ELEMENTARY MECHANICS FROM A MATHEMATICIAN'S VIEWPOINT

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PREFACE

In March of 2004, I had the pleasure of giving eight lectures as part of the

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The Twenty-first Century COE Program at Keio
Integrative Mathematical Sciences:
Progess in Mathematics Motivated by
Natural and Social Phenomena

As explained in Lecture 1, these lectures cover material that I had just finished writing, and which I hope will constitute the first part of a book on Mechanics for Mathematicians. But I tried to present them as course lectures in development, rather than as a finished product, and I benefited immensely from this, since such a presentation, and the reactions of the listeners, raised all sorts of questions that I hadn't thought of, and which will eventually result in a rather thorough rewriting of the original material.

In these notes, I have tried to preserve the informal nature of these lectures, aiming for spontaneity, rather than the more carefully arrangement of material one expects to find in a book, so that they will complement the book (if it ever appears) rather than merely serving as a repetition; however, some parts, especially the notes for the later lectures, are taken rather directly from parts of the book already written. Though the notes do not reproduce the material of the lectures in exactly the same order as they were given, they do cover basically the same material.

I am grateful to Prof. Yoshiaki Maeda for inviting me to give these lectures, and to Prof. Martin Guest of Tokyo Metropolitan University, where two of these lectures were also given. I greatly enjoyed their hospitality and the experience of living in Tokyo.

LECTURE 1

THE HARDEST PART OF MECHANICS (THE FUNDAMENTALS)

These lectures are based on a book that I am writing, or at least trying to write. For many years I have been saying that I would like to write a book (or series of books) called Physics for Mathematicians. Whenever I would tell people that, they would say, Oh good, you're going to explain quantum mechanics, or string theory, or something like that. And I would say, Well that would be nice, but I can't begin to do that now; first I have to learn elementary physics, so the first thing I will be writing will be Mechanics for Mathematicians.

So then people would say, Ah, so you're going to be writing about symplectic structures, or something of that sort. And I would have to say, No, I'm not trying to write a book about *mathematics* for mathematicians, I'm trying to write a book about *physics* for mathematicians; of course, symplectic structures will eventually make an appearance, but the problem is that I could easily understand symplectic structures, it's elementary mechanics that I don't understand.

Then people would look at me a little strangely, so I'd better explain what I mean. When I say that I don't understand elementary mechanics, I mean, for example, that I don't understand this:

Of course, everyone knows about levers. They are so familiar that most of us have forgotten how wonderful a lever is, how great a surprise it was when we first saw a small body balancing a much bigger one. Most of us also know the law of the lever, but this law is simply a quantitative statement of exactly how amazing the lever is, and doesn't give us a clue as to why it is true, how such a small force at one end can exert such a great force at the other.

Now physicists all agree that Newton's Three Laws are the basis from which all of mechanics follows, but it you ask for an explanation of the lever in terms of these three laws, you will almost certainly not get a satisfactory answer. You might be told something about conservation of angular momentum, or perhaps even the principle of virtual work, but I can almost guarantee that as soon as some one starts to give an answer you can almost certainly interrupt them and explain why their answer can't be correct. For, the correct answer **must** begin by saying "First we have to understand rigid bodies", since, after all, the rigidity of the lever is an absolute necessity for it to work, and if one hasn't already analyzed rigid bodies, then one simply isn't in a position to give an explanation of the lever.

Well, we won't get to rigid bodies for a few lectures yet, so it'll be a while before we can give a good answer to this question.

In these lectures I am basically going to cover (portions of) the part of the book that has been written so far. The final book will contain 4 parts:

I. Fundamentals

II. Applications

III. Lagrangian Mechanics

IV. Hamiltonian Mechanics,

with perhaps a fifth part,

V. Abstractions to Lie groups.

I hope to finish this book in about a year. So far, I have written just Part I, and it has taken me nearly a year and a half,[1] but that doesn't mean that my hope is necessarily unrealistic. After all, Part I is the hard part, all about the basic physical ideas, while the remaining parts are basically mathematics.

We are going to be considering the foundations of mechanics by starting right at the source, Newton's *Philosophiæ Naturalis Principia Mathematica* or Mathematical Principles of Natural Philosophy, in English, or simply The Principia.

Now this book is one of the great classics, probably the greatest book in all of physics, but that doesn't mean that some one should try learning physics from it! Like many a classic, it is basically unreadable. To begin with, it's in Latin, and I don't even think there is a Japanese translation. Fortunately, there is a very good recent English translation

Newton, *The Principia*. Translated by Bernard Cohen and Anne Whitman.

and nearly half of this hefty tome is a guide to reading the actual translation; see [N-C-W] in the References at the end of these notes for further details. Any quotations from the Principia are taken from this book.

Newton obviously wrote the Principia with Euclid's *Elements* in mind. In fact, after the usual beginning stuff, the title page, various prefaces for various editions, both by Newton and his editor, and a long ode to Newton written by Halley (which I haven't read, but have grave doubts of being very good), the very first words of the Principia proper are

[1] Actually about 2 years, when you count the rewriting that I felt was necessary after giving these lectures.

DEFINITIONS

Definition 1 *Quantity of matter is a measure of matter that arises from its density and volume jointly.*

Now of course anyone can see that this definition isn't very useful, in fact obviously circular, since density is usually defined as mass per unit volume. In [N-C-W], and elsewhere, you can find long discussions about this, but we don't really need to be concerned. Newton is basically just trying to get started, and he will tie things together better as he goes along. In fact, this definition is immediately followed by further discussion:

Definition 1 *Quantity of matter is a measure of matter that arises from its density and volume jointly.*

If the density of air is doubled in a space that is also doubled, there is four times as much air, and there is six times as much if the space is tripled. The case is the same for snow and powders condensed by compression or liquefaction, ... Furthermore, I mean this quantity whenever I use the term "body" or "mass" in the following pages.

The three dots here indicate omissions that aren't really very important (you'll have to take my word for this), but there's also a portion that has been temporarily shaded out. This part, by contrast, is very important, but it's very confusing that Newton put it right here, and we'll come back to it later!

Naturally, these additional remarks don't clarify the concept very much, but at least they help a little, and in particular Newton has now introduced the important term "mass" instead of the awkward phrase "quantity of matter". It's something of a distraction that he also uses the word "body", because nowadays we think of a body as some object, that has a particular mass, so we won't follow Newton's usage at all, but will always use the word "mass", whatever in the world that is eventually going to mean.

The next definition is a lot simpler,

Definition 2 *Quantity of motion is a measure of motion that arises from the velocity and the quantity of matter jointly.*

and we merely want to point out that nowadays this is what we call "momentum".

The third definition, though just as vague as the others, finally gives us some idea of what "mass" is really supposed to mean.

Definition 3 *Inherent force of matter is the power of resisting by which every body, so far as it is able, perseveres in its state either of resting or of moving uniformly straight forward.*

This force is always proportional to the body [i.e., mass] and does not differ in any way from the inertia of the mass except in the manner it which it is conceived. . . .

This is certainly a strange way of speaking—nowadays, we don't speak of the "force of matter"—but, in short, what Newton is saying is that "mass" is basically what we call "inertia", a measure of how hard it is to get something moving. If I were teaching an introductory mechanics course for physics students I would provide two balls like these [two balls made from styrofoam semi-spheres, identical in appearance], give the first a slight push, so that it would start rolling rapidly, and then ask a brawny-looking student to repeat this with the second. [The second ball was filled with heavy chain, and barely moved when pushed a great deal harder.] This little experiment shows that the second ball has a much greater mass than the first (although the student, if asked, would probably incorrectly conclude that the second ball weighed a lot more than the first, which is something quite different, that we'll get to in a bit).

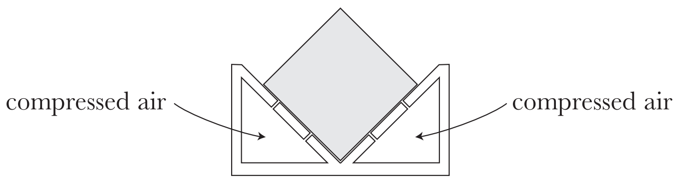
Of course, all of this is merely vaguely descriptive, and the remainder of this first section of the Principia, in which Newton adds a few other definitions doesn't help much.

So it's time to turn to the second section, the axioms:

AXIOMS, OR THE LAWS OF MOTION

Law 1 *Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by forces impressed.*

Of course, Newton didn't call this Newton's First Law, and, as we will see later on, he explicitly credits the law to Galileo. Galileo didn't exactly demonstrate this law, but instead explained why everyday experiences might seem to contradict it. Not too long ago the law might be illustrated dramatically by sliding an object along a glass table with dry ice evaporating from it, forming a cushion of gas that practically eliminates friction. Recently, the invention of the "air-trough" (see reference [N-L]) has provided a nice way to illustrate Newton's first law in the classroom:



side view of a block sliding on an air trough

One gives the block just the tiniest nudge, and watches it glide at constant velocity to the end of the track.

In an elementary physics course, considerable discussion about coordinate systems might be required here, but we'll simply point out that we are basically assuming that there are certain coordinate systems ("inertial systems") in which this law holds, and that any system moving at uniform velocity with respect to an inertial system obviously has the same property. So we pass on immediately to

Law 2 *A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.*

If some force generates any motion, twice the force will generate twice the motion, ...

Once again, we have that obnoxious shaded region, which we'll unveil at the appropriate time.

Of course, nowadays we simply state the second law as

$$\mathbf{F} = m\mathbf{a},$$

where **a** is the acceleration of our body. Newton speaks in terms of (discrete) "changes in motion" because he often thinks in terms of what we would call "instantaneous" forces, exerted only for a very short time.

This law might seem to be virtually meaningless, since we haven't said how to measure mass m, nor how to measure force **F**. But there's really more to it than appears, as we will see by skipping to the discussion ("Scholium") that Newton gives at the end of this section of the Principia, where he begins by acknowledging Galileo (again, some material has been shaded out, to be unveiled at a later date):

> **Scholium** The principles I have set forth are accepted by mathematicians and confirmed by experiments of many kinds. By means of the first two laws [redacted] Galileo found that the descent of heavy bodies is in the squared ratio of the time [redacted] as experiment confirms, except insofar as these motions are somewhat retarded by the resistance of the air.

In the first and second editions of the Principia, this is all that Newton says, and it might seem rather mysterious—what does squared ratio of the time have to do with the second law? As we will see once again in a future lecture, Newton's exposition often suffered from a defect common to many very bright people—he often didn't realize that what was obvious to him might not be obvious to others—and it was only in the third edition that he added a more detailed explanation:

> **Scholium** The principles I have set forth are accepted by mathematicians and confirmed by experiments of many kinds. By means of the first two laws [redacted] Galileo found that the descent of heavy bodies is in the squared ratio of the time [redacted] as experiment confirms, except insofar as these motions are somewhat retarded by the resistance of the air. When a body falls, uniform gravity, by acting equally in individual equal particles of time, impresses equal forces upon that body and generates equal velocities; and in the total time it . . . generates a total velocity proportional to the time. And the spaces described in proportional times are as the velocities and the times jointly, that is, in the squared ratio of the times.

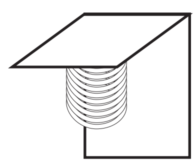
Well, this is the sort of explanation that makes you glad that you aren't trying to learn mathematics in the 17th century! What Newton is saying is the following: Suppose our body starts at rest. The force of gravity, which we think of as an instantaneous force acting at time 0, changes the downward velocity by a small increment v. So after a small increment of time t the body has fallen by an amount vt. Then its velocity receives an additional increment v, giving it velocity $2v$ so after the next increment t of time it falls by the amount $2vt$. In the next increment of time it falls by the amount $3vt$, then by the amount $4vt$, etc. So at the end of a large number N of such increments it has fallen $(1 + 2 + 3 + \cdots + N)$ times vt, which is close to $N^2vt/2$, or to $vT^2/2$, where T is the total time it has fallen. Nowadays, of course, we just say, If $s'' = a$ for a constant a, then $s' = at$, and thus $s = \frac{1}{2}at^2$.

Even after all this explanation, you might wonder what this has to do with the second law, since it seems to be a purely mathematical result about second derivatives. So it's important to go back and look at one particular phrase in Newton's argument:

When a body falls, *uniform gravity*, by acting ...

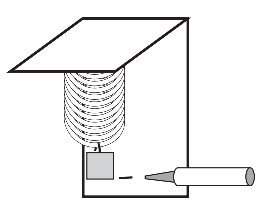
What does Newton mean here by "uniform gravity"? You might think he means that gravity produces the same acceleration on the body no matter what velocity it already has (and this is indeed one of the things he is assuming), but Newton is also appealing (implicitly, it is true) to an important fact that we can test experimentally.

To perform this experiment, we are going to use the following apparatus.



Now as soon as you see this large spring, you might be thinking: oh, we're going to invoke Hooke's law. But we aren't going to be using anything of the sort; notice that's no scale here behind the spring, only a blank piece of wood. We only care about one thing: experience shows us that it takes a greater force to stretch a string by a large amount than is required to stretch it by a small amount—we are completely uninterested in the particular law involved (which we could investigate later, if we ever figured out how to measure force).

For the first part of the experiment, we attach a block to our spring, and use this nice felt pen to mark on the wood how far down the spring has been pulled.



Of course, this isn't a high-tech experiment, it's a Tokyu Hands experiment.[1] But we could easily imagine a much more refined experiment, with a very strong spring, very accurate ways of measuring its displacement, etc.

For the sake of time, I'll simply describe the second part of the experiment, instead of having us actually perform it. For this part of the experiment, we would all go upstairs to the room above this one. Perhaps there's some one there now, well, we'll just ask them if we can take a moment's time to perform a simple experiment. And this simple experiment is a repeat of the experiment that we have just done here: we attach the block to our spring, and see far down the spring has been pulled. It appears that the spring has been pulled down exactly the same amount. (At least within the accuracy of our Tokyu Hands experiment.)

The next part of the experiment is one that probably no one except a tourist like myself might be willing to perform, taking the same measurement at the top of Tokyo Tower. Once again, it seems that the spring is stretched by exactly the same amount.

Now *this* is what Newton means when he speaks of "uniform gravity": a force that is the same no matter how high up we go (of course, that's not really true for the force of gravity, but it's true to a very good approximation for the sort of distances above the earth's surface that we are concerned with). The point is that we can at least say when two forces are *equal* without having to specify a way of measuring them.

Thus, the experimental evidence that the force of gravity is equal at all reasonable distances from the earth's surface is consistent with the equality of *acceleration* along the path of a falling object, which is essentially equivalent to Galileo's observation that "the descent of heavy bodies is in the squared ratio of the time".

It may seem a long way from "equal forces produce equal accelerations" to "force equal mass times acceleration", but that's mainly because we still haven't given an "operational definition" of mass, describing how it is to measured. The important point is that the results of our little experiment actually suggest a way of producing such an operational definition.

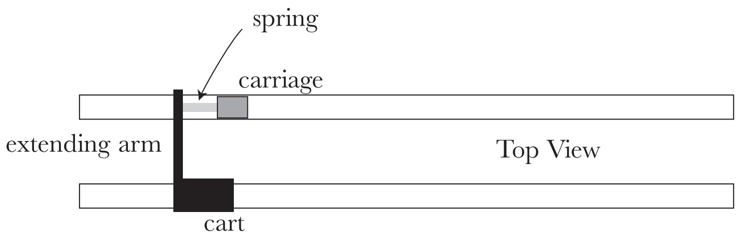
We'll begin with a definition that is conceptually very straightforward, although it would be rather awkward to use in practice.

First we want to have a very long air-trough, with a carriage, of negligible mass, in which we can place a body whose mass we want to measure. Parallel to this air-trough we have a track with a little cart that can be pulled along the track with any desired acceleration a; for simplicity, let's imagine that we merely have to turn a knob on an instrument panel to vary a, without worrying about the clever mechanism that would be required to produce this effect. Of course,

[1] Tokyu Hands is a large do-it-yourself and hobby store where all the paraphernalia for this lecture was purchased.

our track, and the air-trough, will also have to be very long if we expect to pull the cart with constant acceleration a for any reasonable amount of time.

The cart is provided with an extending arm that can be placed behind the carriage on the air-trough so that the carriage is moved in tandem with the cart. But instead of placing this arm directly behind the carriage, we will put a nice strong spring between them.



Now let's choose a particular body B_0 that we want to be our "unit mass", so that we will assign it mass $m = 1$. We place this body in the carriage and pull our cart with some convenient constant acceleration a_0. Initially, of course, the carriage will not move with the same acceleration, because the spring will compress somewhat, so that the carriage won't move exactly in tandem with the cart. But we very quickly reach the point where the spring is no longer compressing, or at any rate the length of the spring is constant within the limits of accuracy of our measurements. We carefully measure this final length, and call it L_0.

Note, by the way, that this whole set-up is *dependent on our original experimental observation* that equal forces produce equal acceleration: the compression of the spring measures the force that is being applied to the carriage, and once the carriage is moving with a constant acceleration, the force applied to it must be constant.

Now let's take some other body B whose mass m we want to determine. We place B in the carriage instead of B_0 and once again pull our cart with constant acceleration a_0, and observe the (final) length of the compressed spring. It probably isn't L_0 any more, so we try adjusting the acceleration a in order to make it become L_0: if the spring was compressed more, to a length $< L_0$, we try an acceleration $< a_0$, if it was compressed to a length $> L_0$ we try an acceleration $> a_0$. After lots of trial and error, we finally find an acceleration a_1 which compresses the spring to exactly the length L_0. We now define

$$\text{mass } m \text{ of } B = \frac{a_0}{a_1}.$$

This definition makes the law $\mathbf{F} = m\mathbf{a}$ work for any particular fixed $\mathbf{F}$, and much experimentation would show that it works just as well for any other $\mathbf{F}$; in other words, if we repeated this whole process using a different spring, and thus a different L_0, we would still end up assigning the same masses to all bodies.

Then, of course, we can use the equation in reverse, as a way of measuring force, by seeing what acceleration is produced on a body of some known mass m.

Before proceeding further, two points should be made. First, our original little experiment is certainly consistent with $\mathbf{F} = m\mathbf{a}$, but it would hardly seem to be very conclusive. After all, how do we know that the correct law isn't something like $\mathbf{F} = m\mathbf{a} + k\mathbf{a}'$ for some constant k, so that third derivatives, or even higher derivatives, are involved? I don't know of any experiments to directly test this, but of course there is an enormous body of experience that attests to it: the force of gravity isn't constant over large distances, so all the calculations that keep satellites in motion, guide space ships to the moon and land them, etc., present a great deal of evidence.

A second point to ponder is that we always take for granted that mass is "additive": if we take bodies of mass m_1 and m_2 and join them together (for example, by placing them together in the carriage on our air-trough), then the new object should have mass $m_1 + m_2$. In terms of our operational definition, this hardly seems clear: it says that if a_1 is the acceleration that the first body must be subjected to in order to compress the spring to length L_0, while a_2 is the acceleration that the second body must be subjected to in order to compress the string the same amount, then to obtain the same compression for the two objects together, they must be subject to an acceleration a satisfying

$$\frac{1}{a} = \frac{1}{a_1} + \frac{1}{a_2}$$

to obtain the same compression. At first glance, this might seem to be a strange fact that could only be verified by experiments with large numbers of varying masses, so it is interesting to try to figure out what reasonable body of basic experimental facts would lead one to the additivity of mass in a more reasonable manner, an exercise you might like to think about before the next lecture.

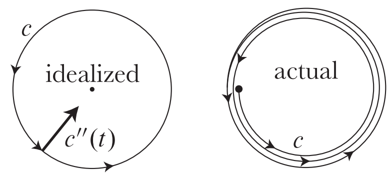
Finally, here is a less direct, but more convenient operational definition of mass, based on the mathematician's and physicist's common view that a straight line is just a circle of infinite radius.

Instead of using an air-trough, we simply attach our body B to the end of a very stiff spring that is being rotated horizontally with some large constant "angular frequency" α, so that B moves along the circle

$$c(t) = R(\cos\alpha t, \sin\alpha t)$$

for some radius R. This radius R will be somewhat larger than the unstretched length of the spring, because B actually begins moving along a spiral, pulling the spring out, though its path soon becomes indistinguishable from a circle. For the acceleration we simply have

(A) $\quad c''(t) = -R\alpha^2(\cos\alpha t, \sin\alpha t),$



so that the acceleration always points directly inward, and has magnitude $R\alpha^2$. This means that the force $\mathbf{F}$ that the spring exerts on B also always points directly inward and has constant magnitude.

Declaring B to be our unit of mass amounts to saying that

$$\text{(a)} \qquad |\mathbf{F}| = 1 \cdot R\alpha^2.$$

To determine the mass m of any other body, we attach it to the end of our spring and vary the angular frequency with which we rotate it until we arrive at an angular frequency β for which our body is moving along a circle

$$c(t) = R(\cos\beta t, \sin\beta t)$$

of the *same* radius R. Now we should have

$$\text{(b)} \qquad |\mathbf{F}| = m \cdot R\beta^2,$$

with the $|\mathbf{F}|$ in equations (a) and (b) having the *same* value, since in both cases the spring has been stretched by the same amount. In other words, we can determine m by

$$m = \alpha^2/\beta^2.$$

We've ignored the effect of gravity on these bodies, but that would become negligible in comparison to the force of our stiff spring when α is large, or we might imagine the measurements being made in outer space.

I'm sure that the basic mechanism for this definition could be greatly refined. Instead of a spring, one might whirl a tube filled with mercury, and measure the compression of this mercury column, etc. But I don't think any one has ever actually produced a mechanism of this sort. In fact, as far as I know, no one has ever measured the mass of *anything* accurately. This statement obviously requires a bit of explanation!

Let's consider these two balls, which we used previously in a little "experiment" to illustrate the concept of mass, and again enlist the aid of the student who performed the experiment, and quite possibly stated, incorrectly, that the experiment showed that the second ball had a greater weight than the first (rather than a greater mass). We could then continue the experiment by taking each of the balls off the table, in turn, and having the student hold it. Almost no effort would be required to hold up the first ball [as you can see when I hold it up], while considerably more effort would be required for the second ball [as you can now see me straining a bit to hold up the second ball]. This second experiment shows that the *weight* of the second ball—a measure of the gravitational force exerted on it by the earth—is much greater than the weight of the first.

In the first experiment, the effect of this gravitational force was irrelevant (except insofar as it affected friction), because the table was counteracting this force, so we were simply comparing the *mass* of the two bodies, while the second experiment directly compares the *weight* of the two bodies. So one of the first things that every beginning physics student needs to learn is the difference between mass and weight. For modern students, it's probably especially easy to illustrate this difference by considering what happens in a freely moving space craft where everything has no weight at all, so that these two balls would simply float where ever they were placed, but where *moving* one of the balls would be a lot more difficult than moving the other.

On the other hand, once we've clarified this distinction, we have to confuse the students once again, by pointing out that the *relative* masses of two bodies seem to be the same as their relative weights—everyday experience would certain have led us to conclude, without even thinking about it, that the second ball, so much harder to set in motion, would also be much harder to lift.

In terms of the second law, we can make a much more specific correlation: Since the weight of an object, of mass m, say, is the force $\mathbf{F}$ of gravity on it, the law $\mathbf{F} = m \cdot \mathbf{v}'$ means that the ratio $\mathbf{F}/m$ of weight to mass is simply the acceleration that an object undergoes under free fall. Thus, proportionality of weight to mass is equivalent to the assertion that all bodies fall with the same acceleration, the famous fact usually attributed to Galileo.

Aristotle, as we all know, had claimed that large bodies fall faster than smaller ones, and people apparently believed this for many centuries afterward, but it never made much sense. Even before Galileo's experiment, a man named J.-B. Benedetti (1530-1590) had pointed out how absurd this would be (see [M], Chapter II, section I, 2). After all, suppose we drop a brick like this one [luckily, it was a fake brick, so it didn't hurt the floor], and then take two of these bricks, tape them together, and drop them side by side: why in the world would the two bricks fall faster than each individual brick? But Benedetti still thought that denser bodies would fall more rapidly than less dense ones, and Galileo is usually credited with being the first to realize that even bodies with differing compositions, like wood and iron, fall at the same rate. Modern physicists would choose something like aluminum and gold, which have such different ratios of protons and neutrons, because, after all, we're really saying that *everything*, even each of the elementary particles, falls at the same rate!

Newton, of course, would never had made Aristotle's first mistake, and certainly realized that it was a question of whether bodies of differing compositions fall at the same rate. Although Galileo had performed experiments to determine this, Newton wanted much more accurate results, and he informs us of this right at the beginning of the Principia.

Recall that the Principia begins with a definition that we previously showed with some material blocked out.

Definition 1 *Quantity of matter is a measure of matter that arises from its density and volume jointly.*

> If the density of air is doubled in a space that is also doubled, there is four times as much air, and there is six times as much if the space is tripled. The case is the same for snow and powders condensed by compression or liquefaction, ... Furthermore, I mean this quantity whenever I use the term "body" or "mass" in the following pages.

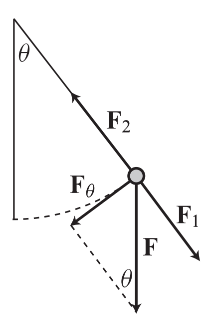
Adding in the remaining material we have

Definition 1 *Quantity of matter is a measure of matter that arises from its density and volume jointly.*

If the density of air is doubled in a space that is also doubled, there is four times as much air, and there is six times as much if the space is tripled. The case is the same for snow and powders condensed by compression or liquefaction, ... Furthermore, I mean this quantity whenever I use the term "body" or "mass" in the following pages. It can always be known from a body's weight, for—by making very accurate experiments with pendulums—I have found it to be proportional to the weight, as will be shown below.

This was certainly a confusing place for Newton to place this remark (just as he is introducing the notion of mass, which he wants to distinguish from weight!), and it must have been all the more confusing when he added the phrase "as will be shown below". Normally, one would expect this phrase to refer to material at the bottom of the page, or perhaps just a few pages later on. But Newton's "shown below" actually refers to material about 400 pages later in his book!

