) many volume-preserving diffeomorphisms of $\mathbb{R}^{2n}$ that do so. For that matter if we had defined $Z^{2n}(A)$ slightly differently the theorem would be false: given any a,A, for sufficiently small $\lambda>0$ the map $\phi(\vec{x},\vec{y})=(\lambda\vec{x},\lambda^{-1}\vec{y})$ is a symplectomorphism that maps $B^{2n}(a)$ into $\{(\vec{x},\vec{y})|\pi(x_1^2+x_2^2)\le A\}$. So it is essential that the cylinder $Z^{2n}(A)$ was defined using the variables x_1,y_1, not x_1,x_2.

To start the proof of Theorem 3.6.1, first note that here will be some $N>0$ such that $\phi(B^{2n}(a))\subset int([-N,N]^{2n})$, so if we let $T^{2n}_N=\frac{\mathbb{R}^{2n-2}}{2N\mathbb{Z}^{2n-2}}$ it suffices to show that (for any N) if there is an embedding $\phi\colon B^{2n}(a)\to B^2(A)\times T^{2n}_N$ with $\phi^*\omega_N=\omega_0$ then $a\le A$. (The point of doing ths is to make the codomain compact.) Here ω is the sympletic form on $\mathbb{R}^2\times T^{2n}_N$ that pulls back by the quotient map to the standard symplectic form on $\mathbb{R}^{2n}$.

For any $b>0$ let $S^2(b)$ denote the symplectic manifold whose underlying smooth manifold is S^2 and has symplectic form whose integral over S^2 is b. (It's a basic symplectic geometry exercise to show that this determines $S^2(b)$ up to symplectomorphism. Of course for a concrete version one could use an appropriate multiple of the Fubini-Study form from Example 2.5.9.) Whenever $a<b$, $B^{2n}(a)$ (with its standard symplectic structure) can be seen as a codimension-zero submanifold-with-boundary of $S^2(b)$ The key lemma is:

LEMMA 3.6.2. *For any positive real numbers b,N, let J be any almost complex structure that is compatible with the product symplectic form Ω on $S^2(b)\times T^{2n-2}_N$. Choose any point $x_0\in S^2(b)\times T^{2n-2}_N$. Then there is a J-holomorphic map $u\colon \mathbb{C}P^1\to S^2(b)\times T^{2n-2}_B$ such that $u_*[\mathbb{C}P^1]=[S^2\times\{pt\}]$ and $x_0\in \mathrm{Im}(u)$.*

PROOF. (Sketch) Let $\pi_1\colon S^2(b)\times T^{2n-2}_N\to S^2$ and $\pi_2\colon S^2(b)\times T^{2n-2}_N\to T^{2n-2}_N$ for the projections, and choose points $s_1,s_\infty\in S^2$ such that the elements $\pi_1(x_0),s_1,s_\infty$ are all distinct. Identify $\mathbb{C}P^1$ with $\mathbb{C}\cup\{\infty\}$ in the standard way, so we have three distinguished points $0,1,\infty\in\mathbb{C}P^1$.

For any $J\in\mathscr{J}(S^2(b)\times T^{2n-2}_N,\Omega)$, consider the space

$$\mathscr{M}_J=\left\{u\colon \mathbb{C}P^1\to S^2\;\middle|\;\begin{array}{c}\bar\partial_J u=0,\ u_*[\mathbb{C}P^1]=[S^2\times\{pt\}],\\ u(0)=x_0,\ \pi_1(u(1))=s_1,\ \pi_2(u(\infty))=s_\infty\end{array}\right\}.$$

