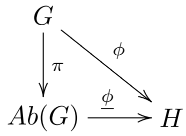
Math 8200, Spring 2011: Problem Set 5. Due Tuesday, March 8

I. Recall that if G is a group a *commutator* in G is an element of the form $xyx^{-1}y^{-1}$ where $x, y \in G$. The *commutator subgroup* $[G, G]$ of G consists of all elements which can be written as products of commutators (this is easily seen to be a normal subgroup of G; you need not prove this). The *abelianization* of G is the group

$$Ab(G) = \frac{G}{[G, G]}.$$

(a) Prove that $Ab(G)$ is abelian, and satisfies the following universal property: if $\phi\colon G \to H$ is a group homomorphism where H is any *abelian* group, then there exists a unique homomorphism $\underline{\phi}\colon Ab(G) \to H$ such that the diagram



is commutative, where $\pi\colon G \to Ab(G)$ is the quotient projection.

(b) Prove that if $G_1, \ldots, G_n$ is any finite collection of groups then there is an isomorphism

$$Ab(G_1 * \cdots * G_n) \cong Ab(G_1) \times \cdots \times Ab(G_n).$$

(Suggestion: find a universal property that both sides satisfy, and argue that any two groups satisfying this universal property are isomorphic.)

(c) Prove that if $n \neq m$ then the free group on n generators is not isomorphic to the free group on m generators. (Hint: Show that their abelianizations are not isomorphic by appealing to the fundamental theorem of finitely generated abelian groups.)

(d) Prove that $\mathbb{C} \setminus \{0, 1, 2\}$ is not homotopy equivalent to $\mathbb{C} \setminus \{0, 1\}$.

II. Hatcher Section 1.2: 1, 2, 3, 11