Newton's experiments involved pendulums, which was the 17th century high-tech way of avoiding friction. We'll consider the physics of the pendulum in greater detail later on (after all, a pendulum is basically a rigid body), but for now let's simply settle for the usual elementary physics analysis, which involves decomposing the gravitational force $\mathbf{F}$ of magnitude gm (where g is the acceleration due to gravity) on the bob into a force $\mathbf{F}_1$ in the direction of the pendulum



string and another force $\mathbf{F}_\theta$ tangent to the path of the bob. The string is also exerting a force, $\mathbf{F}_2$, on the bob, which is assumed to point along the direction of the string. We must have $\mathbf{F}_2 = -\mathbf{F}_1$, since we assume that the bob stays at a constant distance from the pivot point, keeping the string taut but not stretching it out. Thus, the net force on the bob is $\mathbf{F} + (-\mathbf{F}_1) = \mathbf{F}_\theta$, and consequently the acceleration of the bob, tangent to the circular path, has magnitude

$$a_\theta = g \sin\theta. \tag{1}$$

If we consider θ as a function of time, and let l be the length of the string (the radius of the circle on which the pendulum bob moves), then equation (1) yields

$$\theta'' + \frac{g}{l}\sin\theta = 0.$$

Physicists virtually always restrict their attention to the case of "small oscillations" ($\theta \sim 0$), so that they can replace $\sin\theta$ by θ and get an equation that they can solve. But the information we need, and that Newton relied on, doesn't require that consideration, and is a pure similarity argument.

For convenience, we choose the origin O to be the point from which the pendulum hangs. Then for any $\alpha > 0$ the path

$$\gamma(t) = \alpha \cdot c\big(t/\sqrt{\alpha}\big)$$

follows a circle with radius α times the radius of the path c, but with the time reparameterized by the factor $1/\sqrt{\alpha}$. So the angle $\vartheta(t)$ for γ satisfies

$$\vartheta(t) = \frac{\theta(t)}{\sqrt{\alpha}},$$

and it follows that

$$\vartheta'' + \frac{g}{\alpha l}\sin\vartheta = 0.$$

This means that γ gives the path of a pendulum bob with length α times that of the original, and we easily conclude that the period of the pendulum described by γ is $\sqrt{\alpha}$ times the period of the pendulum described by c: the period of a pendulum is proportional to the square root of its length.

A similar argument, left to you, shows that if the acceleration g were replaced by $g \cdot \alpha$, then the period of the pendulum would become $1/\sqrt{\alpha}$ times the original period; so the periods T_1 and T_2 of a pendulum bob undergoing different accelerations g_1 and g_2 are related by

$$\frac{g_2}{g_1} = \frac{{T_1}^2}{{T_2}^2}. \tag{$*$}$$

Newton phrased this result in a somewhat different way. If two objects, of masses m_1 and m_2, have gravitational accelerations g_1 and g_2, and we denote their *weights* by $W_1 = g_1 m_1$ and $W_2 = g_2 m_2$, then $(*)$ can be written as

$$\frac{m_1}{m_2} = \frac{W_1}{W_2} \cdot \frac{{T_1}^2}{{T_2}^2}.$$

This result first occurs in the Principia about 300 pages from the beginning, a mere 100 pages before Newton's description of his pendulum experiments, and I think it will be amusing to quote it as it appears:

Proposition 24 *In simple pendulums whose centers of oscillation agree equally distant from the center of suspension, the quantities of matter are in a ratio compounded of the ratio of the weights and the squared ratio of the times of oscillation* . . . .

For the velocity that a given force can generate in a given time in a given quantity of matter is as the force and the time directly and the matter inversely. . . . Now if the pendulums are of the same length, the motive forces in places equally distant from the perpendicular are as the weights, and this if two oscillating bodies describe equal arcs and if the arcs are divided into equal parts, then, since the times in which the bodies describe single corresponding parts of the arcs are as the times of the whole oscillations, the velocities in corresponding parts of the oscillations will be to one another as the motive forces and the whole times of the oscillations directly and the quantities of matter inversely; and thus the quantities of matter will be as the forces and the times of the oscillations directly and the velocities inversely. But the velocities are inversely as the times, and thus the times are directly, and the velocities are inversely, as the squares of the times, and therefore the quantities of matter are as the motive forces and the squares of the times, that is, as the weights and the squares of the times. Q.E.D.

I haven't the slightest idea what any of this means! But I'm almost certain that it amounts to the similarity argument we have given. Aren't you glad that you aren't a mathematician of the 17th century!?

About 100 pages later Newton describes how he tested

$$\frac{g_2}{g_1} = \frac{{T_1}^2}{{T_2}^2}, \tag{$*$}$$

and thus the proportionality of weight to matter, with equal weights of "gold, silver, lead, glass, sand, common salt, wood, water, and wheat". Each pair of materials to be tested was enclosed within one of two rounded, equal-sized wooden boxes. For the wood bob he simply filled the inside of the box with more wood, but for the gold bob he suspended the gold at the center of the box; he then hung each of the two boxes by eleven-foot cords, which "made pendulums exactly like each other with respect to their weight, shape, and air resistance."

He then started them swinging close to each other from the same height, noting that "they kept swinging back and forth together with equal oscillations for a very long time. … And it was so for the rest of the materials. In these experiments, in bodies of the same weight, a difference of matter that would be even less than a thousandth part of the whole could have been clearly noticed."

Of course, once we know that weight is exactly proportional to mass, there's no longer any need to measure mass accurately. To compare the mass of two objects, we simply have to compare their weights, and weights can easily be measured very accurately, with a balance scale. A balance scale is a special sort of lever, one in which the lever arms have the same length, but we don't need to understand the lever to use the balance scale, since symmetry implies that two objects that balance at the *same* distance from the pivot point must have the same weight (just to be on the safe side, we would probably interchange the two objects for a second check).

Clever experiments have allowed Newton's result to be verified to much greater accuracy in later times, and you can read in physics books about the 19th century Eötvös experiment (see, for example, [Fr]) and the 1964 experiment of Roll, Krotkov, and Dicke (described in [M-T-W]) which verified proportionality of weight to mass within 1 part in 10^{11} for gold and aluminum. The only thing I would like to point out here is that in both of these experiments, as in Newton's original experiment, we never actually measure the mass of anything! We only measure weights.

At this point, I think I would have to reassure the students in our elementary physics class by saying:

You may be starting to feel somewhat overwhelmed. First we insisted on distinguishing between mass and weight. Then we claimed that this distinction didn't really seem to matter, because weight was proportional to mass. Finally, we noted that we established this fact to great accuracy without even being able to measure mass to great accuracy! Don't worry! Every one before Newton was just as confused as you were! Newton was the first person to really make the distinction between mass and weight, and his decision to make mass and force the basic concepts in terms of which others should be defined—and to choose the first two laws as the basis for deducing other results—was one of Newton's main achievements. That's why we still speak of classical mechanics as Newtonian Mechanics.

The first two laws, on which we have spent so much time, involve individual bodies, but say nothing about the interactions between different bodies. This information is given by Newton's third law, and since all of mechanics rests on Newton's three laws, this last one must be quite special.

In fact, it is usually stated in a way that almost guarantees its misuse in philosophical and political discourse:

Law 3 *To any action there is always an opposite and equal reaction*

But Newton's statement was much more specific

Law 3 *To any action there is always an opposite and equal reaction; in other words, the actions of two bodies upon each other are always equal and always opposite in direction.*

Thus, the third law always involves *two different* bodies, each exerting a force on the other. Most misuses and invalid analogues of the third law ignore this basic fact that the two actions in question are exerted on two different bodies.

Although I've made a snide comment about philosophers' and political theorists' misuse of the third law, mathematicians and scientists themselves rarely appreciate the true significant of this law and its consequences. In a typical first-year physics course the law is simply stated, with one or two examples, and then earnestly applied to the solution of mechanics problems, with every one carefully drawing force diagrams to specify all the different reaction forces involved. Almost no mention is even made of the experimental evidence for this law!

There is, however, one consequence of the third law that is always mentioned. Consider the collision of two objects: B_1, having mass m_1, and B_2, having mass m_2. During the collision, B_2 will be exerting a force $\mathbf{F}_{12}$ on B_1, while B_1 will be exerting a force $\mathbf{F}_{21}$ on B_2 (the first subscript indicates the body on which the force acts). We should really write $\mathbf{F}_{12}(t)$ and $\mathbf{F}_{21}(t)$ because these forces may vary with time; in fact, they presumably vary in an incredibly complicated way, depending on the particular way that the two bodies are compressed, spin, vibrate, undulate, bobble, etc. But we always have $\mathbf{F}_{12}(t) = -\mathbf{F}_{21}(t)$, so for all times t we have

$$\begin{aligned}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2)'(t) &= (m_1\mathbf{v}_1)'(t) + (m_2\mathbf{v}_2)'(t)\\ &= \mathbf{F}_{12}(t) + \mathbf{F}_{21}(t)\\ &= 0.\end{aligned}$$

Thus, no matter how complicated the collision may be, the "total momentum" $m_1\mathbf{v}_1 + m_2\mathbf{v}_2$ is constant. Or, as we like to say, momentum is conserved.

This result appears in the Principia as a Corollary of the three laws, and we will quote it here, together with the proof, just to give another example of the flavor of mathematical writing at that time:

Corollary 3 *The quantity of motion, which is determined by adding the motions made in one direction and subtracting the motions made in the opposite direction, is not changed by the action of bodies on one another.*

For an action and the reaction opposite to it are equal by law 3, and thus by law 2 the changes which they produce in motions are equal and in opposite directions. Therefore, if motions are in the same direction, whatever is added to the motion of the fleeing body

> will be subtracted from the motion of the pursuing body in such a way that the sum remains the same as before. But if the bodies meet head-on, the quantity subtracted from each of the motions will be the same, and thus the difference of the motions made in opposite directions will remain the same.

The separate consideration of the case of two bodies moving toward each other and the case of one body overtaking another seems awfully awkward to us, but, as we will soon see, Newton may have had a specific reason for making the distinction clear.

This corollary of third law, usually presented as a neat consequence in physics classes, is actually the experimental evidence upon which Newton originally relied. In fact his Scholium tells us

Scholium The principles I have set forth are accepted by mathematicians and confirmed by experiments of many kinds.

> ...
>
> ... Sir Christopher Wren, Dr. John Wallis, and Mr. Christiaan Huygens, easily the foremost geometers of the previous generation, independently found the rules of the collisions and reflections of hard bodies, and communicated them to the Royal Society at nearly the same time.

This short sentence, off-handedly mentioning the near simultaneous communication of the same results, has a bit of history behind it.

Descartes seems to be the first person to have stated the law of Conservation of Momentum, offering the reasoning that since God made the Universe with a certain amount of momentum in it, obviously He would not allow this momentum to be changed (I admit that those aren't his exact words, but they are not an unfair paraphrase; see [D], Chapter 4, section 3 for an exact statement). One might feel that although this would not be the sort of argument to be accepted in modern physics journals, at least he did state the law. Unfortunately, however, Descartes didn't even state the law correctly, because he didn't think of momentum as a vector quantity, or even—for the case of bodies colliding along a straight line, the special case that people tended to restrict their attention to in those days—as a *signed* quantity, so he was really assuming that the absolute value of momentum was conserved. He then deduced from his principle a long series of consequences about colliding bodies, almost all of which are incorrect. Moreover, Descartes knew that his deductions didn't seem to accord with experiment, but blithely dismissed this as due to experimental errors!

Naturally, not every one was so convinced, so the Royal Society had asked for research into the proper rules, and this was the impetus for the communications of Wren, Wall, and Huygens.

You might assume that, unlike Descartes, these three men simply carried out experiments to see what actually happened, but that's not quite the case. Newton goes on to say

> ... Sir Christopher Wren, Dr. John Wallis, and Mr. Christiaan Huygens, easily the foremost geometers of the previous generation, independently found the rules of the collisions and reflections of hard bodies, and communicated them to the Royal Society at nearly the same time, ... But Wren additionally proved the truth of these rules before the Royal Society by means of an experiment with pendulums ...

Although Wallis and Huygens didn't do experiments, the "thought experiments" of Huygens happen to be extremely interesting, and we will consider them at a later time. But right now we want to briefly discuss those experiments with pendulums.

As we've already mentioned, pendulums were the 17th century way of practically eliminating friction, and they have an additional virtue: By observing how high a body swings past the lowest point, one can figure out the velocity that it had at that lowest point. Thus, Wren could determine both the velocities of two bodies just as they collided at the bottom of their pendulum arcs with the velocities that they obtained right after the collision.

After mentioning Wren's experiments, Newton proceeds to give details of his own experiments, designed to check the results with much greater accuracy, and he spends three entire pages describing them (these experiments are completely different from the ones used to verify proportionality of weight and mass, which occur much later, though mentioned at the very beginning of the Principia).

Newton is actually being rather modest in saying that "the principles I have set forth are accepted by mathematicians ... " because, as far as the third law is concerned, it had only appeared in this form, as conservation of momentum, and it was Newton who then worked backward to formulate his third law. Moreover, Newton took the truly audacious step of generalizing this law, based only on one particular physical phenomenon, the (totally unknown) forces involved in the collision of bodies, to arbitrary forces. One of the forces that Newton was particularly concerned about, of course, was gravity, but it will be quite interesting to first consider another force with which we are all familiar, namely, magnetism.

I'd like to consider a simple experiment, but I'll have to be content with describing it, rather than performing it, because it would require an air-trough, as well as rather strong magnets, stronger than I could find at Tokyu Hands, and a few other bits of modern technology, like a strobe light.

We simply place a magnet and a piece of iron on the air-trough, release them, and make accurate measurements of their positions over very many small intervals of time. This allows us to determine their velocities at these times, and check that the total momentum at all such times is the same.

But we can also make a convincing case without doing any measurements at all, by considering what happens when we release the magnet and the piece of iron simultaneously (so that they have total momentum 0 at that moment). I would really like to try this experiment with a class of naive physics students, after having first shown some collisions when the two are not released simultaneously, just to confuse the issue a bit. It would be interesting to present two different scenarios, one in which the magnet is small, and the piece of iron quite large, and one in which the situation is reversed, and ask students what they think will happen. I don't know, of course, because I haven't tried this (sociology) experiment, and the results would certainly depend on the sophistication of the students involved, but I wouldn't be surprised if many students would suspect that the heavier body would end up pushing the lighter one.

Of course, once we start thinking along the lines of conservation of momentum, we see that when the magnet and iron collide, no matter what their relative masses are, they must come to a *dead stop*, since their total momentum must always be the same as at the beginning, namely 0. Moreover, this one observation should suffice to convince us of conservation of momentum in this experiment, without making any of the intermediate measurements: Since the total momentum at the moment of collision is 0, it must have been very close to 0 just before the collision. But we can obtain a whole range of velocities just before collision simply by varying the initial distance between the magnet and iron—or, equivalently, by varying the strength of the magnet. Thus the total momentum must always be zero!

This experiment could just as well have been carried out using two magnets, but I chose a magnet and a piece of iron to point out the *unintuitive* aspect of the third law, which I would emphasize to a physics class by another simple pair of experiments.

Let us once again start with a magnet and a piece of iron held in place, and then simply release the iron, so that it hurtles toward the magnet. What does this experiment show? Well, obviously, that the magnet exerts a force on the iron; or, in more formal terms, that iron undergoes an acceleration in the presence of a magnet. Now we repeat the experiment, this time keeping the iron fixed, and releasing the magnet, which goes rushing toward the iron. Again we ask, what does this experiment show? Quite likely, many would say that it just shows the same thing, that a magnet exerts a force on the iron. But, of course, it doesn't show anything of the sort, since we don't observe the iron moving. What it shows is that *iron exerts a force on the magnet*. But we tend not to think in those terms. Normally we think of a piece of iron as a hunk of matter pretty much like any other piece. The magnet is something special, it exerts a force on the iron. But, in addition, the iron becomes special at the same time, something we may not usually acknowledge. If we take one of those cute little magnets that people use to hold notes on a refrigerator door, and hold it near the refrigerator, we don't notice the refrigerator moving toward the magnet! Instead, we feel the magnet being pulled toward the refrigerator. Nevertheless, people don't usually go around saying that refrigerators attract magnets. What's truly amazing, of course, is that not only does iron attract magnets, but it does so in exactly the right amount so that conservation of momentum holds.

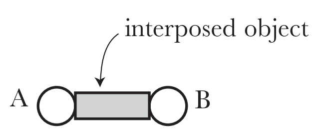
Remarkably enough, our modern understanding of magnetism happens to make this wonderful reciprocity quite understandable: the individual atoms of the iron each act as magnets, except that they are oriented randomly, and the magnet causes them to align, so that the iron now acts as a magnet also. So, ultimately, it's all a matter of iron atoms attracting *each other*, and we have a completely symmetric situation.

That same argument, of course, would make it quite clear why two objects that have been given static electric charges should exert forces of equal magnitude on each other, since ultimately it's all due to the mutual repulsive forces between electrons.

Strangely enough, it's actually the most common example, the repulsive force of colliding bodies, that now seems the most mysterious. Presumably this instance of the third law has to reduce to the third law operating on any pair of particles. So it must hold, for example, for a proton and a neutron, where the situation seems decidedly unsymmetric. In fact, it starts to be clear that if one really understood the third law, one would basically understand all of atomic physics.

I should mention that the little experiment involving a magnet and piece of iron that I have concocted, and which I feel would make a great teaching tool, was inspired by something Newton said, but hardly something for which I would praise him; in fact, it may be the silliest thing that Newton ever said (or, at any rate, the silliest scientific statement he ever made).

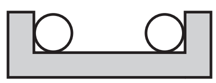
After his description of his pendulum experiments, which involved the *repulsive* force of collisions, Newton also wanted to say something about *attractive* forces (since he had gravity in mind). So after three pages describing his careful experimentation, he immediately adds the following paragraph, for which I have provided a quick picture:



> I demonstrate the third law of motion for attractions briefly as follows. Suppose that between any two bodies A and B that attract each other any obstacle is interposed so as to impede their coming together. If one

> body A is more attracted toward the other body B than that other body B is attracted toward the first body A, then the obstacle will be more strongly pressed by body A than by body B and accordingly will not remain in equilibrium. The stronger pressure will prevail and will make the system of the two bodies and the obstacle move straight forward in the direction from A toward B and, in empty space, go on indefinitely with a motion that is always accelerated, which is absurd and contrary to the first law of motion.

Thus, after three pages of careful experiment, Newton provides a one paragraph theoretical argument, and this argument is patently nonsense! The first law is concerned with the force on one body, not on a "system" consisting of more than one body. Moreover, the whole argument depends on the fact that the "interposed" object is rigid, so that it keeps A and B separated, and as you might imagine, our analysis of rigid bodies will presuppose the third law. Finally, we might note that the same argument could just as well be made to work for repulsive forces:



What's even more amazing is that Newton actually described an experiment made to test this idea, using vessels floating on water instead of an air-trough to reduce friction:

> I have tested this with a lodestone and iron. If these are placed in separate vessels that touch each other and float side by side in still water, neither one will drive the other forward, but because of the equality of the attraction in both directions, they will sustain their mutual endeavors toward each other, and at last, having attained equilibrium, they will be at rest.

It seems reasonable to assume that Newton did indeed actually perform this experiment, in view of his careful description of so many other experiments, but it must be one of the weirdest negative experiments every performed. Did any one really think that the two vessels would go zooming off in one direction!

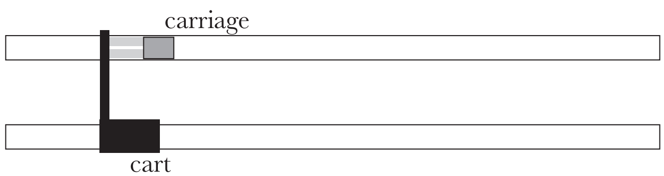
Although this episode must be written off as a strange anomaly in Newton's thought, simply removing the vessels that separate the magnet and iron suggests the more interesting experiment that we have performed before, and, I hasten to add, later lectures will confirm the brilliance of Newton's thought, and show how far ahead of all his contemporaries he was.

LECTURE 2

FURTHER REMARKS ON THE FUNDAMENTALS

In the previous lecture I pointed out that our operational definition of mass didn't seem to provide any clear reason why mass should be additive, and challenged people to find a reasonable explanation of this phenomenon. That was a challenge that I myself wasn't prepared to meet at the time, but I think the following would be a good answer.

First we would need experimental evidence establishing, in effect, that "forces are additive". Suppose that our cart needs to be given the acceleration a_1 in order to compress the spring to length L_0 when some object is placed in the carriage of the air-trough. And now suppose that instead of inserting the single spring, we insert two springs of the same construction. What we will find is that



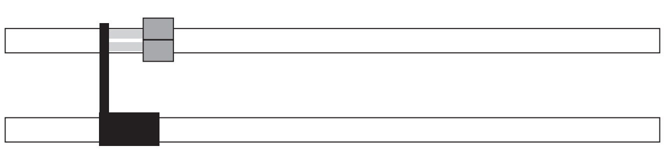
the cart must be given the acceleration $\alpha_1 = 2a_1$ in order for the two springs to be compressed simultaneously to the length L_0. Although this might seem clear if you think in terms of the usual $\mathbf{F} = m\mathbf{a}$ law, it is basically an experimental fact that we have to test, along with similar tests for three springs, four springs, etc.

Another way of expressing this is to say that the mass of our body can be determined by

$$m = \frac{a_0}{\alpha_1/2}$$

when we use two springs instead of one on an object whose mass we are trying to determine (a_0 is still the convenient acceleration that we used on our "unit mass" to determine L_0, with just one spring).

On the other hand, suppose that we place two copies of our object on the carriage, with one spring behind each. Presumably the cart will require an



acceleration of a_1 to compress the two springs to length L_0, since we can simply think of this as two copies of the original experiment carried out side-by-side. If we now think of this as two springs behind the one object consisting of two copies of our original object, it follows that the mass of this new object is just

$$\frac{a_0}{a_1/2} = 2\frac{a_0}{a_1} = 2m,$$

and of course we could just as well repeat the argument for other multiples, and eventually reason our way to the general rule.

I'd also like to examine the "thought experiments" that Huygens adduced in support of conservation of momentum, not only as an interesting historical curiosity, but also because it connects rather closely to the whole notion of "symmetry", which gets rather abused in modern thought.

First consider two identical bodies, say two steel balls, moving toward each other with equal speeds, i.e., with velocities $\mathbf{v}$ and $-\mathbf{v}$. In this simple situation it is obviously reasonable to assume, on the basis of symmetry, that their rebound velocities will also be negatives of each other, $\mathbf{w}$ and $-\mathbf{w}$, so that conservation of momentum holds: it is 0 both before and after the collision.

Now let us imagine the same experiment as observed in a coordinate system that is moving with uniform velocity $\mathbf{u}$ with respect to us, like a boat moving with respect to the shore, to take Huygens' example. ([M], Chapter III, section IV contains a reproduction of the delightfully quaint illustration that appears in Huygens' book "De Motu Corporum ex Percussione" of 1703; this figure also appears in [Fr], which gives a fairly detailed description of the following argument). In this coordinate system, the objects are moving with the initial velocities

$$\mathbf{v}_1 = \mathbf{v} + \mathbf{u} \qquad \text{and} \qquad \mathbf{v}_2 = -\mathbf{v} + \mathbf{u},$$

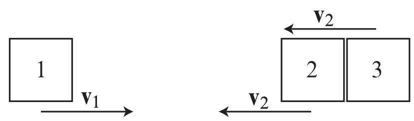
while their rebound velocities are

$$\mathbf{w}_1 = \mathbf{w} + \mathbf{u} \qquad \text{and} \qquad \mathbf{w}_2 = -\mathbf{w} + \mathbf{u},$$

so $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{w}_1 + \mathbf{w}_2$ $(= 2\mathbf{u})$. Since we can obtain any pair $\mathbf{v}_1, \mathbf{v}_2$ by choosing the appropriate $\mathbf{u}$ and $\mathbf{v}$, we find that in the coordinate system of the boat, moving uniformly with respect to the shore, conservation of momentum holds for two identical bodies approaching each other with *arbitrary* velocities. Of course, we could just as well interchange the role of the boat and the observer on shore, to reach the same conclusion for our observer on shore.

Rather than following the succeeding course of Huygens' arguments, we will add some considerations from The Feynman Lectures on Physics, Volume 1 [Fey]. Let's use steel cubes for convenience, and suppose that glue has been applied to opposing faces so that they will stick together when they meet. Symmetry dictates that when they approach each other with the same velocity and then stick together, they will end up at rest, so that conservation of momentum holds. Huygens' argument then implies that a collision with initial velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ results in a "double cube" moving with velocity $\frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2)$.

We can apply these results to the case of one cube with velocity $\mathbf{v}_1$ colliding with two other cubes that have glue applied to opposing faces, but are moving in tandem, separated by a tiny distance, with velocity $\mathbf{v}_2$. Immediately after



cube 1 and cube 2 collide, they have velocities $\mathbf{w}_1$ and $\mathbf{w}_2$ satisfying

(a) $$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2.$$

A moment later, cube 2 collides with and sticks to cube 3, after which the resulting double cube moves with velocity $\mathbf{w}_{12}$ satisfying

(b) $$2\mathbf{w}_{12} = \mathbf{w}_2 + \mathbf{v}_2.$$

It follows that

$$\begin{aligned}
\text{initial total momentum} &= m\mathbf{v}_1 + 2m\mathbf{v}_2 \\
&= m(\mathbf{v}_1 + \mathbf{v}_2) + m\mathbf{v}_2 \\
&= m(\mathbf{w}_1 + \mathbf{w}_2) + m\mathbf{v}_2 && \text{by (a)} \\
&= m\mathbf{w}_1 + m(\mathbf{w}_2 + \mathbf{v}_2) \\
&= m\mathbf{w}_1 + 2m\mathbf{w}_{12} && \text{by (b)} \\
&= \text{final total momentum.}
\end{aligned}$$

Imagining the tiny distance decreased to 0, we conclude that conservation of momentum holds for a collision of a cube with a double cube, and we can easily generalize the argument for any multiple cube colliding with any other.