Evidently it suffices to show that, for every J, $\mathscr{M}_J$ is nonempty. (The conditions on $u(1),u(\infty)$ are introduced to eliminate reparametrization symmetry, as suggested at the end of Example 3.5.5.) These spaces fit into a simple modification of the Banach manifold setup described before Proposition 3.4.1: for $k\ge1,p>2$ we have a bundle $\mathscr{E}^{k-1,p}\to W^{k,p}(\mathbb{C}P^1,M)$ of which $\bar\partial_J$ is a section, and we can also consider the "evaluation map" $e\colon W^{k,p}(\mathbb{C}P^1,M)\to(S^2\times T^{2n-2}_N)\times S^2\times S^2$ defined by

$$e(u)=(u(0),\pi_1(u(1)),\pi_1(u(\infty))).$$

By locally trivializing the bundle $\mathscr{E}^{k,p}$ using parallel transport as before, we see that the intersection of $\mathscr{M}_J$ with a small neighborhood $\mathscr{U}$ of an element $u\in\mathscr{W}^{k,p}(u^*TM)$ is given by the preimage of the point $(0,(x_0,s_1,s_\infty)) \in \mathcal{E}_u^{k-1,p} \times \left((S^2 \times T_N^{2n-2})\right) \times S^2 \times S^2$ under the map $u \mapsto (\mathcal{P}^{\tilde\nabla}\bar\partial_J u, e(u))$. This map has linearization

$$\mathcal{D}_u\colon W^{k,p}(u^*TM) \to \mathcal{E}_u^{k-1,p} \times T_{x_0}(S^2 \times T_N^{2n-2}) \times T_{s_1}S^2 \times T_{s_\infty}S^2$$
$$\xi \mapsto (D_u\xi, \xi(0), \pi_{1*}\xi(1), \pi_{2*}\xi(\infty))$$

where D_u is the operator from Proposition 3.4.1, which was shown to be Fredholm in Theorem 3.4.3.

Since the codomain of $\mathcal{D}_u$ is the product of the codomain of D_u with a $(2n+4)$-dimensional vector space, it is not hard to see from Theorem 3.4.7 that $\mathcal{D}_u$ is also Fredholm and that $ind(\mathcal{D}_u) = ind(D_u) - (2n+4)$. According to (74), we have $ind(D_u) = 2(n + \langle c_1(T(S^2 \times T_{2n-2})), [S^2 \times \{pt\}]\rangle) = 2(n + \langle c_1(TS^2), [S^2]\rangle) = 2(n+2)$ since $c_1(TS^2)$ coincides with the Euler class of TS^2 and so evaluates on the fundamental class to give the Euler characteristic of S^2, *i.e.*, 2. Thus $ind(\mathcal{D}_u) = 0$, so if $\mathcal{D}_u$ is surjective at all elements of $\mathcal{M}_J$ then $\mathcal{M}_J$ will be a zero-dimensional manifold. Let us call an almost complex structure J *regular* if $\mathcal{D}_u$ is surjective at all $u \in \mathcal{M}_J$.

Let's see how this works when our almost complex structure J on $S^2 \times T_N^{2n-2}$ is the "product complex structure" J_0, obtained by acting separately on TS^2 and TT_N^{2n-2} by the standard complex structures on these manifolds (where S^2 is identified with $\mathbb{C}P^1$ and T_N^{2n-2} is viewed as a quotient of $\mathbb{C}^{n-1}$). We can then write any map $u\colon \mathbb{C}P^1 \to S^2 \times T_N^{2n-2}$ as $u = (\pi_1 \circ u, \circ\pi_2 \circ u)$, and $\bar\partial_{J_0} u = 0$ if and only if both $\pi_1 \circ u\colon \mathbb{C}P^1 \to S^2$ and $\pi_2 \circ u\colon \mathbb{C}P^1 \to T_N^{2n-2}$ are holomorphic with respect to the standard complex structures. The statement that $u_*[\mathbb{C}P^1] = [S^2 \times \{pt\}]$ is equivalent to the statement that $(\pi_1 \circ u)_*[\mathbb{C}P^1] = [S^2]$ and $(\pi_2 \circ u)_*[\mathbb{C}P^1] = 0$. So in this case $\bar\partial_{J_0} = 0$ if and only if (again identifying S^2 with $\mathbb{C}P^1$) $\pi_1 \circ u$ is a Möbius transformation and $\pi_2 \circ u$ is constant (as Corollary 3.5.4 shows that a holomorphic null-homologous map from a compact domain must be constant). From this we see that

