Math 8200, Spring 2011: Problem Set 9. Due Thursday, April 14

I. Hatcher Section 2.1, problems 11, 12, 14, 15, 16

II. a) Let $f\colon C_* \to D_*$ be a chain map between two chain complexes (C_*, ∂_C) and (D_*, ∂_D). For $n \in \mathbb{Z}$ define

$$(\mathcal{C}(f))_n = C_{n-1} \oplus D_n$$

and

$$\partial^{(n)}_{\mathcal{C}(f)}\colon\ (\mathcal{C}(f))_n \to (\mathcal{C}(f))_{n-1}$$
$$(c,d) \mapsto \left(-\partial_C^{(n-1)} c, f(c) + \partial_D^n d\right).$$

Prove that these definitions make $((\mathcal{C}(f))_*, \partial_{\mathcal{C}(f)})$ into a chain complex, and that we have a long exact sequence

$$\cdots \to H_{n+1}(\mathcal{C}(f)_*) \to H_n(C_*) \to H_n(D_*) \to H_n(\mathcal{C}(f)_*) \to H_{n-1}(C_*) \to \cdots.$$

Show moreover that the map $H_n(C_*) \to H_n(D_*)$ is the same as the map induced by f on homology. (Notational suggestion: define a new chain complex $C[-1]_*$ by $C[-1]_n = C_{n-1}$ and with boundary operator equal to the negative of ∂_C; this may make it easier to directly adapt results from class to the problem.) In homological algebra, the chain complex $\mathcal{C}(f)_*$ is called the *mapping cone* of f.

b) There is also a notion of mapping cone in topology. If X is a space let $CX = \frac{[0,1]\times X}{\{1\}\times X}$ (*i.e.*, CX is constructed form $[0,1]\times X$ by collapsing $\{1\}\times X$ to a point). Note that CX is contractible, using $F(s,[1-t,x]) = [1-st,x]$. If $f\colon X \to Y$ is a continuous map, the mapping cone of f is by definition the space

$$C_f = \frac{Y \coprod CX}{f(x) \sim [0,x] \text{ if } x \in X}.$$

Using the Mayer-Vietoris sequence, prove that, similarly to (a), there is a long exact sequence for $n > 0$

$$\cdots \to H_{n+1}(C_f) \to H_n(X) \to H_n(Y) \to H_n(C_f) \to H_{n-1}(X) \to \cdots$$

where the map $H_n(X) \to H_n(Y)$ is the one induced by f on homology. Hint: It may be helpful to make use of the mapping cylinder M_f of f which was defined early in the course; in particular recall that M_f deformation retracts to Y.