This clever argument might require supplementary considerations to deal with steel cubes stacked differently, let alone with objects of arbitrary shape, but the real problem is that it applies only to two objects made of the same "homogeneous" material. As soon as we consider objects made of different materials we are at an impasse. Given a steel cube and an aluminum cube of the same mass, approaching each other with velocities $\mathbf{v}$ and $-\mathbf{v}$, no symmetry argument allows us to conclude that the rebound velocities are also negatives of each other; having the same mass simply means that they are given the same acceleration by a given force, it says nothing about why they should react symmetrically in this situation. And that is the whole mystery of the third law.

The third law seems so reasonable, and is accepted so uncritically by physics students, precisely because it rather misleadingly seems to be simply a statement of symmetry (and philosophers and political theorists appeal to it as expressing a sort of moral symmetry).

I want to conclude this lecture with one seemingly minor point about the fundamentals of mechanics that has so far been omitted. Newton's statement concerning conservation of momentum, previously quoted, was

Corollary 3 *The quantity of motion, which is determined by adding the motions made in one direction and subtracting the motions made in the opposite direction, is not changed by the action of bodies on one another.*

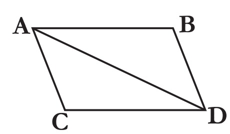
while Corollary 1, which we skipped, says

Corollary 1 *A body acted on by* [*two*] *forces acting jointly describes the diagonal of a parallelogram in the same time in which it would describe the sides if the forces were acting separately.*

Nowadays the vector space structure of $\mathbb{R}^n$ is so ubiquitous and appears so natural that we might unhesitatingly aver that the effect of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting simultaneously must simply be the standard vector sum $\mathbf{F}_1 + \mathbf{F}_2$. But that can't simply follow from the fact that we've decided to represent forces by vectors! In fact, the whole reason for introducing vector addition in the first place was because it represented the "addition" of forces.

As a matter of fact, we can say, without even reading Newton's proof, that it can't possibly be correct, since, after all, the three laws on which it is supposedly based are concerned with single forces acting on bodies—they simply say nothing at all about two forces acting simultaneously.

When we do look at Newton's proof, this is what we find:



Let a body in a given time, by force M alone impressed in A, be carried with uniform motion from A to B, and, by force N alone impressed in the same place, be carried from A to C; then complete the parallelogram ABDC, and by both forces the body will be carried in the same time along the diagonal from A to D. For, since force N acts along the line AC parallel to BD, this force, by law 2, will make no change at all in the velocity toward the line BD which is generated by the other force. Therefore, the body will reach the line BD in the same time whether force N is impressed or not, and so at the end of that time will be found somewhere on the line BD. By the same argument, at the end of the same time it will be found somewhere on the line CD, and accordingly it is necessarily found at the intersection D of both lines. And, by law 1, it will go with [uniform] rectilinear motion from A to D.

Even before we reach any questionable steps, we see from the very first phrases that Newton is framing this proof in terms of impulsive forces, since he states that the forces M and N individually produce a *uniform* motion on the object. Moreover, at the very end of the argument he implicitly assumes that the combination of the two impulsive forces *must also be an impulsive force*, so that the object moves with uniform motion from the initial point A to the final point D. The remaining part of the argument is the most dubious of all, with its argument that the force N "will make no change at all in the velocity toward the line BD which is generated by the other force".

In defense of Newton, we ought to unveil the material that was hidden in our original presentation of the second law:

Law 2 *A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.*

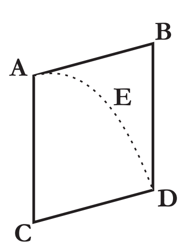
If some force generates any motion, twice the force will generate twice the motion, ... And if the body was previously moving, the new motion (since motion is always in the same direction as the generative force) is added to the original motion if that motion was in the same direction or is subtracted from the original motion if it was in the opposite direction or, if it was in an oblique direction, is combined obliquely and compounded with it according to the directions of both motions.

This almost sounds like a statement of Corollary 1. Actually, it will also help to go back to Newton's Scholium, and unveil the material that was deleted *there*, when we quoted his acknowledgment of Galileo:

Scholium The principles I have set forth are accepted by mathematicians and confirmed by experiments of many kinds. By means of the first two laws and the [first corollary] Galileo found that the descent of heavy bodies is in the squared ratio of the time and that the motion of projectiles occurs in a parabola, as experiment confirms, except insofar as these motions are somewhat retarded by the resistance of the air.

Newton even provides a little picture in the discussion that follows a bit later:

For example, let body A by the motion of projection alone describe the straight line AB in a given time, and by the motion of falling alone describe the vertical distance AC in the same time; then complete the parallelogram ABDC, and by the compounded motion the body will be found in place D at the end of the time; and the curved line AED which the body will describe will be a parabola which the straight line AB touches at A and whose ordinate BD is as AB^2.

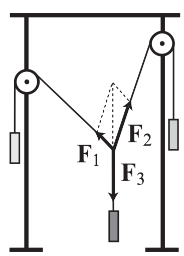


Here, of course, we are considering, on the one hand, an impulsive force, which gives the object its uniform horizontal motion, and, on the other hand, the force of gravity, which gives the object its non-uniform vertical motion. And indeed this really illustrates only that the action of a force on an object is independent of the object's uniform velocity, which was Galileo's basic observation.

Basically, Newton is saying that Corollary 1 holds if we think of one of the two forces having already been applied, and then trying to sucker us into concluding that it holds when they are applied simultaneously. Of course, from a modern quantum mechanical point of view this might seem extremely reasonable, because the each force is presumed to come about by interactions with myriad special particles, and one could ignore the infinitesimal probability of two such interactions occuring exactly at the same time.

But if we stick to the classical picture, then from a strictly logical point of view, it is not even clear that two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting simultaneously should have the same effect as any other single force $\mathbf{F}$: while it's true that the combined forces must end up producing an acceleration of *some* sort on each object, that acceleration might not be proportional to the mass of the object, even though the accelerations produced by $\mathbf{F}_1$ and $\mathbf{F}_2$ individually are.

Physicists nowadays seem resigned to the stance of regarding the parallelogram law as just another law based on observation, and mechanisms like this may be used to illustrate it in classroom settings.



$\mathbf{F}_3 = -(\mathbf{F}_1 + \mathbf{F}_2)$
at equilbrium

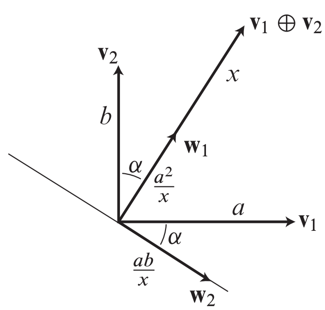
In [M], Chapter I, section III, there is a picture of a much more elaborate mechanism for illustrating the parallelogram law, but even it would probably only yield one or two decimals of accuracy. Since the parallelogram law is presumably an experimental fact, and since virtually *everything* in physics depends on it, one might expect it to be tested to great precision, like the precise experiments to test the proportionality of weight and mass.

But no one ever mentions such experiments! I think that's probably because every one thinks that somehow the result must really be a theorem, and Newton is certainly not the only who tried to present it as such. Numerous mathematicians (and I mean real mathematicians, not circle-squarers) have provided "proofs". The first was Bernoulli (though it was Jean Bernoulli, not his more famous brother, of the Bernoulli numbers), followed by Laplace (of the Laplace transform) in his great work [L], and by Poisson (of the Poisson integral) in [P] and by Hamilton (of Hamiltonian mechanics and the Hamilton-Jacobi equation) in [H].

If you try to read these proofs—supposedly mathematical proofs of a non-mathematical fact—you will see that they are all shrouded in a somewhat impenetrable veil of unstated assumptions, making it all the harder to read them, and of course in those days even mathematical results were stated in strange ways, and the proofs are nowadays hard to read. Since the proofs, no matter how complicated, or how elegant, all share the same fatal flaws, I'll give pride of place to Bernoulli's proof, and present the main idea, based on the account in [M] (not bothering with some details, since the only point of presenting it is to demolish it).

We restrict our considerations to $\mathbb{R}^2$, so that we are only treating forces in one plane. As we have already indicated, the first basic assumption is that two forces $\mathbf{v}, \mathbf{w}$ acting together have the same effect as some other force. Thus, we are assuming that for each pair $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ we have another element $\mathbf{v} \oplus \mathbf{w} \in \mathbb{R}^2$. We presumably shouldn't object to assuming that $\mathbf{v} \oplus \mathbf{w} = \mathbf{w} \oplus \mathbf{v}$, and also that $\mathbf{v} \oplus \mathbf{v} = 2\mathbf{v}$ (an equation inherent in the very discussion at the beginning of this lecture). More generally, of course, we could assume that any k-fold sum $\mathbf{v} \oplus \cdots \oplus \mathbf{v}$ makes sense, and has the value $k\mathbf{v}$, without worrying about the order in which the operations $\oplus$ are performed, but for the moment we will leave aside the question of what other assumptions would be reasonable. Our goal is to show that $\mathbf{v} \oplus \mathbf{w}$ is simply the usual vector sum $\mathbf{v} + \mathbf{w}$.

We begin by considering two perpendicular vectors, $\mathbf{v}_1$ of length a, and $\mathbf{v}_2$ of length b, and let x be the length of $\mathbf{v}_1 \oplus \mathbf{v}_2$. Let $\mathbf{w}_2$ be the vector on the line



perpendicular to $\mathbf{v}_1 \oplus \mathbf{v}_2$ with length $\frac{ab}{x}$, and let $\mathbf{w}_1$ be the vector along $\mathbf{v}_1 \oplus \mathbf{v}_2$ of length $\frac{a^2}{x}$. We now have

$$\begin{aligned}
\text{length } \mathbf{w}_1 &= \tfrac{a}{x} \cdot \text{length } \mathbf{v}_1 \\
\text{length } \mathbf{w}_2 &= \tfrac{a}{x} \cdot \text{length } \mathbf{v}_2 \\
\text{length } \mathbf{v}_1 &= \tfrac{a}{x} \cdot \text{length}(\mathbf{v}_1 \oplus \mathbf{v}_2),
\end{aligned}$$

and the angle α from $\mathbf{w}_2$ to $\mathbf{v}_1$ equals the angle from $\mathbf{w}_1$ to $\mathbf{v}_2$. This means that there is an orthogonal map T—involving a rotation through the angle α, together with a reflection—such that

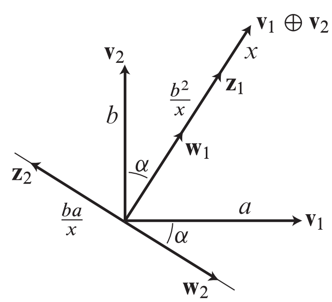
$$\begin{aligned} \tfrac{a}{x}T(\mathbf{v}_1) &= \mathbf{w}_1 \\ \tfrac{a}{x}T(\mathbf{v}_2) &= \mathbf{w}_2 \\ \tfrac{a}{x}T(\mathbf{v}_1 \oplus \mathbf{v}_2) &= \mathbf{v}_1. \end{aligned}$$

From this we conclude that

$$\mathbf{w}_1 \oplus \mathbf{w}_2 = \mathbf{v}_1. \tag{1}$$

Here we are using the assumption, almost never explicitly made, that we should have $T(\mathbf{v} \oplus \mathbf{w}) = T(\mathbf{v}) \oplus T(\mathbf{w})$ for all *orthogonal maps* T, but this is a perfectly reasonable assumption to add, as it merely expresses our experience that the laws of physics seem to be independent of orientation.

Now, similarly, we consider vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ of lengths $\frac{ba}{x} = \frac{ab}{x}$ and $\frac{b^2}{x}$,



respectively, and conclude that we have

$$\mathbf{z}_1 \oplus \mathbf{z}_2 = \mathbf{v}_2. \tag{2}$$

Since $\mathbf{w}_2 = -\mathbf{z}_2$, equations (1) and (2) give

$$\mathbf{w}_1 \oplus \mathbf{z}_1 = \mathbf{v}_1 \oplus \mathbf{v}_2. \tag{3}$$

But $\mathbf{w}_1$ and $\mathbf{z}_1$ lie along $\mathbf{v}_1 \oplus \mathbf{v}_2$, so the length of $\mathbf{v}_1 \oplus \mathbf{v}_2$ is the sum of the lengths of $\mathbf{w}_1$ and $\mathbf{z}_1$, which means that

$$\frac{a^2}{x} + \frac{b^2}{x} = x \implies a^2 + b^2 = x^2 \implies x = \sqrt{a^2 + b^2},$$

and thus $\mathbf{v}_1 \oplus \mathbf{v}_2 = a\mathbf{e}_1 \oplus b\mathbf{e}_2$ has length $\sqrt{a^2 + b^2}$.

In other words, the length of $a\mathbf{e}_1 \oplus b\mathbf{e}_2$ is the same as the length of $a\mathbf{e}_1 + b\mathbf{e}_2$, and Bernoulli has thus demonstrated that $\mathbf{v}_1 \oplus \mathbf{v}_2$ has precisely the length you would expect it to have, in the special case that $\mathbf{v}_1$ and $\mathbf{v}_2$ are perpendicular.

He then proceeds by involved arguments to prove the complete result, for the general case.

The one point that is usually ignored is that in our quick trip from equations (1)-(2) to (3), we had to use associativity of $\oplus$, which is likewise used in all the other proofs that have been fashioned. But if we assume associativity of $\oplus$, then everything is essentially trivial: Consider the map

$$(a, b) = a\mathbf{e}_1 + b\mathbf{e}_2 \longmapsto a\mathbf{e}_1 \oplus b\mathbf{e}_2.$$

If $\oplus$ is associative then this map will be linear. But it takes $\mathbf{e}_1$ to $\mathbf{e}_1$ and $\mathbf{e}_2$ to $\mathbf{e}_2$, so it must be the identity. Q.E.D.

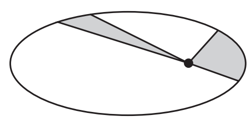
So in the end, I really don't know what to say about the parallelogram law. I think we do have to resort to the modern view that it is an experimental fact, and then just wonder why no one has ever done an experiment to test it!

LECTURE 3

HOW NEWTON ANALYZED PLANETARY MOTION

After the "Definitions" and "Axioms" sections of the Principia, we get to Book 1, "The Motion of Bodies". This begins with a preliminary section that basically treats elements of calculus in a geometric guise, and then Newton immediately starts the next section with "Kepler's second law", though he doesn't mention Kepler's name.

This law, actually the first that Kepler discovered, says that the radius vector of a planet sweeps out equal areas in equal time, or equivalently, that the area swept out in time t is proportional to t.



Newton pointed out that this is a consequence of the fact that the gravitational force that the sun produces on the planet it always directed along the line from the planet to the sun, or equivalently, that the acceleration of the planet is always directed toward the sun—the specific magnitude of this force being irrelevant. Or as Newton expressed it,

> Proposition 1. *The areas which bodies made to move in orbits describe by radii drawn to an unmoving center of forces lie in unmoving planes and are proportional to the times.*

This turns out to be extremely easy to prove analytically, especially if we use the cross product of vectors. For simplicity assume that our force is always directed toward the origin O, and let c be any particle. We always have

$$\begin{aligned}(c \times \mathbf{v})' &= (c \times \mathbf{v}') + (c' \times \mathbf{v}) = (c \times \mathbf{v}') + (\mathbf{v} \times \mathbf{v}) \\ &= c \times \mathbf{v}',\end{aligned}$$

so if $\mathbf{v}'$ points along c, we just get $(c \times \mathbf{v})' = 0$, and consequently the relation

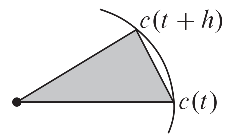
$$(*) \qquad c \times \mathbf{v} = \mathbf{w} \qquad \mathbf{w} \text{ a constant vector.}$$

If $\mathbf{w} = 0$, then $\mathbf{v}(t)$ always points along the line from O to $c(t)$, and our particle must simply be moving along a straight line towards O. If $\mathbf{w} \neq 0$, then, since the inner product satisfies

$$0 = \langle c(t) \times \mathbf{v}(t), c(t) \rangle = \langle \mathbf{w}, c(t) \rangle,$$

we see immediately that $c(t)$ always lies in one plane.

Moreover, $c(t) \times \mathbf{v}(t)$ has a natural interpretation in terms of the area swept out by the radius vector. For small h, this area, $S(t+h)-S(h)$, is approximately



the area of the shaded triangle, and thus approximately

$$\frac{1}{2}\big|c(t) \times [c(t+h) - c(t)]\big|.$$

Consequently, in the limit we have

$$\begin{aligned} S'(t) &= \frac{1}{2} \lim_{h \to 0} \left| c(t) \times \frac{c(t+h) - c(t)}{h} \right| \\ &= \frac{1}{2} |c(t) \times \mathbf{v}(t)|. \end{aligned}$$

Thus, $(*)$ implies that $S'(t)$ is constant, or that $S(t)$ is proportional to t.

We can also simply write c as

$$c(t) = (R(t) \cos\theta(t),\ R(t) \sin\theta(t)),$$

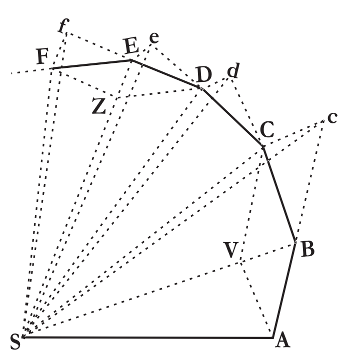
compute $\mathbf{v} = c'$ and observe that

$$|c \times \mathbf{v}| = R^2\theta',$$

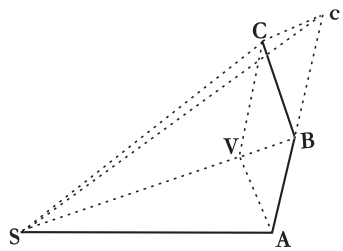
and $\frac{1}{2} \cdot R^2\theta'$ is just the integrand required to compute areas in polar coordinates.

Newton's proof is completely different. It is a a geometric proof, approximating the curve by a polygon, and it is not only simple, but it also seems to show just *why* the proposition is true.

Newton uses the following diagram in his proof



but we only need to consider a small portion of it:



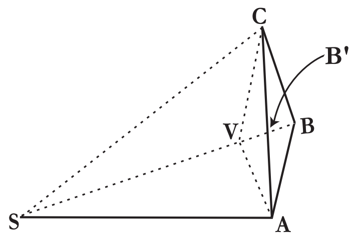
Newton assumes that the particle follows the path $ABCD\ldots$, receiving "impulsive" forces at short equal intervals of time, and that these impulsive forces at B, C, … are always directed toward S, so that the path sweeps out the triangular areas $\triangle SAB$, $\triangle SBC$, … . Newton merely has to point out that, if not for the impulsive force at B, the particle would move to c, with $Bc = AB$. In this case, it would sweep out the triangle $\triangle SBc$, which has the same area as $\triangle SAB$ (since they have equal bases, and the same height). The impulsive force applied at B will instead send the particle to C, which will be at the diagonal of the parallelogram formed by Bc and a line BV pointing along SB, since we are assuming that the force is directed toward S. This means that Cc is parallel to BV, and this in turn means that $\triangle SBc$ has the same area as $\triangle SBC$ (since these triangles have the common base SB and the same height above that base). In short, the area of $\triangle SAB$ is the same as the area of $\triangle SBC$, and so on, all along the path!

It is also noteworthy that Newton expressly states the converse of Proposition 1 (the proof being pretty much the same):

> Proposition 2. *Every body that moves in some curved line described in a plane and, by a radius drawn to a point, … describes areas around that point proportional to the times, is urged by a centripetal force tending toward that same point.*

Newton's proof of Kepler's second law is sometimes presented in elementary physics books, but then we are told that the rest of Newton's arguments won't be given because they require many abstruse properties of conic sections which are unfamiliar to us nowadays. But this is rather insincere. While it is quite understandable that a geometric proof would use many geometric properties of conics, the real mystery is how an hypothesis about inverse square forces is going to be related to geometric properties of conic sections. Moreover, although we won't pursue Newton's argument in its entirety, it turns out that Newton's *strategy* for the proof is extremely clever, far too clever for most of his contemporaries (and even for some people today).

To see how Newton relates the forces to the geometry, we need only follow, with slight modifications, a few steps that he adds a bit later on. In the diagram for Proposition 1, breaking up the motion into small intervals h of time, consider the segment BB' from B to the midpoint B' of the diagonal AC. This

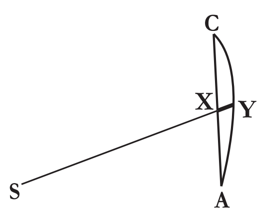


is half of BV, which represents the displacement due to the central force of magnitude F at B, and this distance is just $\frac{1}{2}Fh^2$. So, in the limit,

$$\lim_{h\to 0} \frac{1}{h^2} BB' = \tfrac{1}{4}F.$$

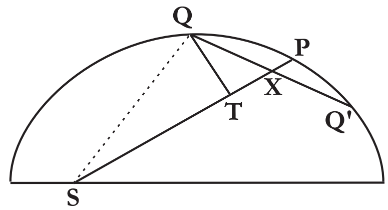
Or as Newton phrases it,

> If . . . a body revolves in any orbit about an immobile center and describes any just-nascent arc in a minimally small time, and if the sagitta of the arc is understood to be drawn so as to bisect the chord and, when produced, to pass through the center of forces, the centripetal force in the middle of the arc will be as the sagitta directly and as the time twice inversely.



Sagitta is an old fashion term [from the Latin for arrow], and Newton's statement explicitly indicates that he is referring to the segment XY of the line through S and the midpoint X of the chord AC. The fraction $\frac{1}{4}$ doesn't appear in Newton's statement because the result is phrased as a proportion: the ratio of the centripetal forces at points A and A' is the same as the ratio of the limits $\lim_{h\to 0} XY/h^2$ for arcs starting at A and A', respectively.

Newton next considers a point P at time t on a curved path around the center S, and two points Q, Q' on the path, at two nearby times $t - h$ and $t + h$. Then the force at P is proportional to

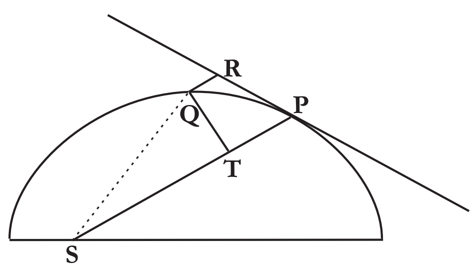


$$\lim_{h \to 0} \frac{PX}{h^2}.$$

But, by Proposition 1, h is proportional to the area of the (curved) triangle SPQ, and thus, in the limit, to $QT \times SP$, where QT is perpendicular to SP. Thus, finally, the force at P is proportional to

$$\lim_{h \to 0} \frac{PX}{(SP)^2 \times (QT)^2}.$$

Newton, however, actually presents a figure that has a tangent line drawn at P, and the line QR drawn parallel to SP, with the assertion that the force at P



is proportional to

$$\text{(F)} \qquad \lim_{h \to 0} \frac{QR}{(SP)^2 \times (QT)^2}.$$

Of course, QR is not actually equal to PX, but it is apparently obvious to Newton that it is equal to second order so that the limit still holds (something that I have not had the courage to try to confirm).

And now Newton is all prepared to show that the orbit of an object moving under an inverse square force is a conic section. Newton begins in a way that might seem strange to us, by proving a partial *converse* of this assertion:

> *Let a body revolve in an ellipse; it is required to find the law of the centripetal force tending toward a focus of the ellipse.*

In other words, given a path c lying along an ellipse, if c'' always points towards one focus of the ellipse, Newton is going to show that

$$|c''(t)| = \frac{k}{d(t)^2} \quad \text{for some constant } k,$$

where $d(t)$ is the distance from $c(t)$ to the focus.

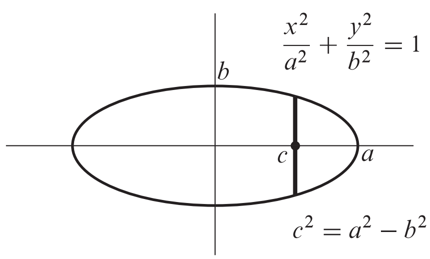
The relation (F) is the key to all this. In fact, in view of (F), our assertion is equivalent to saying that for an ellipse we have

$$\lim_{h\to 0} \frac{QR}{(QT)^2} \quad \text{is a constant.}$$

This limit has nothing to do with forces, and is completely determined by the shape of the ellipse. It could even be computed by a double application of L'Hôpital's Rule: If $F(h)$ denotes QR and $G(h)$ denotes $(QT)^2$, then we have $\lim_{h\to 0} F'(h) = \lim_{h\to 0} G'(h) = 0$, and

$$\lim_{h\to 0} \frac{QR}{(QT)^2} = \frac{F''(0)}{G''(0)};$$

when the quite unpleasant calculation is carried through, it turns out that $F''(0)/G''(0)$ is independent of the point P, and in fact $= a/2b^2$ for the ellipse shown below; the reciprocal, $2b^2/a$, is the length of the classical *latus rectum* of the ellipse, the segment cut off by the ellipse on the vertical line through one of the foci.



Newton proves exactly this result geometrically, and the proof is indeed long, complicated, and depends on numerous results about the ellipse. For a complete exposition of this proof see [N-C-W], pp. 325–330.

Newton then gives a similar proof for a body moving on a hyperbola, and finally a proof for a body moving on a parabola.

And immediately after this, stated as a corollary, comes the result we were anticipating:

> Corollary 1. From the last three propositions it follows that if any body P departs from the place P along any straight line PR with any velocity whatever and is at the same time acted upon by a centripetal force that is inversely proportional to the square of the distance of places from the center, this body will move in some one of the conics having a focus in the center of forces; and conversely.

In other words, Newton is saying that this Corollary, which constitutes the converse of the previous three propositions, automatically follows from them! In the first edition of the Principia, this is *all* that Newton wrote, again failing to comprehend that not every one was as bright as he was. And people who should have known better, like Bernoulli, actually seemed to think that Newton had made some sort of logical error, and even today there are people, who should know better, who have suggested that perhaps Newton didn't quite understand the difference between a proposition and its converse!

So in the second edition, Newton added a bit more, and in the third yet another bit.