(79) $\mathcal{M}_{J_0}$ consists of exactly one element $u_{J_0} = (u_S, u_T)$,

where $u_S\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ is the unique Möbius transformation sending $0, 1, \infty$ to $\pi_1(x_0), s_1, s_\infty$ respectively, and u_T is the constant map to $\pi_2(x_0)$.

In fact it's not hard to see that $\mathcal{D}_{u_{J_0}}$ is surjective; since we already determined that $\mathcal{D}_{u_{J_0}}$ has index zero it suffices to show that $\mathcal{D}_{u_{J_0}}$ is injective.[5] Because J_0 is the product complex structure, an element of $\ker \mathcal{D}_{u_{J_0}}$ is given by a pair (ξ_S, ξ_T) where ξ_S is a holomorphic section of (the holomorphic line bundle) $u_S^*TS^2$ and ξ_T is a holomorphic section of $u_T^*T_N^{2n-2}$, satisfying the additional properties that $\xi_S(0) = \xi_S(1) = \xi_S(\infty) = 0$ and that $\xi_T(0) = 0$ in order to preserve the conditions on $u(0), u(1), u(\infty)$. Now u_T is a constant map, so a holomorphic section of $u_T^*T_N^{2n-2}$ is the same thing as a holomorphic map $\mathbb{C}P^1 \to \mathbb{C}^{n-1}$, any one of which is constant, so the condition that $\xi_T(0) = 0$ implies that $\xi_T = 0$ whenever $(\xi_S, \xi_T) \in \ker \mathcal{D}_{u_{J_0}}$. As for ξ_S, one can see in various ways that a holomorphic section of the bundle u_S^*TM that has more than two zeros must vanish identically. From an algebraic geometry perspective this is because u_S^*TM has degree two and any zero of a not-identically-zero section contributes positively to the degree. For a more elementary argument, one can find a section X of u_S^*TM vanishing to order one exactly at 0 and ∞ (for instance, take a vector field generating a one-parameter family of Möbius transformations each of which fixes s_0, s_∞, and then pull this vector field back by the holomorphic map u_S), and then the ratio $\frac{\xi_S}{X}$ is (after removal

[5]Alternatively, if one didn't want to appeal to the index computation (74), one could show that $\mathcal{D}_{u_{J_0}}$ is surjective by showing that the adjoint of $\mathcal{D}_{u_{J_0}}$ is injective. This would involve first computing this adjoint in terms of sections of various holomorphic line bundles over $\mathbb{C}P^1$ and then applying a similar analysis to the argument that $\mathcal{D}_{u_{J_0}}$ is injective that we give presently.

of singularities) a well-defined holomorphic function $\mathbb{C}P^1 \to \mathbb{C}$ which vanishes at 1. But the only holomorphic functions $\mathbb{C}P^1 \to \mathbb{C}$ are constants, so in fact $\xi_S = 0$. So indeed $\ker \mathscr{D}_{u_{J_0}} = \{0\}$, which implies that J_0 is a regular almost complex structure in the sense defined earlier.

