Math 8200, Spring 2011: Problem Set 10. Due Thursday, April 28

1. Prove that the connecting homomorphism $\delta\colon H_{n+1}(E_*) \to H_n(C_*)$ in the long exact sequence associated to a short exact sequence $0 \to C_* \to D_* \to E_* \to 0$ of chain complexes is "natural" in the following sense: Suppose we have two short exact sequences of chain complexes, and chain maps between them, which fit into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_* & \longrightarrow & D_* & \longrightarrow & E_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C'_* & \longrightarrow & D'_* & \longrightarrow & E'_* & \longrightarrow & 0. \end{array}$$

Then the diagram

$$\begin{array}{ccc} H_{n+1}(E_*) & \xrightarrow{\delta} & H_n(C_*) \\ \downarrow & & \downarrow \\ H_{n+1}(E'_*) & \xrightarrow{\delta'} & H_n(C'_*) \end{array}$$

is also commutative.

2. Prove (without looking at your book) the five lemma: Suppose we have a commutative diagram of abelian groups

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ V & \longrightarrow & W & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

where the rows are exact and the vertical arrows are homomorphisms. Then:

(i) If f_2 and f_4 are surjective while f_5 is injective then f_3 is surjective.
(ii) If f_2 and f_4 are injective while f_1 is surjective then f_3 is injective.

In particular if the four outer arrows are isomorphisms then the inner arrow is too.

3.

(a) Prove that if $f\colon (X, A) \to (Y, B)$ has the property that both $f\colon X \to Y$ and $f|_A\colon A \to B$ are homotopy equivalences then f induces an isomorphism $H_n(X, A) \to H_n(Y, B)$.
(b) Give an example of spaces X, A, Y, B obeying the conditions of (a) but such that there is no $g\colon Y \to X$ such that $g|_B\colon B \to A$ is a homotopy equivalence. (Suggestion: Have B be dense in Y.)

4. Where D^n is the closed unit n-disc, suppose that we have a surjective continuous map $f\colon D^n \to X$ to a Hausdorff space such that $f|_{D^n \setminus \partial D^n}$ is a homeomorphism onto an open subset of X which is disjoint from $f(\partial D^n)$. Where $Y = f(\partial D^n)$ with the subspace topology induced from X, prove that X is homeomorphic to the adjunction space $Y \cup_f D^n$ formed by attaching D^n to Y along f.

5. Compute (justifying all of your steps, including any appeals to problem 4 above) the homology of the quaternionic projective space

$$\mathbb{H}P^n = \frac{\mathbb{H}^{n+1} \setminus \{\vec{0}\}}{\vec{v} \sim \lambda\vec{v} \text{ if } \lambda \in \mathbb{H} \setminus \{\vec{0}\}}.$$