> Corollary 1. From the last three propositions it follows that if any body P departs from the place P along any straight line PR with any velocity whatever and is at the same time acted upon by a centripetal force that is inversely proportional to the square of the distance of places from the center, this body will move in some one of the conics having a focus in the center of forces; and conversely. For if the focus and the point of contact and the position of the tangent are given, a conic can be described that will have a given curvature at that point. But the curvature is given from the given centripetal force and velocity of the body; and two different orbits touching each other cannot be described with the same centripetal force and the same velocity.

Probably Newton should have added even a bit more, because there are still people who can't see the argument, which we'll give explicitly. For simplicity we simply work in $\mathbb{R}^2$, and we choose the origin O as the point toward which the force is directed. Given a point P, and a tangent vector $\mathbf{v}$ at P, we want to find a curve $c = (c_1, c_2)$ with $c(0) = P$ and $c'(0) = \mathbf{v}$ satisfying

$$(*) \qquad c''(t) = \frac{k}{|c(t)|^2} \cdot \frac{-c(t)}{|c(t)|},$$

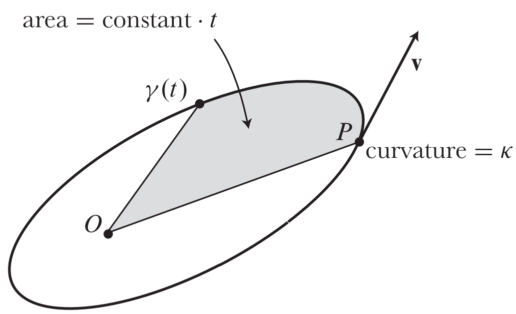
where k is a given constant (the factor $-c(t)/|c(t)|$ is just a unit vector pointing from $c(t)$ to the origin).

Since we know $c'(0)$, and $(*)$ gives us $c''(0)$, we know what the curvature κ of c at 0 should be, since this is given by

$$(**) \qquad \kappa = \frac{c_1{}'(0)c_2{}''(0) - c_2{}'(0)c_1{}''(0)}{(c_1{}'^2(0) + c_2{}'^2(0))^{3/2}},$$

a formula known to Newton and his contemporaries, though they probably defined curvature in terms of the osculating circle.

Now consider a conic section K having O as a focus, which passes through P, and is tangent to $\mathbf{v}$ at P, and whose curvature at P is this κ, assuming for the moment that such a conic section exists. Consider a curve γ with $\gamma(0) = P$, and which traverses K in such a way that the areas cut out by radii from O



is proportional to the time. Such curves are determined up to a multiplicative change of parameter; by choosing the appropriate multiplicative constant, we can arrange for $\gamma'(0) = \mathbf{v}$. According to our converse Proposition 2, we have

$$\gamma''(t) = \frac{\bar{k}}{|\gamma(t)|^2} \cdot \frac{-\gamma(t)}{|\gamma(t)|}$$

for some $\bar{k}$. But we must have $k = \bar{k}$, since we chose γ so that its curvature at $\gamma(0) = P$ would be the κ given by $(**)$.

Thus, γ is a solution of our differential equation $(*)$, and by uniqueness (which of course Newton and all his contemporaries implicitly assumed) it is the only possible solution.

(The only slight lacuna in this argument is the existence of a conic section with the required curvature, and Newton gives a geometric solution to this problem a little later on.)

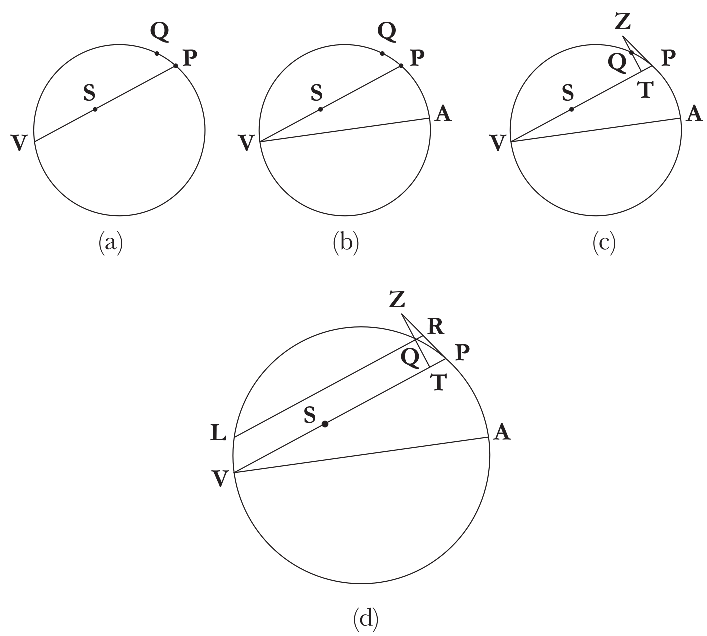
Moral: It's dangerous to be a lot smarter than every one else.

Of course, Newton's argument might strike us as a little weird, starting as it does with the converses of the result we want, but a *geometric* proof almost has to be of this nature: it's a lot easier to start with a geometric object, an ellipse, or hyperbola, or parabola, and deduce a formula for forces, than it would be to start with the formula for forces and somehow conjure up these geometric arguments.

I should also add that Bernoulli was finally reduced to admitting the correctness of Newton's argument (see [N-H-T], Volume 7, pp. 77-79), defending his own analytic proof as being more direct. But, in fact, Newton also has an analytic proof later on in the Principia, as Proposition 41, although it is presented completely geometrically, with a terrifyingly complex figure. You can find this stated on page 529 of [N-C-W] and starting on page 334 of this book's preliminary guide to reading the Principia there is a complete "translation" into modern calculus. See also [Gu] for a thorough treatment of the disputes that arose.

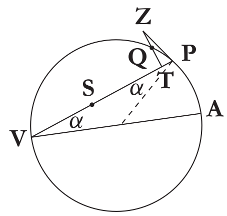
Although relation (F) was so crucial for analyzing inverse square forces, Newton did not immediately apply it to that task, but first showed other ways in which it could be used, which are also rather interesting. The analysis is purely geometric, and for the remainder of this lecture you may think that you're back in geometry class again, and a really hard one at that (Newton and his contemporaries were really good at geometry!).

First Newton asks what central force at a point S will cause a body to move on the circumference of a circle when S is *not* the center of the circle. Newton considers a point P on the circle, with a nearby point Q. Then (a) we draw the line from P to S, intersecting the circle at V. Next (b) we draw the diameter



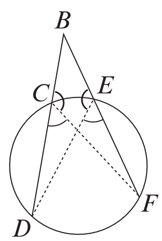
VA, and then (c) we draw the line TQ perpendicular to SP, and extend it until meets the tangent line to the circle at P in the point Z. Finally, (d) we draw the line LR parallel to VP intersecting the circle at L and the tangent line at R.

In (c) we also draw the radius to P, which is perpendicular to ZP and con-



clude that $\angle TZP = \alpha$, so that the right triangle ZTP is similar to the right triangle VPA

We also need to recall the elementary geometry theorem that for two secant lines through a point, we have the relation $BC \cdot CD = BE \cdot BF$, and the obvious



consequence when one of the secant lines is actually a tangent line ($E = F$).

Now, since LR is parallel to VP in (d), we have

$$\begin{aligned}\frac{RP}{QT} &= \frac{ZP}{ZT}\\ &= \frac{AV}{PV}, \quad \text{by similar triangles.}\end{aligned}$$

Squaring, and applying the above mentioned geometric theorem to the secant RL and the tangent RP, we then have

$$\frac{QR \cdot RL}{(QT)^2} = \frac{(AV)^2}{(PV)^2},$$

or

$$\frac{QR}{(QT)^2} = \frac{1}{RL} \cdot \frac{(AV)^2}{(PV)^2},$$

and thus

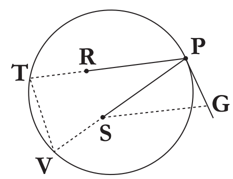
$$\frac{QR}{(SP)^2(QT)^2} = \frac{1}{RL} \cdot \frac{(AV)^2}{(SP)^2(PV)^2}.$$

The left side is the fraction that appears in formula (F), and $RL \to PV$ as $Q \to P$, while AV is a constant, so the force must be inversely proportional to $(SP)^2 \cdot (PV)^3$.

Although this result appears in the first edition of the Principia, the following three remarkable corollaries first appeared in the second edition:

(a) If a particle moves in a circle under a central force directed to a point V *on* the circle, then the force varies inversely as the fifth power of the distance. (An inverse fifth power law is one of the select few, together with the inverse square and inverse cube laws, for which we we can give analytic solutions.)

(b) More generally, suppose that a particle moves on a circle under two different central forces, one with center at R and one with center at S. Let SG be drawn



parallel to RP, intersecting the tangent line to the circle at P in the point G. Then the ratio of the first force to the second is

$$\frac{(RP)^2\cdot(PT)^3}{(SP)^2\cdot(PV)^3} = \frac{SP\cdot(RP)^2}{\dfrac{(SP)^3\cdot(PV)^3}{(PT)^3}} = \frac{(RP)^2\cdot SP}{(SG)^3}.$$

(c) This same result holds for an *arbitrary* orbit under central forces directed toward R and S: if the two forces result in the same orbit, then the ratio of the first force to the second is

$$\text{(R)} \qquad \frac{(RP)^2\cdot SP}{(SG)^3}.$$

For we just have to consider the osculating circle to the orbit at P.

One other situation that Newton analyzed before tackling elliptical orbits under a force directed toward the focus was the question of an elliptical orbit under a force directed *toward the origin.* His treatment of that problem was almost as complicated as his later treatment of a force directed toward the focus, but it's trivial for us to give an immediate solution:

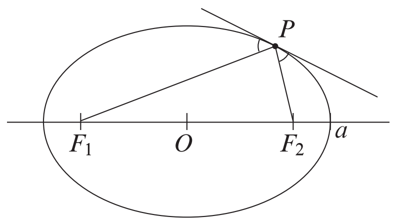
We simply note that a particle moving under the orbit

$$c(t) = (a\cos\alpha t,\ b\sin\alpha t)$$

has $c''(t) = -\alpha^2 c(t)$, so that it is a possible motion under a central force varying *directly* as the distance to the origin, and it is then easy to conclude that the same holds for any ellipse.

This result also appeared in the first edition, perhaps just as a warm-up for tackling the case of a force directed toward the focus, but in the second edition Newton combined this result with the result about the ratio (R) to give an alternative treatment of the focus case.

We will need another property of the ellipse, but it is one that we can easily derive from familiar ones. Recall that the ellipse with major axis $2a$ and foci F_1 and F_2 is defined by the property that $F_1P + PF_2 = 2a$ for all points P. It also



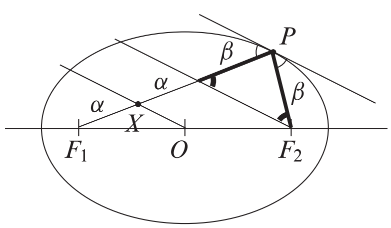
(a)

has the "focal point" property that a light ray starting from F_1 passes through F_2, i.e., that the two angles indicated in (a) are equal.

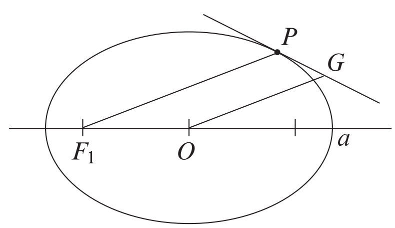
In diagram (b) below we have drawn lines parallel to the tangent line at P through O and F_2. The two angles indicated by thick arcs are equal, so the two thick segments have the same length β. And the two segments with lengths indicated as α are equal because $F_1O = OF_2$. Thus

$$2\alpha + 2\beta = 2a.$$

Finally (c), moving XP over to OG we see that a line through the origin O



(b)



(c)

parallel to the line F_1P always intersects the tangent line through P at a point G with $OG = a$.

From this and the result (R) for the ratio, it is now easy to conclude that the result for a force directed toward a focus of an ellipse follows from the very elementary result for a force directed toward the center.

An analytic reworking of this whole circle of results can be found in Appendix 1 of [A].

LECTURE 4

SYSTEMS OF PARTICLES; CONSERVATION LAWS

This lecture is fairly straightforward, and serves mainly as a prelude for the next, when we finally get to consider rigid bodies.

We will be considering a "system of particles". This means that we have certain particles $c_1, \dots c_K : \mathbb{R} \to \mathbb{R}^3$, with positive masses $m_1, \dots, m_K \in \mathbb{R}$. We also have certain forces, functions

$$\mathbf{F}_i^e : \mathbb{R} \to \mathbb{R}^3,$$

the "external" forces on the particles (for example the force of gravity for a system of particles near the earth), as well as "internal forces"

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji} : \mathbb{R} \to \mathbb{R}^3,$$

where $\mathbf{F}_{ij}$ represents the force exerted by c_j on c_i. In accordance with the third law we are assuming that

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}.$$

In addition, we will soon need the "strong version" of the third law, which says that $\mathbf{F}_{ij}$ points from c_i to c_j, i.e., $\mathbf{F}_{ij}$ is a multiple of $c_i - c_j$. This stronger version of the third law is never stated by Newton, although one might assume that he considered this implicit, on symmetry grounds.

Finally, if we set

$$\mathbf{F}_i = \mathbf{F}_i^e + \sum_j \mathbf{F}_{ij},$$

so that $\mathbf{F}_i$ is the total force on the particle c_i at time t, then we assume, in accordance with the second law, that

$$\mathbf{F}_i = m_i \cdot c_i{}''.$$

The short demonstration in Lecture 1 of conservation of momentum involved just 2 particles and only the internal forces between them, but we can easily state a generalization for an arbitrary system of particles with external forces. If we set $\mathbf{F} = \sum_i \mathbf{F}_i^e$, the total external force, then

$$\mathbf{F} = \left(\sum_i m_i \cdot \mathbf{v}_i \right)'.$$

Note that here we have to regard the various $\mathbf{F}_i^e$ simply as elements of $\mathbb{R}^3$, rather than as tangent vectors at different points of $\mathbb{R}^3$, and similarly for the $\mathbf{v}_i$.

The proof can be left as an easy exercise, but is included in a variant of the formula that is made to look a lot prettier by introducing the **center of mass** of the system $\{c_i\}$, which represents the "average" position of the particles c_i weighted according to their masses:

$$C = \frac{\sum_i m_i \cdot c_i}{\sum_i m_i}.$$

More precisely, we should define the center of mass as the particle consisting of the path C with the mass $M = \sum_i m_i$.

If all $\mathbf{F}_i^e = 0$, so that $\sum_i m_i \cdot \mathbf{v}_i$ is constant, then $C'' = \frac{1}{M}\sum_i m_i \cdot c_i{}'' = \frac{1}{M}\sum_i m_i \cdot \mathbf{v}_i{}' = \frac{1}{M}\big(\sum_i m_i \cdot \mathbf{v}_i\big)'$, so that we also have $C'' = 0$. Thus, C' is constant; in other words, the center of mass moves with uniform velocity. And more generally, If

$$\mathbf{F} = \sum_i \mathbf{F}_i^e$$

is the total external force, then

$$\mathbf{F} = M \cdot C'',$$

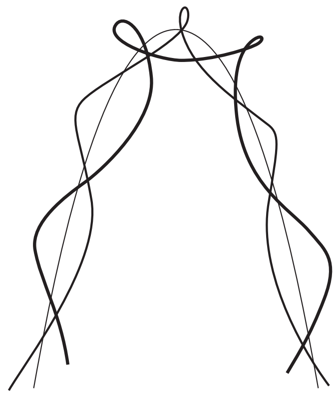
so that the center of mass particle simply moves as if it were acted upon by the total force $\mathbf{F}$.

For the proof we simply note that

$$\begin{aligned} M \cdot C'' &= \sum_i m_i \cdot c_i{}'' \\ &= \sum_i \mathbf{F}_i \\ &= \sum_i \mathbf{F}_i^e + \sum_i \sum_j \mathbf{F}_{ij} \\ &= \sum_i \mathbf{F}_i^e, \end{aligned}$$

since the double sum $\sum_{i,j} \mathbf{F}_{ij}$ vanishes.

Of course, the "particle" C might not be one of the particles in our system. Nevertheless, this result is seldom regarded as particularly "theoretical"—instead it allows us to get a very simple picture of very complex phenomena. For example, if we throw a twirling baton into the air, it will execute a rather complicated motion, but its center of mass moves in a parabola, just like a point mass. A striking illustration may be obtained with a time-exposure photograph taken when a baton is tossed in the air, with lights at the ends and the center of mass, giving a picture like this:



By the way, we usually think of a rod as a “rigid body”, and that might seem to make the above equation even more impressive: in a real rod, with all sorts of complicated intermolecular forces, which make it approximately “rigid”, but not truly so, it is still true that the center of mass moves according to a simple law. But that is a somewhat misleading way of construing the result, since rigidity of the rod is required in order to identify its center of mass with a particular point of the rod, on which we can attach one of the lights.

Newton has a statement of this result as Corollary 4 in his “Axioms” section, but he uses the older term “center of gravity” for “center of mass”.

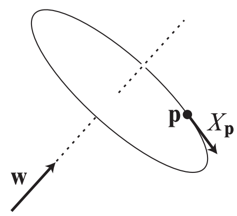
> *The common center of gravity of two or more bodies does not change its state whether of motion or of rest as a result of the actions of the bodies upon one another; and therefore the common center of gravity of all bodies acting upon one another (excluding external actions and impediments) either is at rest or moves uniformly straight forward.*

Notice that Newton explicitly disallows non-zero external forces, thereby foregoing the real interest of the corollary. Moreover, his proof is unbelievably complicated, occupying nearly two pages, referring in addition to a later Lemma that occurs in a completely different context and involves a strangely complicated figure.

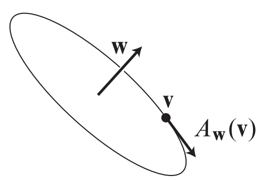
I’m not sure what this all means, except perhaps that simple vector calculus is a lot more convenient than geometry. That is certainly illustrated by a second conservation law, which occurs only implicitly in the Principia.

The cross-product $\times$, which was used at the beginning of the previous lecture basically as a convenient abbreviation for manipulations with determinants, turns out to have a fundamental role in mechanics. Of course, students always wonder why there should be a vector product in 3 dimensions, but not in other dimensions, and every professor seems to have their own answer. I’m going to give my answer now, and even if you don’t like this particular answer, at least it will introduce an important fact.

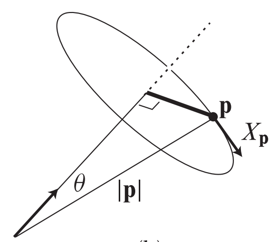
For any vector $\mathbf{w} \in \mathbb{R}^3$, consider the one-parameter family of maps $B(t)\colon \mathbb{R}^3 \to \mathbb{R}^3$, where $B(t)$ is a clockwise rotation through an angle of $t|\mathbf{w}|$ radians around the axis through $\mathbf{w}$ [choosing an orientation $(\mathbf{v}_1, \mathbf{v}_2)$ of the plane perpendicular to $\mathbf{w}$ so that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w})$ is the usual orientation of $\mathbb{R}^3$]. Now consider the vector field generated by this one-parameter family. In other words, for each $\mathbf{p} \in \mathbb{R}^3$ consider the curve $B_{\mathbf{p}}(t) = B(t)(\mathbf{p})$, and then look at the tangent vector $X_{\mathbf{p}}$ of this curve at 0.



To compute $X_{\mathbf{p}}$ geometrically, we note that $X_{\mathbf{p}}$ is clearly perpendicular to both $\mathbf{p}$ and $\mathbf{w}$. Its length is also easy to determine. When $\mathbf{p}$ happens to lie in the plane perpendicular to $\mathbf{w}$, as in (a), the point $\mathbf{p}$ rotates in a circle of



(a)



(b)

radius $|\mathbf{p}|$, and $X_{\mathbf{p}}$ has length $|\mathbf{p}| \cdot |\mathbf{w}|$. More generally (b), the point $\mathbf{p}$ rotates in a circle of radius $|\mathbf{p}| \cdot |\mathbf{w}| \cdot \sin\theta$, where θ is the angle between $\mathbf{w}$ and $\mathbf{p}$. Thus, $X_{\mathbf{p}}$ is just the geometrically defined cross-product $\mathbf{p} \times \mathbf{w}$.

For an analytic determination of $X_{\mathbf{p}} = B_{\mathbf{p}}{}'(0)$, we note that since the $B_{\mathbf{p}}(t)$ are all orthogonal, and $B_{\mathbf{p}}(0) = I$, the derivative $B_{\mathbf{p}}{}'(0)$ is skew-adjoint, with a skew-symmetric matrix M, which we will write in the form

$$M = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Then the vector $X_{\mathbf{p}}$ is the 3-tuple

$$\begin{aligned} X_{\mathbf{p}} &= (p_1, p_2, p_3) \cdot M \\ &= (p_1, p_2, p_3) \cdot \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \\ &= (p_2\omega_3 - p_3\omega_2,\ p_3\omega_1 - p_1\omega_3,\ p_1\omega_2 - p_2\omega_1). \end{aligned}$$

Setting $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, this is just the cross-product $\mathbf{p} \times \boldsymbol{\omega}$. Moreover, $\boldsymbol{\omega}$ is easy to identify, because

$$B_{\mathbf{w}}(t) = \mathbf{w} \text{ for all } t \implies 0 = X_{\mathbf{w}} = \mathbf{w} \times \boldsymbol{\omega},$$

which means that $\boldsymbol{\omega}$ is a multiple of $\mathbf{w}$, and it easy to check, by considering some specially chosen vector, that in fact $\boldsymbol{\omega} = \mathbf{w}$. Thus, one might say that the cross-product $\times$ is special to $\mathbb{R}^3$ because $n = 3$ is the only dimension where $\mathrm{O}(n)$ has dimension n. In any case, what we have shown is that

> The vector fields in $\mathbb{R}^3$ generated by rotations about an axis are of the form $\mathbf{p} \mapsto \mathbf{p} \times \mathbf{w}$ for $\mathbf{w} \in \mathbb{R}^3$.

For a particle c with velocity vector $\mathbf{v}$ we can consider the function $c \times \mathbf{v}$ from $\mathbb{R}$ to $\mathbb{R}^3$. This function is called the **angular velocity** of the particle, and if the mass of the particle is m, the cross-product $\mathbf{L} = c \times m\mathbf{v}$ is called its **angular momentum**. It would be more precise to call these quantities the angular velocity and angular momentum with respect to the origin 0. For any other point P, the angular momentum with respect to P is the cross-product

$$\mathbf{L}_P = (c - P) \times m\mathbf{v}.$$

For a system of particles $(c_1, \ldots, c_K)$ we define the angular momentum $\mathbf{L}$ of the system with respect to 0 as

$$\mathbf{L} = \sum_{i=1}^{K} c_i \times m_i \mathbf{v}_i;$$

here it is naturally necessary to consider all $c_i \times m_i\mathbf{v}_i$ as vectors at a single point, rather than as tangent vectors at different points. And, more generally, we define the angular momentum $\mathbf{L}_P$ with respect to P as

$$\mathbf{L}_P = \sum_{i=1}^{K} (c_i - P) \times m_i \mathbf{v}_i.$$

In particular, suppose we take P to be the center of mass C of the system (this means that we will be considering the angular momentum with respect to different points at different times). Letting $M = \sum_i m_i$, the "mass" of the particle C, we then have

$$\begin{aligned}
\sum_i m_i c_i \times \mathbf{v}_i &= \sum_i m_i (c_i - C) \times \mathbf{v}_i \; + \; \sum_i m_i C \times \mathbf{v}_i \\
&= \mathbf{L}_C + \left[C \times \left(\textstyle\sum_i m_i \mathbf{v}_i\right)\right] \\
&= \mathbf{L}_C + [C \times MC'],
\end{aligned}$$

so that we can write

$$\mathbf{L} = \mathbf{L}_C + (C \times MC').$$

The vector $\mathbf{L}_C$, the angular momentum with respect to the center of mass, is also called the "rotational angular momentum", so our equation says that the total angular momentum $\mathbf{L}$ is the sum of the rotational angular momentum $\mathbf{L}_C$ and the angular momentum of the center of mass with respect to 0.

If instead of a momentum vector we consider an arbitrary force $\mathbf{F}$ at a point c, the cross-product

$$\boldsymbol{\tau} = c \times \mathbf{F}$$

is called the **torque** of the force with respect to 0, while $\boldsymbol{\tau}_P = (c - P) \times \mathbf{F}$ is the torque with respect to P. (Although I have used the physicists' $\mathbf{L}$ for angular momentum, I couldn't bring myself to use the standard $\mathbf{N}$ for torque.)

Similarly, we define the torque of a system of forces on a system of particles; here it is again necessary to consider the individual torques as being vectors at one point, even though we naturally think of the forces as being applied at different points.

All these definitions finally enable us to state:

If our system satisfies the strong form of the third law, then the total torque is the derivative of the total angular momentum,

$$\boldsymbol{\tau} = \mathbf{L}'.$$

For the proof we have

$$\begin{aligned}
\mathbf{L}' = \left(\sum_i c_i \times m_i \mathbf{v}_i\right)' &= \sum_i c_i' \times m_i \mathbf{v}_i + \sum_i c_i \times m_i \mathbf{v}_i' \\
&= 0 + \sum_i c_i \times \mathbf{F}_i \\
&= \sum_i c_i \times \mathbf{F}_i^e + \sum_i \sum_j c_i \times \mathbf{F}_{ij} \\
&= \boldsymbol{\tau} + \sum_i \sum_j c_i \times \mathbf{F}_{ij}.
\end{aligned}$$

The strong form of the third law allows us to write

$$\mathbf{F}_{ij} = \lambda_{ij}(c_i - c_j),$$

with $\lambda_{ij} = \lambda_{ji}$ since $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, so we have

$$\begin{aligned}
\sum_i \sum_j c_i \times \mathbf{F}_{ij} &= \sum_i \sum_j \lambda_{ij}[c_i \times c_i \; - \; c_i \times c_j] \\
&= -\sum_{i,j} \lambda_{ij}[c_i \times c_j].
\end{aligned}$$

This double sum vanishes, since $\lambda_{ij} = \lambda_{ji}$, while $c_i \times c_j = -c_j \times c_i$.

