Teaching Phases and Standard Algorithms in the Common Core

Karen C. Fuson, Professor Emerita, Northwestern University
Sybilla Beckmann, Professor, University of Georgia

This paper is based on the CCSS-M standards, *The NBT Progression for the Common Core State Standards* by The Common Core Writing Team (7 April, 2011), commoncoretools.wordpress.com, and on Fuson, K. C. & Beckmann, S. (Fall, 2012). Standard algorithms in the Common Core State Standards. *National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership, 14 (2)*, 14-30 (which is posted at http://www.math.uga.edu/~sybilla/ as is this talk file). For this handout or if you have difficulties finding either of the above, send me an email karenfuson@mac.com with NCSM in the subject line.

Overview Taken together, the NBT Progression document summarizes that *the standard algorithm* for an operation implements the following mathematical approach with minor variations in how the algorithm is written:

a. decomposing numbers into base-ten units and then carrying out single-digit computations with those units using the place values to direct the place value of the resulting number;
b. using the one-to-ten uniformity of the base ten structure of the number system to generalize to large whole numbers and to decimals.

General methods that will generalize to and become standard algorithms can and should be developed, discussed, and explained initially using a visual model. Helping step methods that clarify the meaning or use of place value, relate easily to parts of visual models, or prevent common errors can be used initially to support student understanding and accuracy. These lead to variations of standard algorithms by dropping steps or writing them in a more efficient way.

Criteria for better variations of how the algorithm is written are:

Given the CCSS emphasis on meaning-making, variations in ways to record the standard algorithm that support and use place value correctly should be emphasized.

Given the centrality of single-digit computations in algorithms, variations that make such single-digit computations easier should be emphasized.

Written methods may involve different kinds of steps, e.g., ungrouping (borrowing) to be able to subtract and the actual subtracting. These kinds of steps can alternate or can be completed all at once. Variations in which the kinds of steps alternate can introduce errors and be more difficult, so methods without such alternations should be emphasized.

Written variations can keep the initial multidigit numbers unchanged, or single-digit numbers can be written so as to change (or seem to change) the original numbers. The former variations are conceptually clearer and so should be emphasized.

Many students prefer to calculate from left to right, consistent with how they read numbers and words. Variations that can be undertaken left to right are helpful to many students, especially initially, so they should be emphasized.

Variations meeting some of these criteria are on the next page.

The learning path

Any method that is taught or used must have a learning path resting on visual models and on explaining the reasoning used. It is not acceptable to teach methods by rote without understanding how place values are used in the methods.

Methods are elicited from students and discussed, but good variations of writing the standard algorithm are introduced early on so that all students can experience them.

Steps in written methods are initially related to steps in visual models.

Experiencing and discussing variations in writing a method is important mathematically.

Students stop making drawings when they are not needed. Fluency is solving without a drawing.

Students drop steps of Helping Step methods when they can move to a short written variation of the standard algorithm for fluency.

Learning Path Teaching-Learning:
Differentiating within Whole-Class Instruction by Using the Math Talk Community

Bridging for teachers and students
by coherent learning supports

Learning
Path

Phase 3: Compact methods for **fluency**

Math Sense-Making
Math Structure

Math Drawings
Math Explaining

Phase 2: Research-based mathematically-desirable and accessible methods in the middle for **understanding and growing fluency**

Math Sense-Making
Math Structure

Math Drawings
Math Explaining

Phase 1: Students' methods elicited for **understanding** but move rapidly to Phase 2

Common Core Mathematical Practices

Math Sense-Making about Math Structure using Math Drawings to support Math Explaining

Math Sense-Making: Making sense and using appropriate precision

1 Make sense of problems and persevere in solving them.
6 Attend to precision.

Math Structure: Seeing structure and generalizing

7 Look for and make use of structure.
8 Look for and express regularity in repeated reasoning.

Math Drawings: Modeling and using tools

4 Model with mathematics.
5 Use appropriate tools strategically.

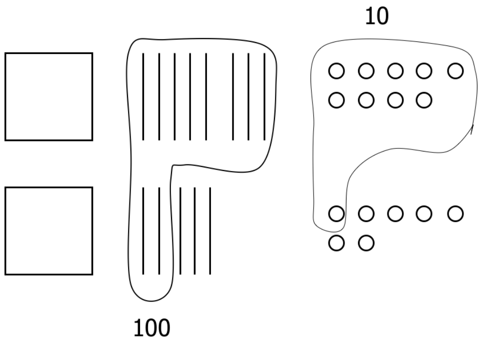
Math Explaining: Reasoning and explaining

2 Reason abstractly and quantitatively.
3 Construct viable arguments and critique the reasoning of others.

The top is an extension of Fuson, K. C. & Murata, A. (2007). Integrating NRC principles and the NCTM Process Standards to form a Class Learning Path Model that individualizes within whole-class activities. National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership, 10 (1), 72-91. It is a summary of several National Research Council Reports.

