Math 8200, Spring 2011: Problem Set 2 (revised). Due Tuesday, February 1

1. Let $f(z) = \sum_{i=0}^{n} a_i z^i$ be any polynomial with coefficients $a_i \in \mathbb{C}$ such that $a_n \neq 0$. Therefore we obtain a loop $\gamma_f\colon S^1 \to \mathbb{C}$ defined by $\gamma_f(t) = f(e^{2\pi i t})$.

(a) Suppose that f has no zeros in the region $D = \{z \in \mathbb{C} | |z| \leq 1\}$. In particular γ_f is then a loop in $\mathbb{C} \setminus \{0\}$, not just in $\mathbb{C}$. By considering the family of polynomials $f_s(z) = f(sz)$ for $s \in [0,1]$ and using that the degree of a loop depends only on its homotopy class, prove that

$$deg(\gamma_f) = 0.$$

(b) Suppose that f has no zeros in the region $R = \{z \in \mathbb{C} | |z| \geq 1\}$. Prove that $deg(\gamma_f) = n$. (Hint: Show that the function $F(s,z) = s^n f(z/s)$ extends to a continuous function from $[0,1] \times R$ to $\mathbb{C} \setminus \{0\}$).

(c) Deduce the Fundamental Theorem of Algebra: Any nonconstant polynomial with complex coefficients must have a zero somewhere in $\mathbb{C}$.

2. Let U and V be spaces, $A \subset U$ a subspace, and $g\colon A \to V$ a continuous map, so we can form the adjunction space

$$U \cup_g V = \frac{U \coprod V}{a \sim g(a) \text{ if } a \in A}.$$

Let $j_U\colon U \to U \cup_g V$ and $j_V\colon V \to U \cup_g V$ be the maps obtained by composing the inclusion of U or V into $U \coprod V$ with the quotient projection $\pi\colon U \coprod V \to U \cup_f V$. Thus where $i\colon A \to U$ is the inclusion we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & V \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j_V} \\ U & \xrightarrow{\;j_U\;} & U \cup_g V \end{array}$$

(a) Prove that j_V is a homeomorphism onto its image $j_V(V) \subset U \cup_g V$. (Said differently, V appears in $U \cup_g V$ as a subset, and you need to show that the resulting subspace topology on V is the same as the original topology on V).

(b) Let Z be another space, and let $h_U\colon U \to Z$ and $h_V\colon V \to Z$ be continuous maps with the property that if $a \in A$ then $h_U(a) = h_V(g(a))$. There is then a unique function $h\colon U \cup_g V \to Z$ characterized by the property that $h_U = h \circ j_U$ and $h_V = h \circ j_V$. Prove that this function h is continuous.

3. If $f\colon X \to Y$ is a homotopy equivalence and V is any space prove that the induced map $f_*\colon [V,X] \to [V,Y]$ of Exercise 1.11 from the last homework is a bijection.

4. Suppose that $f\colon X \to Y$ is a continuous map such that there are functions $g_L\colon Y \to X$ and $g_R\colon Y \to X$ such that $g_L \circ f$ is homotopic to 1_X and $f \circ g_R$ is homotopic to 1_Y. Prove that f is a homotopy equivalence. (In other words, you need to show that g_L and g_R can both be replaced by a single function g. Hint: Consider $g_L \circ f \circ g_R$.)

5. Let $p, q \in S^1$ be two distinct points. Construct a (weak) deformation retraction from $X = (S^1 \times S^1) \setminus \{(q,q)\}$ onto $A = (S^1 \times \{p\}) \cup (\{p\} \times S^1)$.