An easy calculation then shows that, more generally, for any point P,

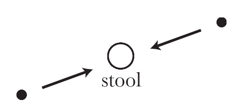
$$\boldsymbol{\tau}_P = \mathbf{L}_P{}'.$$

In particular, of course, if the torque is 0, then angular momentum is conserved. This certainly happens in the special case of a single particle moving under a *central force*, where the external force $\mathbf{F}$ is a multiple of c, so that $\boldsymbol{\tau} = c \times \mathbf{F} = 0$. This was noted by Newton as the first Corollary of his Proposition 1 in the previous lecture.

> Corollary 1. In nonresisting spaces, the velocity of a body attracted to an immobile center is inversely as the perpendicular dropped from that center to the straight line which is tangent to the orbit.

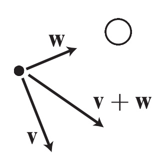
Even the somewhat more general rule that angular momentum is conserved in the absence of external forces was not stated until quite some time afterwards, and this law was known for a long time simply as "the law of areas", or *Flächensatz* in German.

A standard elementary illustration of the law of conservation of angular momentum is provided by a person seated on a rotating stool with arms extended out holding weights, and then increasing the speed of the spin, often quite dra-



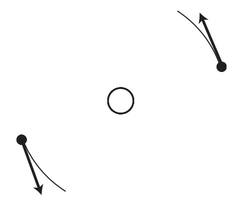
matically, simply by pulling the weights inward. Similarly, ice-skaters speed up their turns by pulling their arms in; divers, starting their dive with a small angular momentum, do rapid somersaults by pulling their arms and knees in; and gymnasts do all sorts of tricks.

By the way, without appealing to conservation of angular momentum we can explain the speed-up as a simple consequence of the parallelogram rule for forces, or even for velocities: the sum of the velocity $\mathbf{v}$ that the weight already has and the velocity $\mathbf{w}$ that it acquires as a result of the inward pull lies along the



diagonal of the rectangle spanned by these two, and consequently has a greater length.

In these examples, we merely altered the given non-zero angular momentum, but something interesting occurs even when we start with angular momentum 0. Moving the weights along a circle in one direction contributes a certain amount



of angular momentum to the system of weights-plus-person, which must be countered by an opposite amount of angular momentum in the system, so the seated person must rotate in the opposite direction. At the end of the motion, when the weights are no longer being rotated, the person will have stopped rotating, but will be facing in a different direction; cats use this mechanism to land on their paws even when dropped from an upside-down position.

In this respect, rotation is quite different from linear motion. A system cannot change its position using only internal forces, and no external forces. On a perfectly frictionless ice surface you can change the direction in which you are facing, but you can't move the position of your center of mass. (Of course, you can forcefully exhale, providing yourself with rocket propulsion, making use of the fact that the air inside your lungs is a part of your system that you aren't attached to; or you could just throw your coat away.)

Conservation of momentum and angular momentum are the two great conservation laws of mechanics. Of course, there's also Conservation of Energy, which receives much greater star-billing in modern physics textbooks. But the general principle of conservation of energy involves much more than mechanics. There is still an important conservation of energy principle for mechanics itself, involving "conservative" forces, but I will skip over that in these lectures, since our main aim is to see how the notion of rigid bodies can be handled, and to investigate other developments to which this treatment leads.

LECTURE 5

RIGID BODIES

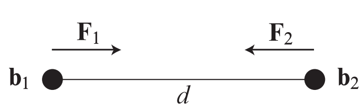
It's finally time to address the problem with which these lectures began: How do we analyze rigid bodies. Since Newton's laws are basically about "particles", it is natural for us to think of a rigid body as a large system of particles, especially since this corresponds to our modern ideas involving atoms. But the analysis is hardly straightforward, because rigid bodies are idealizations, in fact idealizations that don't seem to have any theoretical counterpart.

For a system of particles to constitute a rigid body, the distances between them would have to remain constant, no matter what forces are applied. In special relativity theory this would be impossible in principle: it would mean that applying a force at one end of a rigid body would cause the other end of the rigid body to move *immediately*, so that information would be sent from one of the body to the other instantaneously.

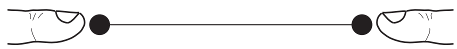
Even in classical mechanics, there is no way for a system of particles to act as a rigid body. The internal forces between two particles of a rigid body must be 0 when they are at their original distance d apart, but become strongly repulsive if the distance is slightly smaller than d and strongly attractive if the distance is slightly larger than d. If two opposing forces are applied to the particles at the end of a rigid body, they will initially push them slightly toward each other, producing a strong repulsive force, which will not only return the particles to their original position, but actually cause them to move slightly further apart; this, in turn, will produce large attractive forces, now moving the particles back toward their initial separation, and slightly beyond, causing the repulsive forces to act again. Thus, we would expect the particles to vibrate around their original separation, which is more or less what actually happens in a real-world rigid rod made of molecules.

Nevertheless, it turns out that we can still ask when a system of particles is acting in a way that we would expect a rigid body to act.

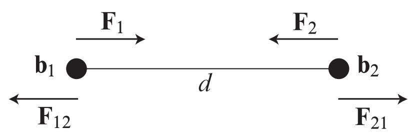
The simplest question we can ask is when a system of particles should be considered to be in equilibrium under a set of external forces. As the very simplest possible example, let's consider a "rigid rod" that consists of just two points $\mathbf{b}_1$ and $\mathbf{b}_2$ (representing two molecules, say) at a distance d apart, and "external" forces $\mathbf{F}_i$ acting on $\mathbf{b}_i$. These forces might be produced, for example,



by some one exerting equal but opposite pressure on both sides of this rod.



If $\mathbf{F}_1 = -\mathbf{F}_2$, then we would expect this rigid rod to be in equilibrium under these forces, and we can justify this expectation by noting that if we consider a force $\mathbf{F}_{21}$ on $\mathbf{b}_2$ equal to $-\mathbf{F}_2$ and a force $\mathbf{F}_{12}$ on $\mathbf{b}_1$ equal to $-\mathbf{F}_1$, then these "internal" forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ do satisfy Newton's third law, and together with the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ they leave our rod, consisting of $\mathbf{b}_1$ and $\mathbf{b}_2$, in equilibrium.



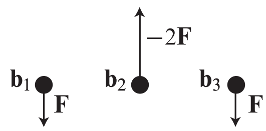
To be sure, this picture becomes quite a bit hazier if we try to imagine how these "internal" forces would arise as the forces $\mathbf{F}_i$ are applied. This would lead us right back to the "realistic" picture of rapidly vibrating molecules. And it won't help to consider the limiting situation as the constraining forces of the molecule are made greater and greater, because this simply causes the molecules to vibrate more and more rapidly—although they will stay closer and closer to their natural separation, their motions will not approach a limit.

So instead, we will consider our abstract rigid rod to be in equilibrium *simply because such forces* $\mathbf{F}_{ij}$ *can be defined*, without worrying about the details of just how these forces would actually arise in practice, for rods that aren't ideally rigid.

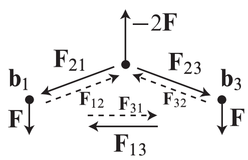
More generally, let us consider a collection of points $\mathbf{b}_1, \dots, \mathbf{b}_K$, which it will sometimes be convenient to regard as a single object, $\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_K)$, as well as a collection of forces $\mathbf{F} = (\mathbf{F}_1, \dots \mathbf{F}_K)$, where we regard $\mathbf{F}_i$ as acting on $\mathbf{b}_i$. Then we can make the following definition:

> The collection of points $\mathbf{b}$ is in **rigid equilibrium** under the forces $\mathbf{F}$ if there exist "internal" forces $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ which are multiples of $\mathbf{b}_i - \mathbf{b}_j$ such that
>
> $$\mathbf{F}_i = -\sum_j \mathbf{F}_{ij}.$$
>
> Much more colloquially, of course, we just say that "the rigid body $\mathbf{b}$ is in equilibrium under the forces $\mathbf{F}$".

So now we have a perfectly meaningful definition, which doesn't require the notion of a rigid body directly. There's only one problem with this definition: it doesn't work. For example, we presumably ought to have equilibrium for the rod shown below, where there are equal forces $\mathbf{F}$ at the ends of the rod together



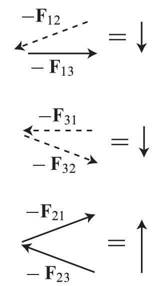
with a force $-2\mathbf{F}$ in the middle. But these forces obviously can't be balanced by forces that are multiples of the vectors $\mathbf{b}_i - \mathbf{b}_j$. Of course, in practice, the rod will bend a bit, and in this situation the necessary "internal" forces will exist.



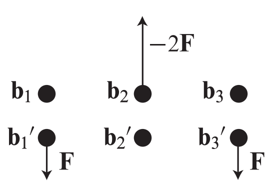
$$\mathbf{F} = -\mathbf{F}_{12} - \mathbf{F}_{13}$$

$$\mathbf{F} = -\mathbf{F}_{31} - \mathbf{F}_{32}$$

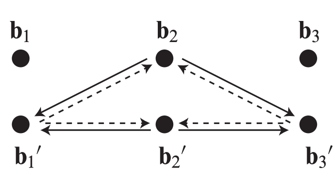
$$-2\mathbf{F} = -\mathbf{F}_{21} - \mathbf{F}_{23}$$



Fortunately, we can stick with our strict theoretical model if we represent the situation by a slightly more realistic figure, with a few extra "molecules", so that

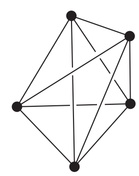


once again the required internal forces will exist.

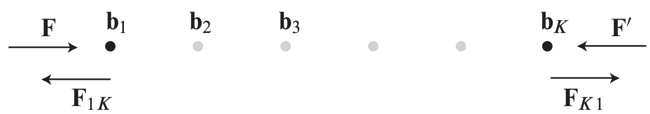


We will normally presume that our particles do not lie on a straight line, or even on a plane, and in realistic situations the number of particles should be much greater, although special cases may be useful for illustration.

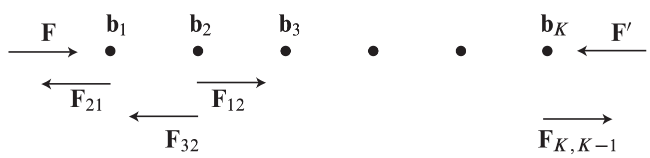
It should be pointed out that these internal forces will almost never be unique. They will be unique if our system consists of just 4 points $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ not on a plane, but as soon as we have more points there will be many possible choices.



In fact the internal forces aren't unique even for the special case of a "rigid rod" consisting of particles $\mathbf{b}_1, \dots, \mathbf{b}_K$ lying on a straight line. Given equal and opposite forces $\mathbf{F}$ and $\mathbf{F}'$ on the ends $\mathbf{b}_1$ and $\mathbf{b}_K$, we could choose just two



forces $\mathbf{F}_{1K}$ and $\mathbf{F}_{K1}$ between $\mathbf{b}_1$ and $\mathbf{b}_K$, essentially ignoring all the particles between them, but it would be more natural to balance $\mathbf{F}$ with a force $\mathbf{F}_{21}$



exerted on $\mathbf{b}_1$ by $\mathbf{b}_2$, requiring an equal but opposite force $\mathbf{F}_{12}$ on $\mathbf{b}_2$, which would in turn be balanced by a force $\mathbf{F}_{32}$ exerted on $\mathbf{b}_2$ by $\mathbf{b}_3$, ... , leading finally to a force $\mathbf{F}_{K,K-1}$ exerted on $\mathbf{b}_K$ by $\mathbf{b}_{K-1}$ that balances $\mathbf{F}'$.

This corresponds much better to our intuitive idea of a rigid rod: the molecules at one end merely influence the nearby molecules, making a tiny sliver of the rod rigid; those molecules in turn influence molecules a little bit further away, making a little bit more of the rod rigid, etc.

Now mathematicians know that it's always rather hard to work with non-unique data, so one always looks for an alternative way of handling such problems. In the case of rigidity this is obtained by considering "rigid motions" of $\mathbf{b}$. By this we simply mean a collection of paths $\mathbf{c} = (c_1, \dots, c_K)$ with $c_i(0) = \mathbf{b}_i$ such that each

$$|c_i(t) - c_j(t)|^2 = \langle c_i(t) - c_j(t),\, c_i(t) - c_j(t)\rangle \quad \text{is constant.}$$

Alternatively, we might think of a rigid motion as a curve $t \mapsto A(t)$ of isometries of $\mathbb{R}^3$, with $c_i(t) = A(t)(c_i(0)) = A(t)(\mathbf{b}_i)$.

Given such a rigid motion, consider the K-tuple of tangent vectors

$$\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_K) = ({c_1}'(0), \dots, {c_K}'(0)) \in (\mathbb{R}^3)^K.$$

Differentiating the equation

$$\langle c_i(t) - c_j(t),\ c_i(t) - c_j(t)\rangle = \text{constant}$$

and evaluating at 0 gives us

$$\langle \mathbf{v}_i - \mathbf{v}_j,\ \mathbf{b}_i - \mathbf{b}_j\rangle = 0. \tag{1}$$

Since the force $\mathbf{F}_{ij}$ is a multiple of $\mathbf{b}_i - \mathbf{b}_j$, this implies that

$$\langle \mathbf{v}_i - \mathbf{v}_j,\ \mathbf{F}_{ij}\rangle = 0.$$

Consequently,

$$\begin{aligned}\sum_{i,j}\langle \mathbf{v}_i, \mathbf{F}_{ij}\rangle = \sum_{i,j}\langle \mathbf{v}_j, \mathbf{F}_{ij}\rangle &= -\sum_{i,j}\langle \mathbf{v}_j, \mathbf{F}_{ji}\rangle \\ &= -\sum_{i,j}\langle \mathbf{v}_i, \mathbf{F}_{ij}\rangle \quad \text{(interchanging } i \text{ and } j\text{)}\end{aligned}$$

and thus

$$\sum_{i,j}\langle \mathbf{v}_i, \mathbf{F}_{ij}\rangle = 0.$$

This in turn means that the external forces $\mathbf{F}_k$ in the condition for rigid equilibrium satisfy

$$\textstyle\sum_k \langle \mathbf{v}_k, \mathbf{F}_k\rangle = -\sum_{k,j}\langle \mathbf{v}_k, \mathbf{F}_{kj}\rangle = 0,$$

or simply

$$\sum_k \langle \mathbf{v}_k, \mathbf{F}_k\rangle = 0. \tag{$*$}$$

Physicists refer to these K-tuples $\mathbf{v} = ({c_1}'(0), \dots, {c_K}'(0))$ for rigid motions $\mathbf{c}$ of $\mathbf{b}$ as "virtual infinitesimal displacements" of $\mathbf{b}$. The word "infinitesimal" in this phrase shouldn't surprise us—it's just the standard physicists' way of referring to tangent vectors. As for the word "virtual" here, it has about as much meaning as it does in the phrase "virtual reality". Basically it refers to the fact that although we have obtained equation $(*)$ under the assumption that our rigid body is in equilibrium, we have done so by considering tangent vectors to "virtual" rigid motions, i.e., motions that our rigid body might have had if it *weren't* in equilibrium.

This can all be expressed in a more familiar, geometric, way by considering the "configuration space" of $\mathbf{b}$, which is the subset $\mathcal{M} \subset (\mathbb{R}^3)^K$ of all points that can be reached from $\mathbf{b}$ at the end of a rigid motion. In other words,

$$\mathcal{M} = \{ (A(\mathbf{b}_1), \dots, A(\mathbf{b}_K)) : A \text{ an orientation preserving isometry of } \mathbb{R}^3 \}.$$

When $\mathbf{b}$ is non-planar, $\mathcal{M}$ is a 6-dimensional manifold diffeomorphic to the set of all orientation preserving isometries A of $\mathbb{R}^3$, and thus to $\mathbb{R}^3 \times \mathrm{SO}(3)$. With this picture, a rigid motion of $\mathbf{b}$ is simply a curve in $\mathcal{M}$, so a virtual infinitesimal displacement $\mathbf{v}$ of $\mathbf{b}$ is simply a tangent vector to $\mathcal{M}$ at $\mathbf{b}$.

We've already found that any such $\mathbf{v}$ satisfies the equation

$$\langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{b}_i - \mathbf{b}_j \rangle = 0. \tag{1}$$

If we define linear functions ϕ_{ij} on $(\mathbb{R}^3)^K$ by

$$\phi_{ij}(\mathbf{v}_1, \dots, \mathbf{v}_K) = \langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{b}_i - \mathbf{b}_j \rangle,$$

this says that

$$\mathcal{M}_{\mathbf{b}} \subset \bigcap_{i,j} \ker \phi_{ij}.$$

At this point we want to state a simple, but absolutely crucial, little lemma:

Lemma. If $\mathbf{b}$ is non-planar, then

$$\mathcal{M}_{\mathbf{b}} = \bigcap_{i,j} \ker \phi_{ij}.$$

Proof. By renumbering, we can assume that $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ are points of $\mathbf{b}$ that do not lie in a plane. There is clearly no loss of generality in assuming that $\mathbf{b}_1 = 0$ [as reflected by the fact that we can replace all $\mathbf{b}_i$ by $\mathbf{b}_i - \mathbf{b}_1$ without changing (1)]. Thus our assumption on $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ amounts to $\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ being linearly independent.

Since we can also replace all $\mathbf{v}_i$ by $\mathbf{v}_i - \mathbf{v}_1$ without changing (1), it follows that

$$\dim\Big(\bigcap_{i,j} \ker \phi_{ij}\Big) = 3 + \dim\Big(\Big\{(0, \mathbf{v}_2, \dots, \mathbf{v}_K) \in \bigcap_{i,j} \ker \phi_{ij}\Big\}\Big).$$

Now for $\mathbf{v}$ with $\mathbf{v}_1 = 0$, a first application of (1) gives

$$\langle \mathbf{v}_i, \mathbf{b}_i \rangle = \langle \mathbf{v}_i - \mathbf{v}_1, \mathbf{b}_i - \mathbf{b}_1 \rangle = 0 \qquad i = 2, 3, 4,$$

and then a second application gives

$$-\langle \mathbf{v}_i, \mathbf{b}_j \rangle = \langle \mathbf{v}_j, \mathbf{b}_i \rangle \qquad i = 2, 3, 4.$$

So if $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation with

$$\mathbf{v}_i = A\mathbf{b}_i \qquad i = 2, 3, 4,$$

then A is skew-adjoint. But the dimension of skew-symmetric 3×3 matrices is 3, so the dimension of $\bigcap_{i,j} \ker \phi_{ij}$ is at most 6, which is the dimension of $\mathcal{M}_{\mathbf{b}}$.

By the way, it shouldn't be too surprising that the mechanics of this proof involved skew-adjoint transformations, since they are the derivatives of orthogonal ones (as we noted in our discussion of the cross-product); given the transformation A of the proof, the isometries e^{tA} would produce the given infinitesimal virtual displacement $\mathbf{v}$.

If we use $\langle\ ,\ \rangle$ for the usual inner product on $(\mathbb{R}^3)^K$, then equation $(*)$ can be written in the simple form

$$\langle \mathbf{v}, \mathbf{F}\rangle = 0;$$

in other words, $\mathbf{F}$ is perpendicular to the tangent space $\mathcal{M}_{\mathbf{b}}$. The inner product of force and distance is generally called work, so this sum is also called the "(virtual) infinitesimal work" done by the forces $\mathbf{F}$ during the (virtual) infinitesimal displacement $\mathbf{v}$. Our little calculation that $\langle \mathbf{v}, \mathbf{F}\rangle = 0$ if $\mathbf{b}$ is in rigid equilibrium under $\mathbf{F}$ is often referred to by physicists as a proof of the "principle of virtual work". In reality, however, when physicists use the principle of virtual work they almost always assume implicitly that it includes the *converse*:

The Principle of Virtual Work. The virtual infinitesimal work $\langle \mathbf{v}, \mathbf{F}\rangle$ equals 0 for all virtual infinitesimal displacements $\mathbf{v}$ of (a non-planar) $\mathbf{b}$ if and only if $\mathbf{b}$ is in rigid equilibrium under $\mathbf{F}$.

Proof. We only have to prove the converse part, that if $\langle \mathbf{v}, \mathbf{F}\rangle = 0$ for all $\mathbf{v}$, then $\mathbf{b}$ is in rigid equilibrium under $\mathbf{F}$. If we consider the linear function Φ on $(\mathbb{R}^3)^K$ defined by

$$\Phi(\mathbf{v}_1, \ldots, \mathbf{v}_K) = \sum_k \langle \mathbf{v}_k, \mathbf{F}_k\rangle,$$

then our hypothesis says that Φ vanishes on $\mathcal{M}_{\mathbf{b}}$, and thus by our lemma,

$$\Phi \text{ vanishes on } \bigcap_{i,j} \ker \phi_{ij}, \qquad\qquad \phi_{ij}(\mathbf{v}_1, \ldots, \mathbf{v}_K) = \langle \mathbf{v}_i - \mathbf{v}_j,\, \mathbf{b}_i - \mathbf{b}_j\rangle.$$

It is a well-known simple result about vector spaces (not necessarily even finite dimensional ones) that in this case there exist constants λ_{ij} with

$$\Phi = \sum_{i,j} \lambda_{ij} \cdot \phi_{ij}.$$

In other words,

$$\sum_k \langle \mathbf{v}_k, \mathbf{F}_k \rangle = \sum_{i,j} \lambda_{ij} \langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{b}_i - \mathbf{b}_j \rangle \qquad \text{all } \mathbf{v}_1, \dots, \mathbf{v}_K \in (\mathbb{R}^3)^K.$$

Choosing all $\mathbf{v}_i$ to be 0 except for the one vector $\mathbf{v}_l$, we thus obtain

$$\begin{aligned}
\langle \mathbf{v}_l, \mathbf{F}_l \rangle &= \sum_j \lambda_{lj} \langle \mathbf{v}_l, \mathbf{b}_l - \mathbf{b}_j \rangle + \sum_i \lambda_{il} \langle -\mathbf{v}_l, \mathbf{b}_i - \mathbf{b}_l \rangle \\
&= \sum_j \lambda_{lj} \langle \mathbf{v}_l, \mathbf{b}_l - \mathbf{b}_j \rangle + \sum_j \lambda_{jl} \langle \mathbf{v}_l, \mathbf{b}_l - \mathbf{b}_j \rangle \\
&= \sum_j (\lambda_{lj} + \lambda_{jl}) \langle \mathbf{v}_l, \mathbf{b}_l - \mathbf{b}_j \rangle \\
&= \Big\langle \mathbf{v}_l, \ \sum_j (\lambda_{lj} + \lambda_{jl})(\mathbf{b}_l - \mathbf{b}_j) \Big\rangle,
\end{aligned}$$

and since this is true for arbitrary vectors $\mathbf{v}_l$ in $\mathbb{R}^3$, we conclude that

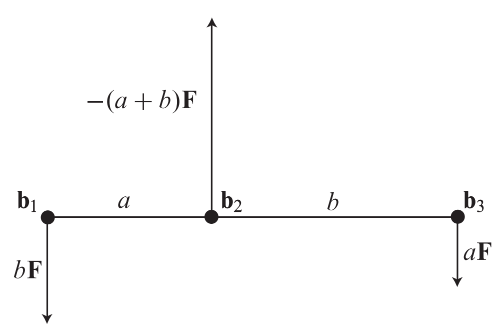
$$\mathbf{F}_l = \sum_j (\lambda_{lj} + \lambda_{jl})(\mathbf{b}_l - \mathbf{b}_j).$$

So we can define

$$\mathbf{F}_{jl} = -(\lambda_{lj} + \lambda_{jl})(\mathbf{b}_l - \mathbf{b}_j)$$

to obtain the required forces.

As an extremely simple example, consider the situation shown below, where the upward force on $\mathbf{b}_2$ balances the two downward forces at points $\mathbf{b}_1$ and $\mathbf{b}_3$, which are at different distances a and b from $\mathbf{b}_2$, with the magnitudes of these forces inversely proportional to those distances. Of course, this is merely a



schematic figure, since it is linear, and we really have to assume that there are other points around, as in our previous examples.

It is easy to see that this collection of points is in rigid equilibrium under these forces:

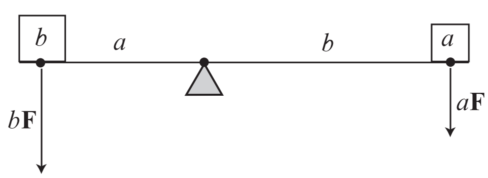
(1) For an infinitesimal displacement given by a vector $\mathbf{z}$ pointing in the vertical direction, the virtual infinitesimal work is 0 because the upward force is the negative of the sum of the two downward forces.

(2) For an infinitesimal displacement given by a vector $\mathbf{z}$ pointing in the horizontal direction, the virtual infinitesimal work is 0 because each individual component is 0.

(3) For an infinitesimal displacement generated by a rotation around $\mathbf{b}_2$ (i.e., around an axis through $\mathbf{b}_2$ perpendicular to the plane of the diagram), the vectors $\mathbf{v}_1$ and $\mathbf{v}_3$ will be in opposite vertical directions, with lengths proportional to the distances a and b. Consequently, the virtual infinitesimal work, involving vectors with length *inversely* proportional to these distances, will be 0. (One can check directly that this is just as true for an infinitesimal displacement generated by a rotation around either $\mathbf{b}_1$ or $\mathbf{b}_3$, but that isn't necessary, since the set of virtual infinitesimal displacements that stay in the plane of the diagram has dimension 3.)

(4) For infinitesimal displacements given by a vector perpendicular to the plane of the figure, or by a rotation through axes perpendicular to our first rotation, the virtual infinitesimal work also works out to be 0; or we can just simplify matters by restricting our attention to the 2-dimensional situation to begin with.

Notice that this provides a fairly good schematic representation of a lever, which of course requires not only a rigid body, but also a *fulcrum*, an immovable point. In practice, this "immobility" is provided in a complicated way by the



connections between the fulcrum and the earth, but it seems reasonable simply to regard this connection as a mechanism that automatically supplies the proper *upward* force to the fulcrum when the downward forces are applied at the ends of the lever.

