Math 8200, Spring 2011: Problem Set 8. Due Thursday, April 7

1. Let Δ^p denote the standard p-simplex in $\mathbb{R}^p$. If $v_0, \ldots, v_p \in \mathbb{R}^n$, prove that the map

$$\Delta^p \to [v_0, \ldots, v_p]$$
$$(t_1, \ldots, t_p) \mapsto \left(1 - \sum_{i=1}^{p} t_i\right) v_0 + \sum_{i=1}^{p} t_i v_i$$

is a homeomorphism if and only if the vectors $\{v_1 - v_0, \ldots, v_p - v_0\}$ are linearly independent in $\mathbb{R}^n$.

2. Let (C_*, ∂) be a finite-dimensional chain complex of vector spaces over some field k (in other words we have k-vector spaces C_n for each $n \in \mathbb{Z}$ where $\oplus_{n\in\mathbb{Z}} C_n$ is finite-dimensional, and "boundary operators" $\partial^{(n)}\colon C_n \to C_{n-1}$ which are k-linear maps such that $\partial^{(n-1)} \circ \partial^{(n)} = 0$). The *Euler characteristic* of (C_*, ∂) is by definition

$$\chi(C_*) = \sum_{n\in\mathbb{Z}} (-1)^n \dim C_n$$

(note that by hypothesis this sum only has finitely many terms). Prove that we also have

$$\chi(C_*) = \sum_{n\in\mathbb{Z}} (-1)^n \dim H_n(C_*).$$

(Hint: Find a relationship between $\dim C_n$, $\dim H_n$, and the ranks of the boundary operators.)

3. Prove that if $X = U \cup V$ where U and V are open and disjoint, then $H_n(X) = H_n(U) \oplus H_n(V)$.

4. If (C_*, ∂) and (D_*, δ) are two chain complexes, a *chain map* $f\colon C_* \to D_*$ is given by, for each $n \in \mathbb{Z}$, a homomorphism $f_n\colon C_n \to D_n$ such that

$$\delta^{(n)} \circ f_n = f_{n-1} \circ \partial^{(n)}$$

for all n. Prove that if $f\colon C_* \to D_*$ is a chain map then for all n we obtain a well-defined homomorphism $f_*\colon H_n(C_*) \to H_n(D_*)$ by setting $f_*[c] = [f(c)]$ where for a cycle x (in either C_* or D_*) we let $[x]$ denote the homology class of x.

5. If $f, g\colon C_* \to D_*$ are two chain maps, a *chain homotopy* from f to g is a collection of homomorphisms $K^{(n)}\colon C_n \to D_{n+1}$ such that, for all n,

$$g_n - f_n = \delta^{n+1} \circ K^{(n)} + K^{(n-1)} \circ \partial^{(n)}.$$

(a) Prove that if there exists a chain homotopy from f to g then the induced maps $f_*\colon H_n(C_*) \to H_n(D_*)$ and $g_n\colon H_n(C_*) \to H_n(D_*)$ are equal.

(b) Give an example of two chain maps f and g between finite-dimensional chain complexes such that there exists a chain homotopy from f to g but f and g have different ranks.