Written Variations of Standard Algorithms

Quantity Model ⟷ Good Variations | Current Common



New Groups Below

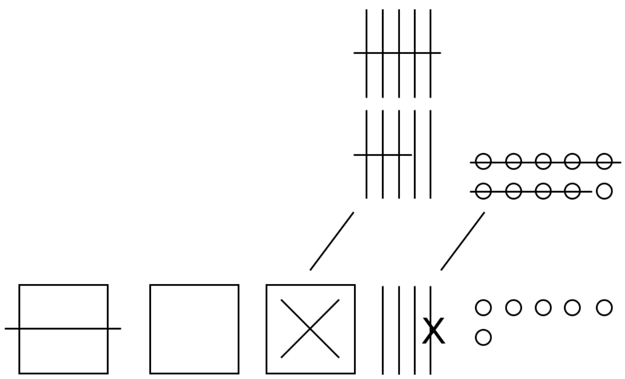
$$\begin{array}{rrrr} & 1 & 8 & 9 \\ + & 1 & 5 & 7 \\ \hline & {\scriptstyle 1} & {\scriptstyle 1} & \\ & 3 & 4 & 6 \end{array}$$

Show All Totals

$$\begin{array}{rrrr} & 1 & 8 & 9 \\ + & 1 & 5 & 7 \\ \hline & 2 & 0 & 0 \\ & 1 & 3 & 0 \\ & & 1 & 6 \\ \hline & 3 & 4 & 6 \end{array}$$

New Groups Above

$$\begin{array}{rrrr} & {\scriptstyle 1} & {\scriptstyle 1} & \\ & 1 & 8 & 9 \\ + & 1 & 5 & 7 \\ \hline & 3 & 4 & 6 \end{array}$$



Ungroup Everywhere First, Then Subtract Everywhere

Left → Right

$$\begin{array}{rrrr} & & 13 & \\ & 2 & \cancel{14} & 16 \\ & \cancel{3} & \cancel{4} & \cancel{6} \\ - & 1 & 8 & 9 \\ \hline & 1 & 5 & 7 \end{array}$$

Right → Left

$$\begin{array}{rrrr} & & 13 & \\ & 2 & \cancel{3} & 16 \\ & \cancel{3} & \cancel{4} & \cancel{6} \\ - & 1 & 8 & 9 \\ \hline & 1 & 5 & 7 \end{array}$$

R → L Ungroup, Then Subtract, Ungroup, Then Subtract

$$\begin{array}{rrrr} & & 13 & \\ & 2 & \cancel{3} & 16 \\ & \cancel{3} & \cancel{4} & \cancel{6} \\ - & 1 & 8 & 9 \\ \hline & 1 & 5 & 7 \end{array}$$

Area Model

	40	+ 3
60	2400	180
+ 7	280	21

Place Value Sections

$$\begin{array}{rr} & 2400 \\ & 180 \\ & 280 \\ + & 21 \\ \hline & {\scriptstyle 1} \\ & 2881 \end{array}$$

Expanded Notation

$$\begin{array}{rcl} 43 & = & 40 + 3 \\ \times\ 67 & = & 60 + 7 \\ \hline 60 \times 40 & = & 2400 \\ 60 \times 3 & = & 180 \\ 7 \times 40 & = & 280 \\ 7 \times 3 & = & 21 \\ \hline & & {\scriptstyle 1} \\ & & 2881 \end{array}$$

1-Row

$$\begin{array}{rr} & {\scriptstyle 1} \\ & {\scriptstyle 2} \\ & 43 \\ \times & 67 \\ \hline & 301 \\ & 258 \\ \hline & 2881 \end{array}$$

Rectangle Sections

	40	+ 3	= 43
67	2881 - 2680	201 201	
	201	0	

Expanded Notation

$$\begin{array}{r} 3 \\ 40 \end{array} \Big] \ 43$$

$$\begin{array}{r} 67\,\overline{)\,2881} \\ -\ 2680 \\ \hline 201 \\ -\ 201 \\ \hline \end{array}$$

Digit by Digit

$$\begin{array}{r} 43 \\ 67\,\overline{)\,2881} \\ -\ 268 \\ \hline 201 \\ -\ 201 \\ \hline \end{array}$$

Learning Path for Multidigit Computation in CCSS

Bridging for teachers and students
by coherent learning supports

Learning
Path

Phase 3: Compact methods for fluency

Students use any good variation of writing the standard algorithm with no drawing to build fluency. They explain occasionally to retain meanings.

Math Sense-Making ↕ Math Drawings
Math Structure ↕ Math Explaining

Phase 2: Research-based mathematically-desirable and accessible methods in the middle for understanding and growing fluency

Students focus on and compare efficient, accurate, and generalizable methods, relating these to visual models and explaining the methods.
They write methods in various ways and discuss the variations.
They may use a helping step method for understanding and/or accuracy.
They choose a method for fluency and begin solving with no drawing.

Math Sense-Making ↕ Math Drawings
Math Structure ↕ Math Explaining

Phase 1: Students' methods elicited for understanding but move rapidly to Phase 2

Students develop, discuss, and use efficient, accurate, and generalizable methods and other methods. They use concrete models or drawings that they relate to their written method and explain the reasoning used.

Note. Students may consider problems with special structure (e.g., 98 + 76) and devise quick methods for solving such problems. But the major focus must be on general problems and on generalizable methods that focus on single-digit computations (i.e., that are or will generalize to become a variation of writing the standard algorithm).