Naturally, a more realistic picture would use a much large number of points, forming a 3-dimensional object. But in any case, our analysis shows, especially when we think of the lever as bending slightly, that it is the internal forces of the lever that make the weights balance; in short, all the "extra force" that one obtains by pushing at a large distance from the fulcrum is supplied by the lever itself, in its effort to preserve rigidity (together with the force that the earth supplies on the fulcrum, to keep it from moving downward).

Now we are ready to apply the same ideas in the more general case. Instead of looking for a condition for equilibrium, we now seek a criterion for a rigid motion $\mathbf{c} = (c_1, \dots, c_K)$ of $\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_K)$ to be consistent with the forces $\mathbf{F} = (\mathbf{F}_1, \dots \mathbf{F}_K)$. We will think of the $\mathbf{F}_i$ as functions on $\mathcal{M} \times \mathbb{R}$, so the forces may depend not only on time, but also on the particular rigid motion that the body has undergone at any particular time.

We now need "internal" forces $\mathbf{F}_{ij}(t) = -\mathbf{F}_{ji}(t)$ with $\mathbf{F}_{ij}(t)$ a multiple of $c_i(t) - c_j(t)$ so that

$$m_i c_i''(t) = \mathbf{F}_i(\mathbf{c}(t), t) + \textstyle\sum_j \mathbf{F}_{ij}(t)$$

or

$$m_i c_i''(t) - \mathbf{F}_i(\mathbf{c}(t), t) = \textstyle\sum_j \mathbf{F}_{ij}(t).$$

The latter equation, which may be regarded as stating that the body is in rigid equilibrium under the forces $\mathbf{F}_i - m_i c_i''$, is often called "d'Alembert's principle" and regarded as the fundamental law—so that, as the physicists like to say, "dynamics reduces to statics" (in the words of [Go], Chapter 1.4). But this really becomes useful only when we apply the principle of virtual work: The conditions on the $\mathbf{F}_{ij}$ imply that

$$\textstyle\sum_i \langle \mathbf{v}_i,\, m_i c_i''(t) - \mathbf{F}_i(\mathbf{c}(t), t)\rangle = 0$$

for all tangent vectors $\mathbf{v}$, and, conversely, the principle of virtual work implies that if this condition holds, then the requisite $\mathbf{F}_{ij}(t)$ exist. This leads us to the following definition:

> **d'Alembert's Principle:** The rigid motion $\mathbf{c}$ is a **rigid solution** for the forces $\mathbf{F}$, or, more colloquially, "$\mathbf{c}$ is a possible motion of the rigid body $\mathbf{b}$ under the forces $\mathbf{F}$", if for each t,
>
> $$\sum_i \langle \mathbf{F}_i(\mathbf{c}(t), t) - m_i c_i''(t),\ \mathbf{v}_i\rangle = 0$$
>
> for all tangent vectors $\mathbf{v}$ at $\mathcal{M}_{\mathbf{c}(t)}$.

If we agree to let $m\mathbf{c}$ denote $(m_1 c_1, \dots, m_K c_K)$ and similarly for $m\mathbf{c}''$, and also let $\langle\ ,\ \rangle$ denote the usual inner product on $(\mathbb{R}^3)^K$, then we can write

$$\langle \mathbf{F}(\mathbf{c}(t), t) - m\mathbf{c}''(t),\ \mathbf{v}\rangle = 0 \qquad \text{for all } \mathbf{v} \text{ tangent to } \mathcal{M}_{\mathbf{c}(t)},$$

or, if we are willing to tolerate a little ambiguity in our notation, simply

$$(**) \qquad \langle \mathbf{F} - m\mathbf{c}'',\ \mathbf{v}\rangle = 0 \qquad \text{for all } \mathbf{v} \text{ tangent to } \mathcal{M}_{\mathbf{c}}.$$

Our condition amounts to a system of second order differential equations for vector-valued functions: If we choose a local coordinate system $x^1, \dots, x^6$ on the 6-dimensional manifold $\mathcal{M}$, then we only have to verify $(**)$ for $\mathbf{v} = \partial/\partial x^i$, giving us 6 equations for the vector-valued functions $\mathbf{c}^i = x^i \circ \mathbf{c}$.

Since $\mathcal{M}$ is basically $\mathbb{R}^3 \times \mathrm{SO}(3)$, we can restate this much more concretely. One 3-dimensional collection of vector fields tangent to $\mathcal{M}$ are those of the form $\mathbf{v}_i = \mathbf{z}$ for a constant vector $\mathbf{z}$. Condition $(**)$ becomes

$$\begin{aligned} 0 &= \sum_i \langle \mathbf{F}_i - m_i c_i'', \mathbf{z}\rangle \\ &= \left\langle \textstyle\sum_i \mathbf{F}_i, \mathbf{z}\right\rangle - \left\langle \textstyle\sum_i m_i c_i'', \mathbf{z}\right\rangle. \end{aligned}$$

Since this must hold for all $\mathbf{z}$, we must have

$$(\mathbf{F}_{\mathrm{rigid}}) \qquad \mathbf{F}_{\mathrm{total}} = \sum_i \mathbf{F}_i = \sum_i m_i c_i'' = MC'',$$

where C is the center of mass, and $M = \sum_i m_i$ is the total mass [recall that $\mathbf{F}_i$ really stands for $t \mapsto \mathbf{F}_i(\mathbf{c}(t), t)$].

We also have to consider the vector fields generated by rotations. As we saw in our discussion of the $\times$-product, these are of the form $\mathbf{v}_i = c_i \times \boldsymbol{\eta}$. Thus, condition $(**)$ becomes

$$\begin{aligned} 0 &= \sum_i \langle \mathbf{F}_i - m_i c_i'', c_i \times \boldsymbol{\eta}\rangle \\ &= \sum_i \langle \mathbf{F}_i, c_i \times \boldsymbol{\eta}\rangle - \sum_i m_i \langle c_i'', c_i \times \boldsymbol{\eta}\rangle \\ &= \sum_i \langle c_i \times \mathbf{F}_i, \boldsymbol{\eta}\rangle - \sum_i m_i \langle c_i \times c_i'', \boldsymbol{\eta}\rangle. \end{aligned}$$

Since this must hold for all $\boldsymbol{\eta}$, we must have [with $\mathbf{F}_i$ standing for $t \mapsto \mathbf{F}_i(\mathbf{c}(t), t)$]

$$(\boldsymbol{\tau}_{\mathrm{rigid}}) \qquad \boldsymbol{\tau} = \sum_i c_i \times \mathbf{F}_i = \sum_i m_i c_i \times c_i''.$$

Condition $(\mathbf{F}_{\mathrm{rigid}})$ simply says that the rigid body must move in such a way that the momentum law is satisfied, while condition $(\boldsymbol{\tau}_{\mathrm{rigid}})$ simply says that the rigid body must move in such a way that the angular momentum law is satisfied.

To solve these equations we might begin by writing our rigid motion $\mathbf{c} = (c_1, \dots, c_K)$ of $\mathbf{b}$ in the form

$$\begin{aligned} c_i(t) &= B^{-1}(t)(\mathbf{b}_i) + \mathbf{w}(t) \\ &= B^{\mathbf{t}}(t)(\mathbf{b}_i) + \mathbf{w}(t) \end{aligned}$$

for orthogonal $B(t)$; we've used B^{-1} instead of B to conform to the physicists' convention that $B(t)$ is the rotation that will return the body at time t to its original unrotated position. Since $B^{\mathbf{t}}B = I$, we have

$$\begin{aligned} B^{\mathbf{t}\prime}B &= -B^{\mathbf{t}}B' \\ &= -(B'^{\mathbf{t}}B)^{\mathbf{t}} \\ &= -(B^{\mathbf{t}\prime}B)^{\mathbf{t}}, \end{aligned}$$

so $(B^{\mathbf{t}\prime}B)(t)$ is skew-adjoint, and its matrix can be written as

$$\begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix}.$$

Setting $\boldsymbol{\omega}(t) = \big(\omega_1(t), \omega_2(t), \omega_3(t)\big)$, we then have

$$\begin{aligned} \mathbf{v}_i(t) = c_i{}'(t) &= B^{\mathbf{t}\prime}(t)(\mathbf{b}_i) + \mathbf{w}'(t) \\ &= \big(B^{\mathbf{t}\prime}(t)B(t)\big)\big(B^{-1}(t)(\mathbf{b}_i)\big) + \mathbf{w}'(t) \\ &= (B^{\mathbf{t}\prime}(t)B(t))(c_i(t)) + \mathbf{w}'(t) \\ &= c_i(t) \cdot \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix} + \mathbf{w}'(t) \\ &= [\boldsymbol{\omega}(t) \times c_i(t)] + \mathbf{w}'(t). \end{aligned}$$

There is a natural choice for $B(t)$ and $\mathbf{w}(t)$ in our description of rigid body motion: choose $\mathbf{w}(t) = C(t)$, where $C(t)$ is the center of mass at time t, so that $B(t)$ represents the rotation about the center of mass required to move the body back to its initial unrotated position. We then have

$$\mathbf{v}_i(t) = [\boldsymbol{\omega}(t) \times c_i(t)] + C'.$$

We can now write the angular momentum $\mathbf{A}$ of $\mathbf{c}$ as

$$\begin{aligned} \sum_i m_i c_i \times \mathbf{v}_i &= \sum_i m_i c_i \big[(\boldsymbol{\omega} \times c_i) + C'\big] \\ &= \sum_i \big[m_i c_i \times (\boldsymbol{\omega} \times c_i)\big] + \left(\sum_i m_i c_i\right) \times C' \\ &= \sum_i \big[m_i c_i \times (\boldsymbol{\omega} \times c_i)\big] + \big(MC \times C'\big), \qquad M = \textstyle\sum_i m_i. \end{aligned}$$

Comparing this with the formula on page 51, we see that the quantity

$$\sum_i m_i c_i \times (\boldsymbol{\omega} \times c_i)$$

is the same as the "rotational angular momentum", that is, the angular momentum of $\mathbf{c}$ around its center of mass.

For the moment we only want to consider the basic aspects of solving equations ($\mathbf{F}_{\text{rigid}}$) and ($\boldsymbol{\tau}_{\text{rigid}}$). To simplify matters, we'll ignore the motion of the center of mass, and just look for a solution of the form $c_i = B^{\mathbf{t}}(\mathbf{b}_i)$, essentially describing how the body rotates about the center of mass.

Thus, we are looking for $\boldsymbol{\omega}$ so that ($\boldsymbol{\tau}_{\text{rigid}}$) holds when we have

$$c_i{}' = \mathbf{v}_i = \boldsymbol{\omega} \times c_i.$$

We have

$$\begin{aligned} c_i{}'' &= (\boldsymbol{\omega}' \times c_i) + (\boldsymbol{\omega} \times \mathbf{c}_i{}') \\ &= (\boldsymbol{\omega}' \times c_i) + (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times c_i)), \end{aligned}$$

so equation ($\boldsymbol{\tau}_{\text{rigid}}$) becomes

$$\boldsymbol{\tau} = \sum_i m_i c_i \times (\boldsymbol{\omega}' \times c_i) \; + \; \sum_i m_i (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times c_i)).$$

In terms of the linear function $\mathbf{I}_c \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{I}_c(\boldsymbol{\eta}) = \sum_i m_i c_i \times (\boldsymbol{\eta} \times c_i),$$

we can write this as

$$(*) \qquad \mathbf{I}_c(\boldsymbol{\omega}') = \boldsymbol{\tau} \; - \; \sum_i m_i (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times c_i)),$$

where the right side depends on $c \in \mathcal{M}$ and $\boldsymbol{\omega}$. The only thing we need to check is that we can always solve this for some $\boldsymbol{\omega}'$ as a function of c and $\boldsymbol{\omega}$, thereby obtaining a system of first order equations for $\boldsymbol{\omega}'$, and thus a system of second order equations for the elements of B. In other words, we need to know that the linear transformation $\mathbf{I}_c$ is an isomorphism for all $c \in \mathcal{M}$.

Since we only have to consider $c \in \mathcal{M}$ of the form $c_i = P(\mathbf{b}_i)$ for some orthogonal P, we have

$$\begin{aligned} \mathbf{I}_c(\boldsymbol{\eta}) &= \sum_i m_i P(\mathbf{b}_i) \times (\boldsymbol{\eta} \times P(\mathbf{b}_i)) \\ &= P\Big(\textstyle\sum_i m_i \mathbf{b}_i \times (P^{-1}(\boldsymbol{\eta}) \times \mathbf{b}_i)\Big) \\ &= (P \mathbf{I}_{\mathbf{b}} P^{-1})(\boldsymbol{\eta}). \end{aligned}$$

Thus we only have to check that $\mathbf{I} = \mathbf{I}_{\mathbf{b}}$ is an isomorphism, where

$$\mathbf{I}(\boldsymbol{\phi}) = \sum_i m_i \mathbf{b}_i \times (\boldsymbol{\phi} \times \mathbf{b}_i) \qquad \text{for } \boldsymbol{\phi} \in \mathbb{R}^3.$$

Now for any $\boldsymbol{\psi} \in \mathbb{R}^3$ we have

$$\begin{aligned}
\langle \mathbf{I}(\boldsymbol{\phi}), \boldsymbol{\psi} \rangle &= \sum_i \langle m_i \mathbf{b}_i \times (\boldsymbol{\phi} \times \mathbf{b}_i),\ \boldsymbol{\psi} \rangle \\
&= \sum_i m_i \langle \boldsymbol{\phi} \times \mathbf{b}_i,\ \boldsymbol{\psi} \times \mathbf{b}_i \rangle \\
&= \sum_i m_i \langle \boldsymbol{\psi} \times \mathbf{b}_i,\ \boldsymbol{\phi} \times \mathbf{b}_i \rangle \\
&= \langle \mathbf{I}(\boldsymbol{\psi}), \boldsymbol{\phi} \rangle.
\end{aligned}$$

Thus $\mathbf{I}$ is self-adjoint, and consequently has an orthonormal basis of eigenvectors. Since

$$\langle \mathbf{I}(\boldsymbol{\phi}), \boldsymbol{\phi} \rangle = \sum_i m_i |\boldsymbol{\phi} \times \mathbf{b}_i|^2,$$

the corresponding eigenvalues are all ≥ 0, and in fact they are all > 0 because we are assuming that $\mathbf{b}$ is non-planar, and thus at least one $|\boldsymbol{\phi} \times \mathbf{b}_i| > 0$. Since $\mathbf{I}$ always has positive eigenvalues, it is always an isomorphism, so we can indeed always solve equation $(*)$ for $\boldsymbol{\omega}'$.

The map $\mathbf{I} = \mathbf{I}_{\mathbf{b}}$ is called the **inertia tensor** of $\mathbf{b}$. I should say that this is not how the inertia tensor is usually introduced (in fact, I don't know of any physics text that does introduce it this way), but it's the way it *should* be introduced!

The directions of the eigenvalues of the inertia tensor are called the **principal axes of inertia**, and the corresponding eigenvalues are called the **principal moments of inertia**. As our rigid body moves under the rotations $B^{-1}(t)$, the inertia tensor for $B^{-1}(t)(\mathbf{b})$ is just the composition $B^{-1}(t) \circ \mathbf{I} \circ B(t)$; the principal moments of inertia remain the same for all positions of the rigid body under the motion, while the principal axes of inertia are transformed by the $B^{-1}(t)$.

Aside from the values of $\mathbf{F}_{\text{total}}$ and $\boldsymbol{\tau}$, the principal moments of inertia are the only other data entering into our equations, so, in a sense, the whole motion of the rigid body $\mathbf{b}$ depends only on them. In particular, for motion under no external forces, we obtain exactly the same equations for two rigid bodies of arbitrary shape, provided only that they have the same principal moments of inertia.

We've described the inertia tensor rather abstractly, but it is not hard to write down the matrix of $\mathbf{I}$ with respect to the standard basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, temporarily adopting the notation $\mathbf{b}_i = (x_i, y_i, z_i)$.

I should remind you of the identity

$$\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \langle \mathbf{w}, \mathbf{v} \rangle \mathbf{u} - \langle \mathbf{w}, \mathbf{u} \rangle \mathbf{v}$$

which we can prove by noting that $\mathbf{w} \times (\mathbf{u} \times \mathbf{v})$ is perpendicular to $\mathbf{u} \times \mathbf{v}$, so it is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and the appropriate coefficients are easy to determine using the usual identities for $\times$.

With this identity at hand, we can write

$$\mathbf{I}(\boldsymbol{\omega}) = \sum_i m_i(|\mathbf{b}_i|^2\boldsymbol{\omega} - \langle \mathbf{b}_i, \boldsymbol{\omega}\rangle \mathbf{b}_i),$$

which gives

$$\begin{aligned}\mathbf{I}(\mathbf{e}_1) &= \sum_i m_i\big(|\mathbf{b}_i|^2\mathbf{e}_1 - \langle \mathbf{b}_i, \mathbf{e}_1\rangle \mathbf{b}_i\big)\\ &= \sum_i m_i\big(|\mathbf{b}_i|^2\mathbf{e}_1 - x_i(x_i, y_i, z_i)\big)\\ &= \sum_i m_i\big({y_i}^2 + {z_i}^2, -x_i y_i, -x_i z_i\big),\end{aligned}$$

with similar results for $\mathbf{e}_2$ and $\mathbf{e}_3$. Thus, the matrix of $\mathbf{I}$ with respect to the standard basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is

$$\mathcal{I} = \begin{pmatrix} \sum_i m_i({y_i}^2 + {z_i}^2) & -\sum_i m_i x_i y_i & -\sum_i m_i x_i z_i \\ -\sum_i m_i y_i x_i & \sum_i m_i({x_i}^2 + {z_i}^2) & -\sum_i m_i y_i z_i \\ -\sum_i m_i z_i x_i & -\sum_i m_i z_i y_i & \sum_i m_i({x_i}^2 + {y_i}^2) \end{pmatrix}.$$

The same result obviously holds for any orthonormal basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ if we set $x_i = \langle \mathbf{b}_i, \mathbf{v}_1\rangle$, $y_i = \langle \mathbf{b}_i, \mathbf{v}_2\rangle$, and $z_i = \langle \mathbf{b}_i, \mathbf{v}_3\rangle$.

The diagonal terms of the matrix of $\mathbf{I}$ with respect to an orthonormal basis are the quantities that were classically called the "moments of inertia" of $\mathbf{b}$ about the axes; the off-diagonal terms are sometimes called the "products of inertia". In other words, the moment of inertia I_A of $\mathbf{b}$ about an axis A is

$$I_A = \sum_i m_i {r_i}^2,$$

where r_i is the distance from $\mathbf{b}_i$ to A. If our orthonormal coordinate system happens to point along the principal axes, then these moments of inertia are the principal moments of inertia.

As we might expect, for a rigid body $\mathbf{b}$ a special role is played by the moment of inertia about the axes that pass through the center of gravity $C = (x_C, y_C, z_C)$. More generally, there is a simple relationship between the matrix $\mathcal{I}$ and the matrix $\mathcal{I}'$ of the inertia tensor of $\mathbf{b}$ in a parallel coordinate system whose origin is C. Let

$$x_i = \bar{x}_i + x_C, \qquad y_i = \bar{y}_i + y_C, \qquad z_i = \bar{z}_i + z_C,$$

so that $(\bar{x}_i, \bar{y}_i, \bar{z}_i)$ are the coordinates of the rigid body in this system, and let $M = \sum_i m_i$. When we write $\mathcal{I}$ in terms of the $\bar{x}_i, \bar{y}_i, \bar{z}_i$, any cross term like

$\sum_i m_i \bar{y}_i y_C$ vanishes, since

$$\sum_i m_i \bar{y}_i = \sum_i m_i (y_i - y_C) = \sum_i m_i y_i - M y_C,$$

which is 0 by definition of y_C. Thus, we obtain simply

$$\begin{aligned} \mathcal{I} &= \mathcal{I}' + M \cdot \begin{pmatrix} {y_C}^2 + {z_C}^2 & -x_C y_C & -x_C z_C \\ -y_C x_C & {x_C}^2 + {z_C}^2 & -y_C z_C \\ -z_C x_C & -z_C y_C & {x_C}^2 + {y_C}^2 \end{pmatrix} \\ &= \mathcal{I}' + \mathcal{I}_C, \end{aligned}$$

where $\mathcal{I}_C$ is the matrix of the inertia tensor of the single body C with mass M around the origin of our original coordinate system.

In particular, we have

The Parallel Axis Theorem (**Steiner's Theorem**). If the point P is at distance d from the center of mass C of $\mathbf{b}$, the moment of inertia of $\mathbf{b}$ about any axis through P is Md^2 plus the moment of inertia about the parallel axis through C.

The parallel axis theorem is actually most useful when we make the obvious generalizations to a continuous rigid body B with density ρ. First of all, the total mass M is given by

$$M = \int_B \rho = \int_B \rho(x, y, z)\, dx\, dy\, dz,$$

using x, y, z for the standard coordinate functions on $\mathbb{R}^3$. The center of mass is the vector given by

$$C = \frac{1}{M} \int_B \rho(x, y, z)(x, y, z)\, dx\, dy\, dz,$$

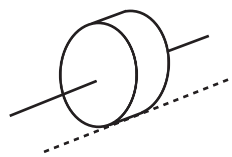
i.e., C is the point of $\mathbb{R}^3$ with coordinates

$$\int_B x \cdot \rho(x, y, z)\, dx\, dy\, dz, \quad \int_B y \cdot \rho(x, y, z)\, dx\, dy\, dz, \quad \int_B z \cdot \rho(x, y, z)\, dx\, dy\, dz.$$

The inertia tensor of B is the linear transformation whose matrix $\mathcal{I}$ with respect to the standard basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is given by

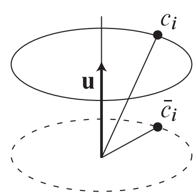
$$\begin{pmatrix} \int_B \rho \cdot (y^2 + z^2)\, dx\, dy\, dz & -\int_B \rho \cdot xy \;\, dx\, dy\, dz & -\int_B \rho \cdot xz \;\, dx\, dy\, dz \\ -\int_B \rho \cdot yx \;\, dx\, dy\, dz & \int_B \rho \cdot (x^2 + z^2)\, dx\, dy\, dz & -\int_B \rho \cdot yz \;\, dx\, dy\, dz \\ -\int_B \rho \cdot zx \;\, dx\, dy\, dz & -\int_B \rho \cdot zy \;\, dx\, dy\, dz & \int_B \rho \cdot (x^2 + y^2)\, dx\, dy\, dz \end{pmatrix}.$$

A straightforward double integration, left to you, enables us to find the moment of inertia of a cylinder about the axis through its center. The parallel axis



theorem then allows us to compute the moment of inertia around a parallel axis that goes through the edge, a result that we will mention in the next lecture.

Moments of inertia play a crucial role when we consider a rigid body whose motion is a rotation about an axis. For a unit vector $\mathbf{u}$ pointing along this axis,



if we decompose each c_i as

$$c_i = \bar{c}_i + \langle c_i, \mathbf{u} \rangle \mathbf{u}$$

where $\bar{c}_i$ is in the plane perpendicular to $\mathbf{u}$, then $|\bar{c}_i|$ is the distance r_i from c_i to the axis. The tangent vector $\bar{c}_i{}'$ is in the same plane as $\bar{c}_i$ and perpendicular to it, and if $\theta(t)$ is the angle through which the body has rotated at time t, then the length of $\bar{c}_i{}'$ is $r_i\theta'$. So

$$\bar{c}_i \times \bar{c}_i{}' = r_i{}^2\theta' \cdot \mathbf{u},$$

and it follows easily that

$$\langle c_i \times c_i{}', \mathbf{u} \rangle = \langle \bar{c}_i \times \bar{c}_i{}', \mathbf{u} \rangle = r_i{}^2\theta',$$

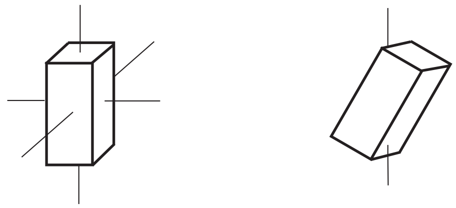
and thus

$$\langle c_i \times c_i{}'', \mathbf{u} \rangle = \langle c_i \times c_i{}', \mathbf{u} \rangle' = r_i{}^2\theta''.$$

So equation ($\boldsymbol{\tau}_{\text{rigid}}$) gives the $\mathbf{u}$ component of $\boldsymbol{\tau}$ as

$$(\boldsymbol{\tau}_{\text{axis}}) \qquad \langle \boldsymbol{\tau}, \mathbf{u} \rangle = I_A \cdot \theta''.$$

Now we ask whether a rigid body can rotate about an axis if there are no external forces. The block shown below can rotate about the three axes of



symmetry. You might naively expect that if it were provided with the right initial push it could also rotate about any other axis that passes through the center of mass of the block, as illustrated in the right hand part of the figure, but a little thought should be able to convince you otherwise (note that the angular momentum vector won't be constant).

In general, if there are no external forces on our rotating body, then we have $\theta'' = 0$ in equation ($\boldsymbol{\tau}_{\text{axis}}$), so that the body rotates with constant angular velocity $\theta' = a$. Since we now have

$$c_i{}' = a\mathbf{u} \times c_i,$$

we get

$$\begin{aligned} c_i{}'' &= a\mathbf{u} \times c_i{}' \\ &= a^2\mathbf{u} \times (\mathbf{u} \times c_i) \\ &= a^2[\langle \mathbf{u}, c_i \rangle \mathbf{u} - c_i], \end{aligned}$$

so that

$$c_i \times c_i{}'' = -a^2\langle \mathbf{u}, c_i \rangle (\mathbf{u} \times c_i).$$

Since there are not external forces, equation ($\boldsymbol{\tau}_{\text{rigid}}$) thus gives

$$0 = \sum_i m_i \langle c_i, \mathbf{u} \rangle (\mathbf{u} \times c_i).$$

Since

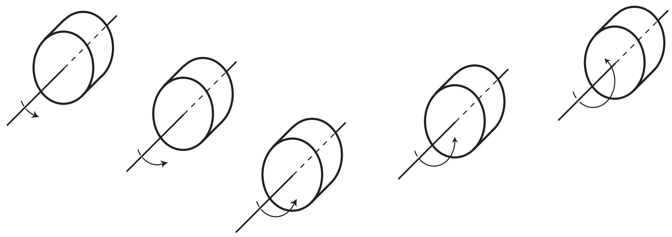
$$\begin{aligned} \mathbf{I}_c(\mathbf{u}) &= \sum_i m_i c_i \times (\mathbf{u} \times c_i) \\ &= \sum_i m_i |c_i|^2 \mathbf{u} - \sum_i m_i \langle c_i, \mathbf{u} \rangle c_i, \end{aligned}$$

this then shows that

$$\mathbf{u} \times \mathbf{I}_c(\mathbf{u}) = 0,$$

so $\mathbf{I}_c(\mathbf{u})$ is a multiple of $\mathbf{u}$, and $\mathbf{u}$ must be an eigenvector of $\mathbf{I}_c$. In general, a rigid body has just three axes around which it can rotate without external forces, and then the angular velocity must be constant.