The key now is to consider one parameter families of almost complex structures $\{J_t\}_{t\in[0,1]}$ beginning at our regular complex structure J_0, and their associated "parametrized moduli spaces"

$$\mathscr{M}_{\{J_t\}} = \{(t,u) \in [0,1] \times W^{k,p}(\mathbb{C}P^1, S^2 \times T_N^{2n-2}) | u \in \mathscr{M}_{J_t}\}.$$

One finds by a similar analysis to what we have seen before that any such space is locally modeled by the zero locus of a Fredholm operator of index equal to $ind(\mathscr{D}_u) + 1$, *i.e.* to 1. If this Fredholm operator is surjective at all elements of $\mathscr{M}_{\{J_t\}}$, then $\mathscr{M}_{\{J_t\}}$ will be a one-dimensional manifold with boundary, the boundary consisting of the points with $t = 0$ or $t = 1$. An argument with the Sard-Smale theorem along the lines suggested at the end of Section 3.4.2 shows that, for any k, there is a C^k-dense set in the space of paths of Ω-compatible almost complex structures $\{J_t\}_{t\in[0,1]}$ beginning at J_0 such that this surjectivity property is satisfied. Thus, for any member of this dense set of paths, $\mathscr{M}_{\{J_t\}}$ is a one-manifold with boundary consisting of the set $\{(0, u_{J_0}))\} \sqcup (\{1\} \times \mathscr{M}_{J_1})$.

The final crucial point is that any such $\mathscr{M}_{\{J_t\}}$ is compact. To explain why, we know from Section 3.5.1 that if a sequence $\{(t_m, u_m)\}_{m=1}^{\infty}$ failed to have a convergent subsequence, then this sequence would necessarily form bubbles. These bubbles would be nonconstant J_{t_m}-holomorphic spheres $v\colon \mathbb{C}P^1 \to S^2 \times T_N^{2n-2}$, each having energy $E(v)$ bounded above by the energy of each of the u_m, namely $\langle [\Omega], [S^2 \times \{pt\}] \rangle = b$. Now elementary fundamental group considerations show that v lifts to a map $\mathbb{C}P^1 \to S^2 \times \mathbb{C}^{n-1}$, so we must have $v_*[\mathbb{C}P^1] = k[S^2 \times \{pt\}]$ for some $k \in \mathbb{Z}$. But $E(v) = \int_{\mathbb{C}P^1} v^*\Omega = \langle [\Omega], v_*[\mathbb{C}P^1] \rangle$ is both positive (since v is nonconstant) and bounded above by b, so the only possibility for bubbling would be that $k = 1$, in which case v would be the only bubble and the almost-everywere limit u of the u_m would be constant. In fact this last possibility is prevented by the incidence conditions that $u_m(0) = x_0$, $\pi_1(u_m(1)) = s_1$, and $\pi_1(u_m(\infty)) = s_\infty$, since $\{u_m\}$ converges to u on the complement of its set of bubble points, and so if there is only one bubble point then u would take distinct values at at least two of the three points $0, 1, \infty$. So u cannot be constant, in view of which no bubbling can occur and $\mathscr{M}_{\{J_t\}}$ is compact.

It thus follows that for any path $\{J_t\}$ in our dense set, $\mathscr{M}_{\{J_t\}}$ is a *compact* 1-manifold with boundary equal to $\{(0, u_{J_0}))\} \sqcup (\{1\} \times \mathscr{M}_{J_1})$. But a compact 1-manifold with boundary can never have exactly one boundary point; thus $\mathscr{M}_{J_1}$ is nonempty.

This holds for arbitrary almost complex structures J_1 that arise as endpoints of paths $\{J_t\}$ in our C^k-dense set. Now if $J \in \mathscr{J}(S^2 \times T_N^{2n-2}, \Omega)$ is a completely arbitary compatible almost complex structure, then since Proposition 3.5.3 implies the existence of paths in $\mathscr{J}(S^2 \times T_N^{2n-2}, \Omega)$ from J_0 to J, we can approximate one such path by a sequence of paths in our dense set, and this yields a sequence of almost complex structures J_m with $J_m \to J$ and each $\mathscr{M}_{J_m}$ nonempty, say with $u_m \in \mathscr{M}_{J_m}$. The same compactness argument as that given two paragraphs ago shows that the u_m have a subsequence that converges to an element of $\mathscr{M}_J$, finally proving that $\mathscr{M}_J$ is nonempty. □