Finally, for use in the next lecture, I want to point out that equation ($\boldsymbol{\tau}_{\text{axis}}$) holds even in the more general case where the motion of a rigid body is the result of combining rotation about an axis A with a motion of this axis parallel



to itself. In fact, this just changes the c_i to

$$c_i + \alpha\mathbf{u} + \mathbf{v},$$

for some functions α and $\mathbf{v}$, with $\mathbf{v}$ always perpendicular to $\mathbf{u}$. It is then easy to check that

$$\langle (c_i + \alpha\mathbf{u} + \mathbf{v}) \times (c_i{}'' + \alpha''\mathbf{u} + \beta''\mathbf{v}), \mathbf{u}\rangle = \langle (c_i \times c_i{}''), \mathbf{u}\rangle,$$

because each of the other terms in the expansion is 0.

LECTURE 6

CONSTRAINTS

Though many problems of elementary physics involve a rigid body, this rigid body is usually subject to some "constraint", like a block that slides along an inclined plane, or a wheel that rolls along it. Simplest of all is the pendulum, a rigid body (the pendulum bob) constrained to move in a circular arc by a string attached to some fixed point.

We analyzed a pendulum in the first lecture, but that treatment was necessarily somewhat incomplete, because a pendulum is actually an abstraction, just like a rigid body. If we release a raised pendulum bob (conveniently regarded simply as a particle, or point mass), then it will start to fall downwards, rather than along the arc of the circle, stretching the string a bit, and then this stretching will cause the string to exert an upward force that pulls the string a bit above the arc of the circle, and so on.

The circular arc along which the pendulum bob supposedly moves plays the role of the "configuration space" $\mathcal{M}$ in our treatment of rigid bodies—it represents the set of all positions that the pendulum bob can reach under the constraint that it remains at a fixed distance l from the pivot point (we are also assuming that the pendulum bob is moving in a plane, to make everything a lot easier).

In addition to the external force $\mathbf{F} = gm$ of gravity on our pendulum bob of mass m, there is an "internal" force $\mathbf{F}_1$ on the pendulum bob that is exerted along the string, and the law of motion is

$$mc'' = \mathbf{F} + \mathbf{F}_1$$

or

$$mc'' - \mathbf{F} = \mathbf{F}_1.$$

But the string is always perpendicular to $\mathcal{M}$, so this implies that

$$\text{(a)} \qquad \langle \mathbf{v}, mc'' - \mathbf{F} \rangle = 0$$

for all $\mathbf{v}$ tangent to $\mathcal{M}$, just as in d'Alembert's Principle. If θ is the obvious coordinate system on $\mathcal{M}$, and we allow the usual abuse of notation of letting θ also be $\theta \circ c$ for our particle c, then when we apply this equation to $\mathbf{v} = \partial/\partial\theta$ we get the same equation

$$\text{(P)} \qquad \theta'' + \frac{g}{l}\sin\theta = 0$$

as before.

In the case of our earlier derivation of (the easy direction of) the principle of virtual work for a rigid body, we had to use the equations

$$\langle c_i(t) - c_j(t),\, c_i(t) - c_j(t)\rangle = \text{constant}$$

defining a rigid body to derive the equation

$$\sum_k \langle \mathbf{v}_k, \mathbf{F}_k\rangle = 0,$$

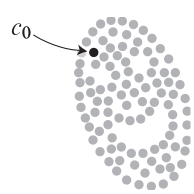
But in the case of the pendulum, the analogous equation (a) followed directly from the nature of the problem—the internal forces were automatically perpendicular to the tangent space of $\mathcal{M}$. In fact, "constraint problems" basically mean ones in which the internal "constraint" forces have this property.

We can sum this up as

> **d'Alembert's Principle for Constraints:** If the constraints on a system confine the system to a configuration space $\mathcal{M}$, and are perpendicular to $\mathcal{M}$, then the motions of the system under the external forces $\mathbf{F}$ satisfy
>
> $$\langle \mathbf{F} - m\mathbf{c}'',\ \mathbf{v}\rangle = 0 \qquad \text{for all } \mathbf{v} \text{ tangent to } \mathcal{M}_{\mathbf{c}}.$$

Another difference between the case of a rigid body and constraint problems is that the internal forces for a rigid body are generally not unique, and d'Alembert's Principle allows us to solve for the motion without having to find the internal forces. On the other hand, "constraint" forces usually are unique, and easy to determine once the problem has been solved.

Let's now consider the "compound pendulum", which is basically a pendulum that is a thin plate, of arbitrary shape, oscillating about a fixed point c_0. Now we have the constraint that our pendulum is a rigid body, together with the constraint that the point c_0 is fixed.



Thus we are assuming that the constraint forces keep the point c_0 at distance 0 from some point P. This is a bit weird, of course, since this constraint force supposedly acts along the line between c_0 and P, which doesn't tell us anything. Nevertheless, even though we can't specify the direction of this constraint force $\mathbf{C}$, we will still have $\langle \mathbf{C}, \mathbf{v}_0\rangle = 0$ for all virtual infinitesimal displacements

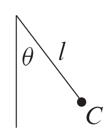
$\mathbf{v} = (\mathbf{v}_0, \ldots, \mathbf{v}_K)$ of our pendulum under this constraint, for the simple reason that $\mathbf{v}_0 = 0$, since the constraint keeps c_0 fixed.

For this problem our configuration space is again a circle, although now we don't think of it as a circle of a particular radius, but simply as the collection of angles θ through which our pendulum can rotate. Restricting ourselves to this configuration space takes care of the constraint that c_0 stays fixed, and we've already analyzed all the other constraints that make the pendulum a rigid body; and we just have to apply our equation ($\boldsymbol{\tau}_{\text{axis}}$) from Lecture 5.

Since the force on particle c_i is $gm_i\mathbf{u}$, where $\mathbf{u}$ is the unit downward vector field, we have

$$\begin{aligned}\boldsymbol{\tau} &= \textstyle\sum_i c_i \times g m_i \mathbf{u} \\ &= g \cdot \left(\textstyle\sum_i m_i c_i\right) \times \mathbf{u} \\ &= gM \cdot C \times \mathbf{u},\end{aligned}$$

where C is the center of mass of the pendulum. This means that $\boldsymbol{\tau}$ points in



the direction of the axis A, and has magnitude

$$gMl\sin\theta,$$

where θ is the angle of C with the vertical, and l its distance from the pivot. So equation ($\boldsymbol{\tau}_{\text{axis}}$) becomes

$$\theta'' + \frac{gMl}{I_A}\sin\theta = 0.$$

Comparing to the formula (P), we see that our pendulum acts precisely like a single bob pendulum whose distance from the pivot is

$$\frac{I_A}{Ml}.$$

If I_C is the moment of inertia about the center of gravity C, then by the parallel axis theorem we have

$$\frac{I_A}{Ml} = \frac{I_C + Ml^2}{Ml} = l + \frac{I_C/M}{l}.$$

Introducing the *radius of gyration* k by

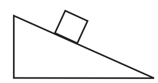
$$I_C = Mk^2,$$

we find that our pendulum acts precisely like a single bob pendulum whose distance from the pivot is

$$l + \frac{k^2}{l}.$$

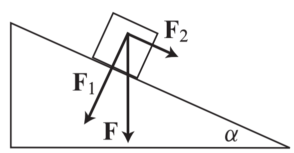
When our rigid body pendulum consists only of the pivot and a single particle at C, we have $k = 0$, but in any other case the pendulum will have a longer period.

A good illustration of the method of configuration spaces is provided by a problem that becomes quite complicated with standard elementary analysis. Elementary physics courses almost always analyze a block sliding down an inclined plane, though it may not always be emphasized that this plane is pre-



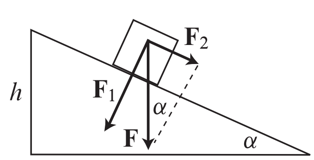
sumed to be immovable, rather than an object that can itself slide horizontally along the floor.

The force $\mathbf{F}$ of gravity on the block is decomposed into a force $\mathbf{F}_2$ parallel to the inclined plane, and a force $\mathbf{F}_1$ perpendicular to the inclined plane, where $\mathbf{F}_2$ supposedly doesn't act on the inclined plane, because we are assuming that



the block slides without friction. We reason, from the third law, that since $\mathbf{F}_1$ is the force of the block on the inclined plane, the inclined plane must exert the force $-\mathbf{F}_1$ on the block. Here is where we are using the hypothesis that the inclined plane is stationery: $\mathbf{F}_1$ determines the acceleration of the block in the direction perpendicular to the inclined plane, but that must be 0 (since the inclined plane is not moving and the block slides along it), so the inclined plane must be exerting a force of $-\mathbf{F}_1$ on the block.

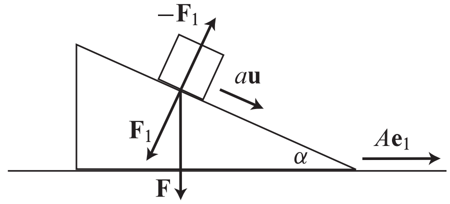
We thus find that the force $\mathbf{F}_2$ parallel to the inclined plane has magnitude $mg \sin\alpha$, so if $c(t)$ is the distance that the block has traveled along the inclined



plane after time t, then $c'' = g \sin\alpha$, and our block slides with a uniform acceleration that is $\sin\alpha$ times its free fall acceleration.

But things are quite a bit more complicated if we allow the inclined plane to be a wedge that slides along the horizontal plane, without friction. The force $\mathbf{F}_1$ that the block exerts on the wedge can no longer be obtained simply by resolving the downward force of gravity $\mathbf{F}$ into forces perpendicular and parallel to the wedge, because our identification of $\mathbf{F}_1$ with the perpendicular component depended on the wedge being fixed.

Choosing the unit vector $\mathbf{e}_1 = (1, 0)$ parallel to the floor and the unit vector $\mathbf{u}$ parallel to the slope of the wedge, we let $A\mathbf{e}_1$ be the acceleration of the wedge, of mass M, along the horizontal plane, while $a\mathbf{u}$ is the acceleration of the block, of mass m, along the wedge, so that $a\mathbf{u} + A\mathbf{e}_1$ is the acceleration of the block in



our inertial system. Note that in our picture, we actually have $A < 0$, so that the arrow $A\mathbf{e}_1$ points in the opposite direction, since the force $\mathbf{F}_1$ causes the block to slide to the left.

Breaking up the equation

$$-\mathbf{F}_1 + \mathbf{F} = m(a\mathbf{u} + A\mathbf{e}_1) \tag{1}$$

for the motion of the block into the components that are parallel and perpendicular to the slope of the wedge gives

$$mg\sin\alpha = ma + mA\cos\alpha \tag{1a}$$

$$|\mathbf{F}_1| - mg\cos\alpha = mA\sin\alpha. \tag{1b}$$

The force on the wedge, of mass M, is $\mathbf{F}_1$ plus the gravitational force downward, plus whatever upward force the horizontal plane must exert to keep the wedge from moving downwards. So A is determined by the horizontal component of $\mathbf{F}_1$:

$$-|\mathbf{F}_1|\sin\alpha = MA. \tag{2}$$

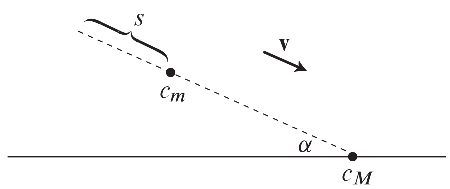
From (1b) and (2) we get

$$A = -g\left(\frac{\sin\alpha\cos\alpha}{\sin^2\alpha + \frac{M}{m}}\right),$$

and then (1a) gives

$$a = g\sin\alpha - A\cos\alpha.$$

To analyze this problem using configuration spaces, we regard our system as consisting of two "particles", the block c_m and the wedge c_M. Since the



wedge c_M always stays on the horizontal axis, we'll simply consider our problem as occurring in $(\mathbb{R}^2)\times\mathbb{R}$, with $((a,b),x)$ representing the particle c_m at the point (a,b), and the particle c_M at x.

Our configuration space $\mathcal{M}$ consists of all $((a,b),x)$ which represent the block c_m resting on the wedge c_M [i.e., for which we have $b = (x-a)\cos\alpha$]. A convenient coordinate system on $\mathcal{M}$ is provided by the coordinate $x \in \mathbb{R}$ giving the position of c_M, together with the distance s of c_m from the top of the wedge.

To determine $\partial/\partial s$, we keep x fixed and vary s, obtaining a curve in $\mathcal{M}$ whose $\mathbb{R}^2$ component moves down the slope of the wedge, while its $\mathbb{R}$ component is fixed, so for the unit vector $\mathbf{u}$ in $\mathbb{R}^2$ parallel to the slope of the wedge, we have

$$\frac{\partial}{\partial s} = (\mathbf{u}, 0).$$

On the other hand, if we keep s fixed and vary x, then we obtain a curve in $\mathcal{M}$ whose $\mathbb{R}^2$ component moves parallel to the first axis along with its $\mathbb{R}$ component, so for $\mathbf{e}_1 = (1,0)$ we have

$$\frac{\partial}{\partial x} = (\mathbf{e}_1, 1).$$

Now if $s(t), x(t)$ are the functions describing the motion of c_m, c_M, we have

$$\begin{aligned} c_m(t) &= s(t)\mathbf{u} + x(t)\mathbf{e}_1 \in \mathbb{R}^2 \\ c_M(t) &= x(t) \in \mathbb{R}, \end{aligned}$$

so

$$\begin{aligned} c_m{}'' &= s''\mathbf{u} + x''\mathbf{e}_1 \in \mathbb{R}^2 \\ c_M{}'' &= x'' \in \mathbb{R}. \end{aligned}$$

The external forces $\mathbf{F}_m$ on c_m and $\mathbf{F}_M$ on c_M are given by

$$\begin{aligned} \mathbf{F}_m &= -mg\mathbf{e}_2 \\ \mathbf{F}_M &= 0, \end{aligned}$$

so our condition for a solution is that

$$\langle -mg\mathbf{e}_2 - mc_m{}'', \mathbf{v}_1\rangle + (0 - Mc_M{}'') \cdot \mathbf{v}_2 = 0$$

for all $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ tangent to $\mathcal{M}$, where $\langle\ ,\ \rangle$ is the usual inner product in the first factor $\mathbb{R}^2$, while the inner product in the second factor $\mathbb{R}$ is just ordinary multiplication. Choosing

$$\mathbf{v} = \frac{\partial}{\partial s} = (\mathbf{u}, 0) \qquad \text{and then} \qquad \mathbf{v} = \frac{\partial}{\partial x} = (\mathbf{e}_1, 1)$$

gives the two equations

$$\begin{aligned}0 &= \langle -mg\mathbf{e}_2 - ms''\mathbf{u} - mx''\mathbf{e}_1, \mathbf{u}\rangle - Mx'' \cdot 0\\ 0 &= \langle -mg\mathbf{e}_2 - ms''\mathbf{u} - mx''\mathbf{e}_1, \mathbf{e}_1\rangle - Mx'' \cdot 1,\end{aligned}$$

which amount to the equations

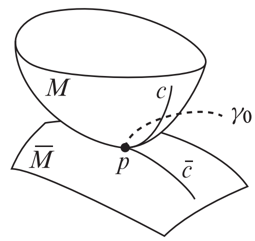
$$\begin{aligned}0 &= mg\sin\alpha - ms'' - mx''\cos\alpha\\ 0 &= -ms''\cos\alpha - mx'' - Mx''.\end{aligned}$$

Solving for x'' and s'' gives us the same results that we obtained previously, when they were called A and a, respectively. The first of the above equations is precisely (1a), while the second is a combination of (1a), (1b), and (2).

Finally, let us consider the problem of rolling wheels. Physics books that discuss rolling in any reasonably detailed way point out that rolling depends on a truly paradoxical fact: a wheel only rolls because of frictional forces, ones that "oppose sliding": a wheel not only displays the characteristics of our abstract rigid body, but it also has the strange feature that it is affected by a frictional force that is exerted only at the (always changing) contact point of the wheel on the inclined plane. To make the picture even more confusing, we can somehow ignore this frictional force because, the physics books note, the path followed by any point on the circumference of the wheel has velocity 0 at the moment it hits the plane!

It is indeed well known that a cycloid, the path followed a point on the circumference of a wheel, has velocity 0 at the point of contact. Physics books seem to regard this as intuitively clear, but for those of us not endowed with the requisite physical intuition, here is a proof, for the general case of one surface rolling on another.

Proposition. Consider two surfaces M and $\overline{M}$ in $\mathbb{R}^3$ that are tangent at a point p. Let c be a curve in M, and $\bar{c}$ a curve in $\overline{M}$ such that $c(0) = p = \bar{c}(0)$, and



such that $c'(0) \neq 0$ is a multiple of $\bar{c}'(0)$. For each t let $A(t)$ be the rigid motion for which

(a) $A(t)(c(t)) = \bar{c}(t)$,
(b) $A(t)(M)$ is tangent to $\overline{M}$ at $\bar{c}(t)$,
(c) $A(t)(c'(t))$ points in the same direction as $\bar{c}(t)$, so that

$$A(t)(c'(t)) = \alpha(t) \cdot \bar{c}'(t) \text{ for some function } \alpha.$$

Also, for each point $c(\tau)$ on the curve c, let γ_τ be the curve that this point follows under these rigid motions,

$$\gamma_\tau(t) = A(t)(c(\tau)).$$

Then for all t we have

$$\alpha(t) = 1 \iff {\gamma_t}'(t) = 0.$$

Consequently, $\alpha(t) = 1$ for all t, so that the lengths of c and $\bar{c}$ are the same on any time interval $[t_0, t_1]$, if and only if all ${\gamma_t}'(t) = 0$, so that γ_t has velocity 0 at the time that it hits $\overline{M}$.

Proof. Write $A(t)$ in the form

$$A(t)(\mathbf{x}) = B(t)(\mathbf{x}) + \mathbf{w}(t) \qquad \mathbf{x} \in \mathbb{R}^3,$$

for orthogonal $B(t)$. Setting $\mathbf{x} = c(t)$ and using $A(t)(c(t)) = \bar{c}(t)$, we see that $\mathbf{w}(t) = \bar{c}(t) - B(t)(c(t))$, so we can write

$$A(t)(\mathbf{x}) = B(t)(\mathbf{x}) + [\bar{c}(t) - B(t)(c(t))].$$

The definition $\gamma_0(t) = A(t)(c(0)) = A(t)(p)$ gives

$$\gamma_0(t) = B(t)(p) + \bar{c}(t) - B(t)(c(t)).$$

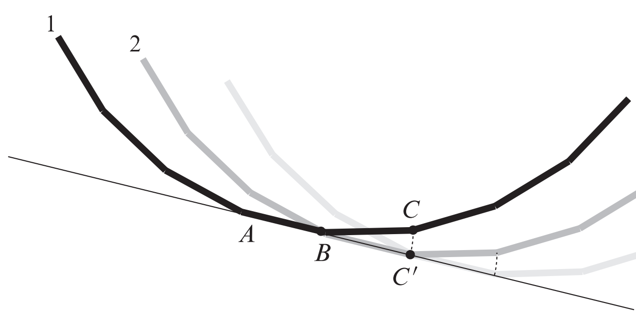
Since $B(0)$ is the identity, differentiation gives

$$\begin{aligned}\gamma_0{}'(0) &= B'(0)(p) + \bar{c}'(0) - B'(0)(p) - c'(0)\\ &= \bar{c}'(0) - c'(0)\\ &= \bar{c}'(0) - \alpha(0)\cdot\bar{c}'(0).\end{aligned}$$

So $\alpha(0) = 1$ if and only if $\gamma_0{}'(0) = 0$.

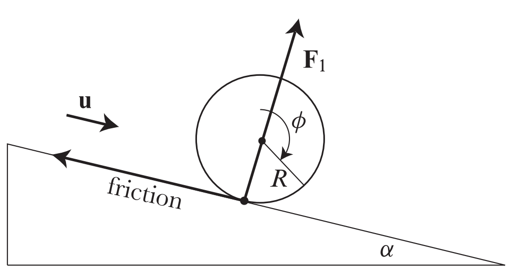
Our hypotheses on $A(t)$ then allow us to use this same argument at any point $c(\tau)$ by considering the reparameterization $t \mapsto t + \tau$.

For the case of a wheel rolling down an inclined plane, we can provide a picture that both reinforces this geometric information and also allows us to see "what is going on" physically. We regard our circular wheel as a polygon with a very large number of sides, and suppose that initially, in position 1, it is lying



on the inclined plane along the segment AB. Now, instead of sliding down the plane, it *rotates* about the point B, reaching position 2 when vertex C hits the inclined plane, at C'. Then it rotates around C', and so forth.

Let's return to the special case of a wheel rolling down an inclined plane, once again assumed to be immovable. Our "wheel" is really supposed to be a cylinder, so that it is forced to roll along a straight line, but the 2-dimensional cross-section picture provides all the interesting information. The standard elementary treatment of this problem is the following:



We consider a wheel of radius R and mass M and uniform density, and let $\phi(t)$ be the angle through which it has rotated after time t. For a unit vector $\mathbf{u}$ pointing down along the inclined plane, we will let $a \cdot \mathbf{u}$ be the acceleration of the center of mass, and let $-f \cdot \mathbf{u}$ be the frictional force along the inclined plane at the contact point of the wheel and the plane. The total force on the wheel is the sum of the downward force of gravity, a constraining force $\mathbf{F}_1$ perpendicular to the plane, which keeps the wheel from moving perpendicularly to the plane, and the frictional force $-f \cdot \mathbf{u}$.

For the acceleration $a \cdot \mathbf{u}$ of the center of mass we have

$$Ma = Mg \sin\alpha - f, \tag{1}$$

since $Mg \sin\alpha$ is the magnitude of the component of the gravitational force along the inclined plane, while the constraining force $\mathbf{F}_1$ is perpendicular to $\mathbf{u}$.

For the rotational motion about the center of mass we can apply equation ($\boldsymbol{\tau}_{\text{axis}}$) to the axis through the center of mass that is perpendicular to the plane of the drawing to get

$$I\phi'' = Rf, \tag{2}$$

where I is the moment of inertia of the wheel about its center of mass.

Finally, the fact that our wheel is rolling tells us that the distance traveled by the center of mass at time t is equal to $R \cdot \phi(t)$, which means that

$$a = R \cdot \phi''. \tag{3}$$

Solving (1)–(3) gives

$$a = g \sin\alpha \frac{1}{1 + \dfrac{I}{R^2 M}},$$

and substituting into (1) gives

$$f = Mg \sin\alpha \cdot \frac{I}{R^2 M + I}.$$

This indicates the amount of frictional force that the inclined plane must be able to produce in order to prevent sliding. As a general rule, the frictional force that a body produces on an inclined plane is proportional to the normal component of the downward gravitational force, i.e., it equals $\mu \cdot Mg \cos\alpha$ for a constant μ, the "coefficient of friction". So to prevent sliding at the angle α we need to have

$$\mu \cdot Mg \cos\alpha = Mg \sin\alpha \cdot \frac{I}{R^2 M + I}$$

or

$$\mu = \tan\alpha \cdot \frac{I}{R^2 M + I}.$$

Thus we need a "perfectly rough" surface, with "$\mu = \infty$" if we want to prevent sliding at any angle.

For a homogeneous disc of radius R and mass M we easily compute that $I = \frac{1}{2}MR^2$, so we have

$$a = \tfrac{2}{3} g \sin\alpha.$$

Thus the wheel rolls down the incline plane at only 2/3 of the speed that a block slides down a frictionless inclined plane.

Even though a rolling wheel involves friction, it is still a natural candidate for treatment by d'Alembert's principle, for the same reason that the standard elementary treatment ignores the effect of friction: If we consider our wheel as a rigid body made up of a large collection of particles, and let $\mathbf{v}$ be any virtual infinitesimal displacement of the wheel, then the inner product $\langle \mathbf{f}, \mathbf{v}_p \rangle$ of the frictional force $\mathbf{f}$ and the velocity $\mathbf{v}_p$ at the point of contact p is always 0 since, as our Proposition shows, $\mathbf{v}_p = 0$.

Thus, the situation is quite like that of the compound pendulum. The only real difficulty is that we have a sort of hybrid between our initial pendulum bob problem, where we considered a single particle acted upon by a constraint force, and the compound pendulum, where almost all our constraints had already been considered in the analysis of rigid body motion. We really need to think of our wheel as representing two different "particles" s and ϕ in $\mathbb{R}$,

$$\begin{aligned} s &= \text{position of the center of mass} \\ \phi &= \text{angle through which wheel has turned,} \end{aligned}$$

having the respective masses

$$\begin{aligned} M &= \text{the total mass of the wheel} \\ I &= \text{the moment of inertia of the wheel.} \end{aligned}$$

We thus have a problem in $\mathbb{R}^2$ that is reduced to a problem on a 1-dimensional submanifold $\mathcal{M} \subset \mathbb{R}^2$ by the rolling condition

$$\phi(t) = s(t)/R.$$

On $\mathcal{M}$ we have the obvious single coordinate s, the distance along the inclined plane, and the corresponding tangent vector to $\mathcal{M}$ represents the pair

$$(1, 1/R).$$

We are looking for a function $s(t)$ such that

$$(s(t), \phi(t)) = (s(t), s(t)/R)$$

satisfies

$$\begin{aligned} 0 &= \langle\, (-\mathbf{F}_s - Ms'',\ 0 - I\phi''),\ (1, 1/R)\,\rangle \\ &= \langle\, (-\mathbf{F}_s - Ms'',\ 0 - Is''/R),\ (1, 1/R)\,\rangle, \end{aligned}$$

where $\mathbf{F}_s$ is the component of the force on the center of mass that is parallel to the inclined plane. Thus we get

$$\begin{aligned} 0 &= Mg\sin\alpha - Ms'' - (Is''/R)\cdot 1/R \\ &= Mg\sin\alpha - Ms'' - Is''/R^2, \end{aligned}$$

which gives the same result

$$s'' = g\sin\alpha\,\frac{1}{1 + \dfrac{I}{R^2 M}}$$

that we obtained previously by combining three equations.