We now apply this lemma to our embedding problem. Suppose we have an embedding $\phi\colon B^{2n}(a) \to Z^{2n}(A)$ with $\phi^*\omega_0 = \omega_0$. As noted earlier, for sufficiently large N we can replace the codomain by $B^2(A) \times T_N^{2n-2}$ and we will still have an embedding. Now let $\epsilon > 0$, and consider the symplectic manifold $(S^2(A+\epsilon) \times T_N^{2n-2}, \Omega)$ where Ω is the same symplectic structure as in Lemma 3.6.2; using the (symplectic) inclusion $B^2(A) \subset S^2(A+\epsilon)$, ϕ is an embedding $B^{2n}(a) \hookrightarrow S^2(A+\epsilon) \times T_N^{2n-2}$ with $\phi^*\Omega = \omega_0$.

Now let J_0 be the standard complex structure on $\mathbb{C}^n \supset B^{2n}(a)$. Because $\phi^*\Omega = \omega_0$, the almost complex structure given by $\phi_* \circ J_0 \circ \phi_*^{-1}$ on $\phi(B^{2n}(a))$ is compatible with (the restriction of) Ω. By the proof of Proposition 3.5.3, we may construct an Ω-compatible almost complex structure J on

$S^2(A+\epsilon)\times T_N^{2n-2}$ whose restriction to $\phi(B^{2n}(a-\epsilon))$ is equal to $\phi_*J_0\phi_*^{-1}$. (Use a partition of unity to construct a Riemannian metric restricting to $\phi(B^{2n}(a-\epsilon))$ as $g_{\phi_*J_0\phi_*^{-1}}$ and input this metric into the map $g\mapsto J_g$ constructed in the proof of Propositon 3.5.3.) Lemma 3.6.2 produces a J-holomorphic curve $u\colon \mathbb{C}P^1\to S^2(b)\times T_N^{2n-2}$ such that $u(0)=\phi(\vec{0},\vec{0})$. The condition on the homology class of u means that $E(u)=A+\epsilon$.

Now $S:=u^{-1}(\phi(B^{2n}(a-\epsilon))^\circ)$ is an open subset of $\mathbb{C}P^1$, and we have a well-defined map $\phi^{-1}\circ u\colon S\to B^{2n}(a-\epsilon)^\circ$ (the superscript $^\circ$ means interior). Because J coincides with $\phi_*\circ J_0\circ\phi_*^{-1}$ on S, this map $\phi^{-1}\circ u$ is in fact J_0-holomorphic. Moreover

$$E(\phi^{-1}\circ u)=\int_S(\phi^{-1}\circ u)^*\omega_0=\int_S u^*\Omega=E(u|_S)\le E(u)=A+\epsilon.$$

By construction, $\phi^{-1}\circ u(0)=(\vec{0},\vec{0})$. If $c<a-\epsilon$ is any regular value of the smooth function $\pi\|\phi^{-1}\circ u\|^2$, then $S_c:=u^{-1}(\phi(B^{2n}(c)))$ is a subsurface-with-boundary of $\mathbb{C}P^1$, such that $\phi^{-1}\circ u|_{S_c}\colon S_c\to B^{2n}(c)$ is a J_0-holomorphic map with $\phi^{-1}\circ u(\partial S_c)\subset\partial B^{2n}(c)$. So by Sard's theorem (or the weaker statement that there exist regular values arbitrarily close to $a-\epsilon$) we conclude:

Proposition 3.6.3. *Suppose that there is an embedding $\phi\colon B^{2n}(a)\to Z^{2n}(A)$ with $\phi^*\omega_0=\omega_0$, and let $\epsilon>0$. Then there is $c>a-2\epsilon$, a compact almost complex 2-manifold with boundary S_c, and a J_0-holomorphic map $v\colon S_c\to B^{2n}(c)$ such that $(\vec{0},\vec{0})\in\mathrm{Im}(v)$, $v(\partial S_c)\subset\partial B^{2n}(c)$, and $E(v)\le A+\epsilon$.*

The key fact now is that a *holomorphic* map $v\colon S_c\to B^{2n}(c)$ with $(\vec{0},\vec{0})\in\mathrm{Im}(v)$ and $v(\partial S_c)\subset B^{2n}(c)$ necessarily has $E(v)\ge c$. So we obtain inequalities $a-2\epsilon<c\le E(v)\le A+\epsilon$, which since ϵ is arbitrary concludes the proof of the non-squeezing theorem modulo this lower bound $E(v)\ge c$ on the energy of a holomorphic map with image passing through the origin and boundary contained in the boundary of a ball of cross-sectional area c.

So we finally sketch the proof of this lower bound $E(v)\ge c$. For $0\le t\le c$ let $S_t=v^{-1}(B^{2n}(t))$. Whenever t is a regular value of the function $\pi\|v\|^2\colon S_c\to\mathbb{R}$, S_t is a submanifold with boundary of S_c, with v restricting as a J_0-holomorphic map of pairs $(S_t,\partial S_t)\to(B^{2n}(t),\partial B^{2n}(t))$. Furthermore, still assuming that t is a regular value of $\pi\|v\|^2$, $v|_{S_t}$ is transverse to $\partial B^{2n}(t)$, so at each point of ∂S_t the outward normal vector is mapped by v_* to a vector pointing outward from $\partial B^{2n}(t)$.

For all $t\in[0,c]$ let $\alpha(t)$ denote the area of S_t as a subset of S_c with the measure given by the pulled back form $v^*\omega_0$. This quantity is well-defined for all $t\in[0,c]$ and gives a monotone increasing function; for those t which are additionally regular values of $\pi\|v\|^2$ it additionally holds that S_t is a smooth manifold with boundary and $\alpha(t)$ is the integral of $(v|_{S_t})^*\omega$ over this manifold with boundary.

Also, for each regular value t of $\pi\|v\|^2$, let $\lambda(t)$ denote the length of the parametrized curve $v|_{\partial S_t}$. Fix such a t, and let us parametrize ∂S_t by a map $\gamma\colon I\to\partial S_t$ where I is a union of circles $\frac{\mathbb{R}}{\ell_j\mathbb{Z}}$ each having angular coordinate τ, consistently with the orientation of ∂S_t, in such a way that $v_*\frac{d\gamma}{d\tau}$ has length 1 with respect to the standard metric g_{J_0}. Thus $\lambda(t)=\sum\ell_j$. Then $-j\frac{d\gamma}{d\tau}$ is an outward normal along ∂S_t, so as noted earlier $v_*\left(-j\frac{d\gamma}{d\tau}\right)$ points outward from $\partial B^{2n}(t)$. Since v is J-holomorphic this latter vector is equal to $-Jv_*\frac{d\gamma}{d\tau}$, and so it also has length 1 and is orthogonal to $v_*\frac{d\gamma}{dt}$. So for small ϵ, flowing out from ∂S_t along the vector field $-\epsilon j\frac{d\gamma}{d\tau}$ extends S_t to a subsurface S_t^ϵ of S_c which is mapped by v inside the ball of radius $\sqrt{\frac{t}{\pi}}+\epsilon+o(\epsilon)$, and the energy of v on the newly-introduced part $S_t^\epsilon\setminus S_t$ is $\epsilon\lambda(t)+o(\epsilon)$. (Here as usual $o(\epsilon)$ is a generic symbol for a quantity depending on ϵ such that $\lim_{\epsilon\to 0}\frac{o(\epsilon)}{\epsilon}=0$.) This proves that, for t a regular value of $\pi\|v\|^2$, we have

$$\liminf_{\epsilon\to 0^+}\frac{\alpha(t+2\sqrt{\pi t}\epsilon)-\alpha(t)}{\epsilon}\ge\lambda(t),$$

and so since the implicit function theorem fairly readily implies that $\alpha(t)$ is differentiable near t when t is a regular value of $\pi\|v\|^2$,

$$\alpha'(t) \geq \frac{\lambda(t)}{2\sqrt{\pi t}} \quad \text{when } t \text{ is a regular value of } \pi\|v\|^2. \tag{80}$$

The final ingredient is then given by a mild adaptation of the classic isoperimetric inequality that in turn bounds $\lambda(t)$ below in terms of $\alpha(t)$. The standard isoperimetric inequality (see [**O**]) says that if C is a compact 2-manifold with boundary and $w\colon C \to \mathbb{R}^2$ is smooth then the length L of the parametrized curve $w|_{\partial C}$ obeys $L^2 \geq 4\pi \int_C w^*\omega_0$. If instead we have a map $w\colon C \to \mathbb{R}^{2n}$ then, denoting by $p_1, \ldots, p_n\colon \mathbb{R}^{2n} \to \mathbb{R}^2$ the projections to the various complex coordinate planes, one has $\int_C w^*\omega_0 = \sum_{i=1}^n \int_C (p_i \circ w)^*\omega_0$ (where ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$ or $\mathbb{R}$ as appropriate). Meanwhile, if we parametrize each component of ∂C by $\gamma_j\colon \mathbb{R}/\mathbb{Z} \to \partial C$ in such a way that $w_*\frac{d\gamma_j}{d\tau}$ is a unit vector at all times, then we have

$$\begin{aligned} Length(w|^2_{\partial C}) &= \sum_j \int_0^1 \|\gamma_j'(\tau)\|^2 d\tau = \sum_i \sum_j \int_0^1 \|(p_i \circ \gamma)_j'(\tau)\|^2 d\tau \\ &\geq \sum_i Length(p_i \circ w|_{\partial C})^2 \end{aligned}$$

where the last inequality follows from the Schwarz inequality (it may not be an inequality because the $p_i \circ w$ probably will not be constant-speed-parametrized). So the standard isoperimetric inequality shows that maps $w\colon C \to \mathbb{R}^{2n}$ continue to obey $Length(w|_{\partial C})^2 \geq 4\pi \int_C w^*\omega_0$. Applying this to our maps $v\colon S_t \to \mathbb{C}^n$ yields

$$\lambda(t)^2 \geq 4\pi\alpha(t) \quad \text{when } t \text{ is a regular value of } \pi\|v\|^2.$$

Combining this with (80) shows that, for any regular value t of $\pi\|v\|^2$, we have $\alpha'(t) \geq \sqrt{\frac{\alpha(t)}{t}}$. But then

$$\frac{d}{dt}\left(\sqrt{\alpha(t)} - \sqrt{t}\right) = \frac{\alpha'(t)}{2\sqrt{\alpha(t)}} - \frac{1}{2\sqrt{t}} \geq 0$$

whenever t is a regular value[6] of $\pi\|v\|^2$. Since Sard's theorem shows that the regular values of $\pi\|v\|^2$ form a full measure set, and since α is monotone increasing for all t (not just for regular values) and $\alpha(0) = 0$, it follows easily that $\sqrt{\alpha(t)} \geq \sqrt{t}$ for all $t \in [0, c]$. In particular $\alpha(c) \geq c$, so the map v produced by Proposition 3.6.3 has energy greater than or equal to c.

As noted earlier, this directly proves that $a - 2\epsilon < A + \epsilon$, which since ϵ is arbitrary completes the proof of the non-squeezing theorem.

[6]Note that the fact that $v(0) = \vec{0}$ implies that the sets S_t are never empty, which implies that $\alpha(t) > 0$ at all regular values, justifying the division by $\sqrt{\alpha(t)}$.

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