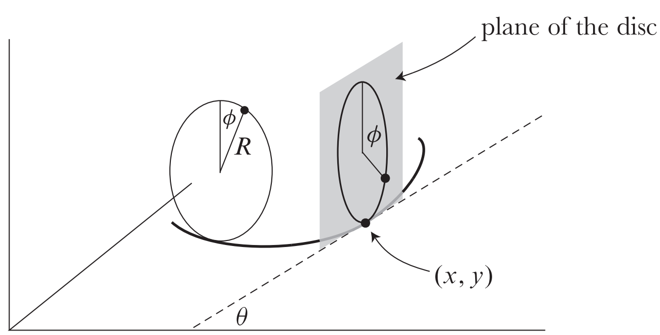
LECTURE 7

HOLONOMIC AND NON-HOLONOMIC CONSTRAINTS

In the last lecture we examined "constraint" problems where the constraints restricted our solutions to lie in a "configuration space" $\mathcal{M}$ that was a submanifold of the larger space for which the problem was originally posed. Our method of solution basically used the obvious principle that if you are looking for the solutions of a differential equation on a manifold $\mathcal{N}$, and you know that the solution lies on a submanifold $\mathcal{M} \subset \mathcal{N}$, then you might as well just consider what the equation says on $\mathcal{M}$, thereby obtaining an equation in fewer variables.

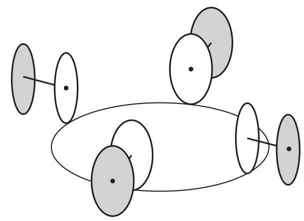
Physicists call such constraints "holonomic". They also mention various sorts of other constraints, like the constraint that particles remain within a given box, which are expressed by inequalities or more complicated conditions, and obviously require special considerations in each individual case. But there is one other very important sort of "non-holonomic" constraints that allows a systematic treatment.

The standard example of this kind of constraint is provided by an upright disc rolling on a plane. The possible positions of the disc are determined by the



coordinates (x, y) of the point at which the disc rests on the plane, the angle ϕ that a fixed point on the disc makes with the vertical, and the angle θ that the plane of the disc makes with the x-axis.

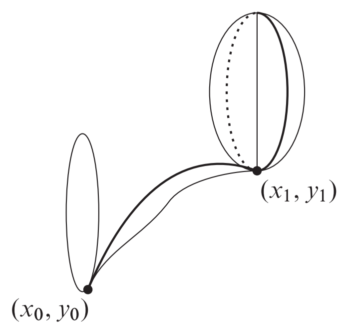
This example is rather idealized. To begin with, in order for the disc to remain upright, we might imagine that it has a companion disc attached to it by an axle. We will want to assume that the axle and the companion disc both have negligible weight, and it is also important that the two discs be able to rotate independently about this axle, so that, for example, the disc can revolve around a circle, with its companion "shadow" disc revolving around a circle of



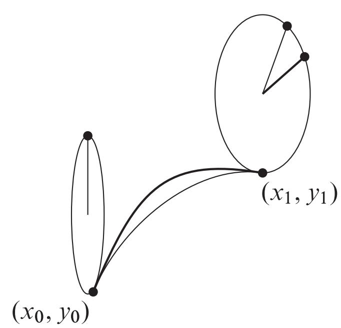
a different radius. In addition, although our disc has to have *some* thickness, we want to imagine it to be so small that it actually can roll along a circle—or, indeed, along any path—rather than being constrained to roll along a straight line.

In the simplest case, where there are no external forces, it is easy to guess from the symmetry of the situation that the disc of mass m and angular momentum $\mathbf{L}$ will roll with constant speed along a circle of radius $m/|\mathbf{L}|$, or along a straight line when $\mathbf{L} = 0$. But that doesn't suggest a general method for solving the problem where there are external forces, for example if we tilt the plane, so that now the force of gravity is only partially offset by the constraining perpendicular force of the plane.

Unfortunately, we are stymied when we try to use our method of configuration spaces to reduce the problem to one in fewer variables. Starting with our disk at a point (x_0, y_0), we can roll it, as in (a) to a nearby point (x_1, y_1) along paths of



(a)



(b)

the same length that all start in the same direction at (x_0, y_0) but reach (x_1, y_1) at different angles. This means that we obtain a whole interval of possible θ values at some particlar value of x_1, y_1, and ϕ_1. Moreover, we can also roll it, as in (b), along paths of different lengths that all have the same direction at both (x_0, y_0) and (x_1, y_1), thereby obtaining a whole interval of possible ϕ values at a particular value of x_1, y_1, and θ_1. Thus, the proper configuration space for this problem is a whole neighborhood of $\mathbb{R}^2 \times S^1 \times S^1$, rather than a lower-dimensional submanifold. (The case of a sphere, or other convex body, rolling on a plane, presents exactly the same problem, with fewer idealizations necessary, but it presents a problem even harder to analyze than the disc.)

This phenomenon is a reflection of a simple fact about the relations between the coordinates of our disc moving in the space with coordinate functions (x, y, ϕ, θ). Letting $x(t), y(t), \phi(t), \theta(t)$ denote the components of the coordinates of the disc, the velocity of the center of mass is $R\phi'$, where R is the radius of the disc, and consequently we have

$$\begin{aligned} x' &= R\phi' \cos\theta \\ y' &= R\phi' \sin\theta. \end{aligned} \tag{1}$$

This means that the tangent vectors of the curve satisfy

$$\begin{aligned} 0 &= dx - R\cos\theta\, d\phi \\ 0 &= dy - R\sin\theta\, d\phi, \end{aligned} \tag{1'}$$

In particular, they therefore satisfy the condition

$$dy - \tan\theta\, dx = 0.$$

This determines a 3-dimensional subspace of all tangent vectors at each point, but this 3-dimensional distribution isn't integrable, as we can easily see from the standard integrability conditions. To apply the differential form version of the Frobenius integrability theorem, for example, we simply note that the 2-form

$$d(dy - \tan\theta\, dx) = \sec^2\theta\, d\theta \wedge dx$$

isn't in the ideal generated by $dy - \tan\theta\, dx$. Equivalently, we can note that the distribution is spanned by the vectors

$$X_1 = \frac{\partial}{\partial x} - \tan\theta \frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial \theta}, \qquad X_3 = \frac{\partial}{\partial \phi},$$

but the bracket

$$\left[\frac{\partial}{\partial x} - \tan\theta \frac{\partial}{\partial y},\ \frac{\partial}{\partial \theta}\right] = -\sec^2\theta \frac{\partial}{\partial y}$$

obviously cannot be written as a linear combination of the three vectors X_1, X_2, and X_3.

Thus, although we have a condition that must be satisfied by tangent vectors to a solution curve, we can't select a 3-dimensional configuration space on which the solution curves must lie. We can only say the following:

> We must have $\langle \mathbf{F} - mc'',\ \mathbf{v}\rangle = 0$
>
> for all $\mathbf{v} \in \ker(dx - R\cos\theta\, d\phi) \cap \ker(dy - R\sin\theta\, d\phi)$.

Here $\mathbf{F}$ is evaluated at $(\mathbf{c}(t), t)$ and c'' is evaluated at t, while $dx - R\cos\theta\, d\phi$ and $dy - R\sin\theta\, d\phi$ are evaluated at $c(t)$.

More generally, if the conditions in (1′) are replaced by the vanishing of certain 1-forms $\omega_1, \dots, \omega_L$, then

$$\text{We must have } \langle \mathbf{F} - mc'', \mathbf{v} \rangle = 0$$
$$\text{for all } \mathbf{v} \in \ker \omega_1 \cap \cdots \cap \ker \omega_L.$$

In terms of the linear functional

$$\Phi(\mathbf{v}) = \langle \mathbf{F} - mc'', \mathbf{v} \rangle$$

this condition says that

$$\ker \Phi \supset \ker \omega_1 \cap \cdots \cap \ker \omega_L.$$

We can now appeal to the very same vector space fact was used in our proof of the Principle of Virtual Work, in Lecture 5, and conclude, applying the argument at each point, that

$$\Phi = \lambda_1 \omega_1 + \cdots + \lambda_L \omega_L$$

for some functions $\lambda_1, \dots, \lambda_L$, known as *Lagrange multipliers*. This leads us to the following criterion for solutions:

> **d'Alembert's Principle for Differential Constraints:** If the constraints on a system require the tangent vector of the motion to lie in the subspace $\ker(\omega_1) \cap \cdots \cap \ker(\omega_L)$, then there are Lagrange multipliers $\lambda_1, \dots, \lambda_L$ such that the motions of the system under the external forces $\mathbf{F}$ satisfy
>
> $$\langle \mathbf{F} - m\mathbf{c}'', \mathbf{v} \rangle = \lambda_1 \omega_1(\mathbf{v}) + \cdots + \lambda_L \omega_L(\mathbf{v})$$
> $$\text{for } \textit{all} \text{ tangent vectors } \mathbf{v} \text{ at } \mathbf{c}.$$

To apply this to the problem of an upright disc rolling on a plane, where we have the relations

$$\begin{aligned} x' &= R\phi' \cos\theta \\ y' &= R\phi' \sin\theta. \end{aligned} \tag{1}$$

and

$$\begin{aligned} 0 &= dx - R\cos\theta\, d\phi \\ 0 &= dy - R\sin\theta\, d\phi, \end{aligned} \tag{1'}$$

we note that, as in the case of the rolling wheel, we are not dealing with a single particle $c(t)$; in the present situation we have to think of the disc as three different "particles", the particle (x, y) with mass M, the particle ϕ with mass I, the moment of inertia of the disc about the axle, and the particle θ with mass I_θ, the moment of inertia of the disc about a diameter. We thus have

$$(*)\quad \langle(-Mx'', -My'', -I_\theta\theta'', -I\phi''), \mathbf{v}\rangle$$
$$= \lambda_1(dx - R\cos\theta\, d\phi)(\mathbf{v}) + \lambda_2(dy - R\sin\theta\, d\phi)(\mathbf{v}) \qquad \text{for all } \mathbf{v}.$$

Taking $\mathbf{v} = \partial/\partial x$, $\partial/\partial y$, $\partial/\partial\theta$, and $\partial/\partial\phi$, this gives us the equations

$$(2x)\qquad -Mx'' = \lambda_1$$
$$(2y)\qquad -My'' = \lambda_2$$
$$(2\theta)\qquad I_\theta\theta'' = 0$$
$$(2\phi)\qquad I\phi'' = \lambda_1 R\cos\theta + \lambda_2 R\sin\theta.$$

Differentiating our original constraint equations (1) gives

$$x'' = R\phi''\cos\theta - R\phi'\theta'\sin\theta$$
$$y'' = R\phi''\sin\theta + R\phi'\theta'\cos\theta,$$

so substituting $(2x)$ and $(2y)$ into (2ϕ) gives

$$(3)\qquad I\phi'' = -MR\big[\big(R\phi''\cos\theta - R\phi'\theta'\sin\theta\big)\cos\theta + \big(R\phi''\sin\theta + R\phi'\theta'\cos\theta\big)\sin\theta\big]$$
$$= -MR^2\phi''.$$

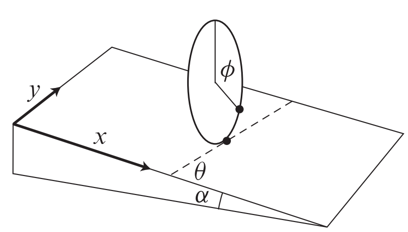
Thus $(I + MR^2)\phi'' = 0$, and ϕ' is constant.

Equation (2θ) shows that θ' is also constant, and if we substitute the two expressions

$$\phi(t) = at + b$$
$$\theta(t) = ct + d$$

into (1) and solve, we find that (x, y) moves along a circle of radius Ra/c for $c \neq 0$, or a straight line if $c = 0$, in which case θ is constant.

We can apply the same analysis to the more interesting case where our disc is rolling down an inclined plane with slope α. The only change to $(*)$ is that



the term $-Mx''$ must be replaced with $Mg\sin\alpha - Mx''$. In the set of equations (2), the only change is that equation ($2x$) is replaced by

$$gM\sin\alpha - Mx'' = \lambda_1.$$

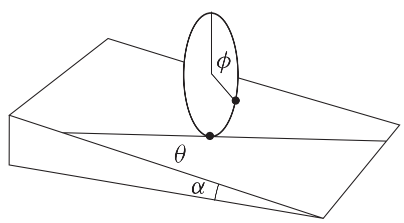
Proceeding as before, (3) then becomes

$$\begin{aligned}(I + MR^2)\phi''(t) &= (MgR\sin\alpha)\cos\theta(t)\\ &= (MgR\sin\alpha)\cos(ct+d),\end{aligned}$$

which, introducing an appropriate constant A, we write simply as

$$\phi''(t) = A\cos(ct+d).$$

In the special case $c = 0$, the angle θ will be constant. This case is essentially just the same as the case of a wheel rolling down an inclined plane: the disc rolls down the inclined plane along the straight line that makes a constant angle with the x-axis.



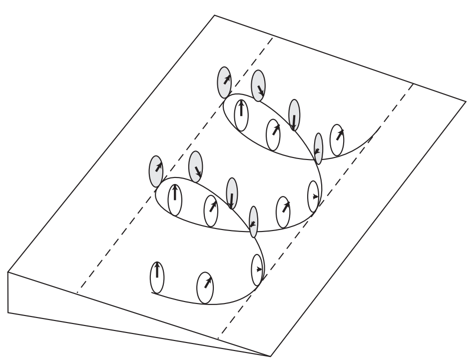
For $c \neq 0$, we might as well take $d = 0$, since this just amounts to changing the point from which ϕ is measured, so there are constants B and C with

$$\phi'(t) = B\sin(ct) + C.$$

In the special case where $C = 0$, equation (1) gives, for a constant D,

$$\begin{aligned}x'(t) &= D\sin(ct)\cos(ct) = \tfrac{1}{2}D\sin(2ct)\\ y'(t) &= D\sin^2(ct) = \tfrac{1}{2}D(1-\cos(2ct)).\end{aligned}$$

Thus x is basically sinusoidal, while y is a sinusoidal motion combined with an additive motion. Somewhat like a pendulum bob, the disc is continually



interchanging potential energy and kinetic energy. At the lowest points of the trajectory, where the kinetic energy is greatest, the disc is moving parallel to the y-axis, where it has just enough kinetic energy to propel it back up to the highest level.

More generally, we have constants E and F with

$$\begin{aligned} x' &= E\sin(2ct) + F\cos(ct) \\ y' &= E(1-\cos(2ct)) + F\sin(ct), \end{aligned}$$

and if you have a nice computer program for producing graphs you can find out what the most general solution looks like.

The Lagrange multiplies λ_i that occur in d'Alembert's principle for differential constraints may seem to have appeared out of the blue, but they can be interpreted in terms of the constraint forces $\mathbf{C}$ on our system S. In fact, consider two systems:

(a) the system S with constraints $\mathbf{C}$ and external forces $\mathbf{F}$,

(b) the system S with no constraints and external forces $\mathbf{F}+\mathbf{C}$.

The systems (a) and (b) obviously have the same solutions. But the solutions to (a) satisfy

$$\langle \mathbf{F} - mc'', \mathbf{v}\rangle = \lambda_1\omega_1(\mathbf{v}) + \cdots + \lambda_L\omega_L(\mathbf{v}) \qquad \text{for all } \mathbf{v},$$

while the solutions to (b) satisfy

$$\langle \mathbf{F} + \mathbf{C} - mc'', \mathbf{v}\rangle = 0 \qquad \text{for all } \mathbf{v};$$

subtracting the first equation from the second, we find that we must have

$$\langle \mathbf{C}, \mathbf{v}\rangle = -(\lambda_1\omega_1 + \cdots + \lambda_L\omega_L)(\mathbf{v}) \qquad \text{for all } \mathbf{v}.$$

By writing out $\lambda_1\omega_1 + \cdots + \lambda_L\omega_L$ in terms of the coordinates, we can then find all the components of $\mathbf{C}$.

For example, in our original problem of the disc rolling on a horizontal plane, where we have

$$\begin{aligned} \lambda_1\omega_1 + \lambda_2\omega_2 &= \lambda_1(dx - R\cos\theta\, d\phi) + \lambda_2(dy - R\cos\sin\theta\, d\phi) \\ &= \lambda_1\, dx + \lambda_2\, dy - (\lambda_1 R\cos\theta + \lambda_2 R\sin\theta)\, d\phi \end{aligned}$$

we find that the components $\mathbf{C}_x$ and $\mathbf{C}_y$ are given by

$$\begin{aligned} \mathbf{C}_x &= \langle \mathbf{C}, \partial/\partial x\rangle = -\lambda_1 \\ \mathbf{C}_y &= \langle \mathbf{C}, \partial/\partial y\rangle = -\lambda_2 \\ \mathbf{C}_\theta &= \langle \mathbf{C}, \partial/\partial \theta\rangle = 0 \\ \mathbf{C}_\phi &= \langle \mathbf{C}, \partial/\partial \phi\rangle = R(\lambda_1\cos\theta + \lambda_2\sin\theta). \end{aligned}$$

Thus, the constraint forces can be found in terms of λ_1 and λ_2, which we can determine from $(2x)$ and $(2y)$ once we've solved explicitly for $x(t)$ and $y(t)$. The x and y components together, the vector $(x''(t), y''(t))$, represents the constraint force on our "particle" $(x(t), y(t))$, and thus the frictional force exerted by the plane to keep the center of mass in its circular orbit. Since the center of mass moves in a circle with constant angular velocity, $(x''(t), y''(t))$ is always perpendicular to the velocity vector $\mathbf{v} = (x'(t), y'(t))$ of the center of mass, as we would expect.

The ϕ component of the constraints,

$$-R(x'' \cos\theta + y'' \sin\theta),$$

represents the constraint force on our "particle" ϕ, the frictional force exerted by the plane against the direction of ϕ; it is the additional frictional force, in the direction of the velocity vector, that is needed to insure that the wheel rolls.

In the case of holonomic constraints, we didn't need to use the Lagrange multipliers λ_i, but we *can* use them, if we want to obtain the constraint forces. For example, consider the rolling wheel problem, which we treated by the method of configuration spaces on page 83. Now we will simply use the coordinates s and ϕ and the relation

$$s'(t) = R\phi'(t)$$

between their derivatives. We then have the following condition for all $\mathbf{v}$:

$$\langle Mg \sin\alpha - Ms'' - I\phi'', \mathbf{v}\rangle = \lambda(ds - Rd\phi)(\mathbf{v}).$$

Taking $\mathbf{v} = \partial/\partial s$ and then $\mathbf{v} = \partial/\partial\phi$ we get

$$\text{(a)} \qquad Mg \sin\alpha - Ms'' = \lambda$$

$$\text{(b)} \qquad -I\phi'' = -R\lambda,$$

so

$$Mg \sin\alpha - Ms'' = \frac{I}{R}\phi''$$

and differentiating the constraint $s' = R\phi'$ gives $s'' = R\phi''$, so this becomes

$$Mg \sin\alpha - Ms'' = \frac{I}{R^2}s'',$$

with the same solution

$$s'' = g \sin\alpha \frac{1}{1 + \dfrac{I}{R^2 M}}$$

as before. Substituting back into (a) then gives

$$\lambda = Mg \sin\alpha \frac{I}{R^2 M + I},$$

which agrees with the formula for the frictional force f on page 82.

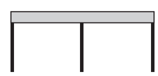
Of course, we can easily formulate and use a "mixed" version of d'Alembert's principle, where some of the constraints restrict our system to lie in a configuration space $\mathcal{M}$, while other constraints restrict the tangent vector of the system to be in the kernels of various 1-forms.

LECTURE 8

STATICALLY INDETERMINATE STRUCTURES

Although we've naturally emphasized constraint problems that have solutions using only our basic consideration of rigid bodies, it is easy to pose problems of quite another sort.

For example, suppose that we have a rigid plank of weight W resting on three identical rigid supports. What are the upward forces exerted by these supports

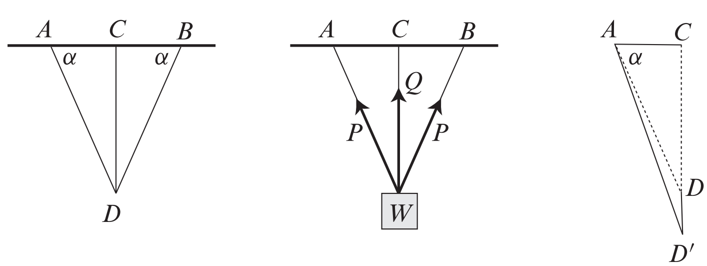


on the plank?

It is easy to see that there simply is no well-determined answer to this question. For example, one solution is that each support provides an upward force of $W/3$, while another possible solution is obtained by simply ignoring the middle support, and thus assigning an upward force of $W/2$ to each of the end supports.

Of course, in reality the solution is not indeterminate—one can actually measure the upward forces and obtain a specific answer; and, in reality, the plank actually *isn't* rigid, but bends a bit. For some reason such problems never seem to be addressed in physics course nowadays, even though they would seem to be wonderful examples of actually applying the basic laws of physics. They are apparently relegated to courses on "material science" or "mechanical engineering", and I only found a simple discussion in the delightfully old-fashioned book [S-G].

Before tackling this problem, let's look at a simpler one, just to see how some simple physics can be applied to problems of this nature. Consider three



(a) (b) (c)

filaments, of the same material, arranged as in (a), so that ABD is an isosceles triangle, and CD is its altitude. A weight W is hung from the end, as in (b), so that the side filaments exert a force of magnitude P along their directions, while the middle filament exerts a force of magnitude Q. These forces actually come about because the filaments stretch slightly as in (c).

We assume that the filaments obey *Hooke's law*: If the filament is fixed at one end and the force $\mathbf{F}$ pulls on the other end, then

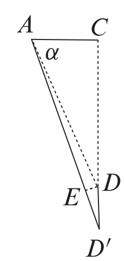
$$|\mathbf{F}| = \lambda \cdot \frac{\Delta l}{l} \qquad \text{for some constant } \lambda,$$

where l is the length of the unstretched filament, and Δl is the increase in length (this holds only for a certain interval of Δl values).

We could use geometry to find the length $AD' - AD$ of the left filament in terms of the change in length $\delta = DD'$, express both P and Q in terms of δ and the constant λ, and then use $2P + Q = W$ to find δ, leading to rather messy formulas for P and Q, in terms of λ. But it is much easier to see what the limiting values are for $\varepsilon \to 0$.

Suppose we choose point E so that $AE = ED$. Then

$$\begin{aligned}\frac{P}{Q} &= \frac{ED'}{AD'} \cdot \frac{CD'}{DD'}\\ &= \frac{ED'}{DD'} \cdot \frac{CD'}{AD'}\\ &= \frac{ED'}{DD'} \sin\alpha.\end{aligned}$$

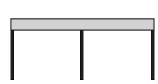


But the angle DED' is $\sim \pi/4$, so the first ratio is $ED'/DD' \sim \sin\alpha$, and we conclude that

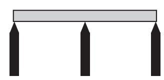
$$\frac{P}{Q} \sim \sin^2\alpha,$$

and using $2P + Q$ we can then solve for P and Q.

Our original plank problem is similar, but rather more complicated. If the plank were perfectly rigid, the supports would all have to be compressed by the

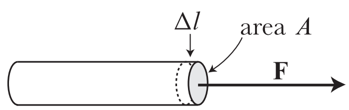


same amount, and thus all have to provide an upward force exactly 1/3 of the weight of the plank. But we are interested instead in analyzing the effect of the plank's bending, which results in different compressions of the supports, and thus different upward forces. In fact, we aren't interested in the extremely tiny compression of the supports at all, only in the upward forces that they will have to provide to balance the bent plank. For simplicity we consider "knife edge" supports, which touch the plank along a line, appearing as a single point in our



2-dimensional section.

Hooke's law also holds for a plank or rod, except that it is stated a bit differently, because we want to take into account the cross-section A, which was



essentially assumed to be 0 for the case of a filament. So we consider the ratios

$$\text{stress } \sigma = \frac{|\mathbf{F}|}{A}, \qquad\qquad \text{strain } \varepsilon = \frac{\Delta l}{l},$$

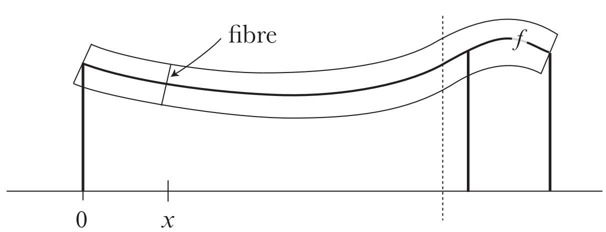
and write Hook's law in the form

$$\sigma = E\varepsilon$$

for a constant E, the *modulus of elasticity*. Of course, A actually changes a bit when the force $\mathbf{F}$ is applied, but the change is so minute that it is disregarded. It should also be noted that the modulus of elasticity for stretching might not be the same as that for compression (concrete is supposed to be an example), but we will be only consider the case where they are the same.

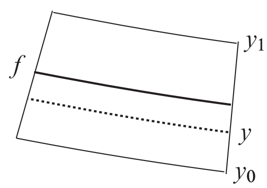
For steel, $E \sim 29 \times 10^6$psi (pounds per square inch), while for molybdenum, $E \sim 49 \times 10^6$psi. For wood E varies between $\sim .6 \times 10^6$psi and $\sim 1.7 \times 10^6$psi, depending on the type and grade of wood, the direction of the load, etc.

Now consider a long plank supported by various knife edge supports. The plank is actually going to sag a small amount, so that viewing it head on we see something like the picture shown below. To the left of the dotted line, there is a compressing stress along the top and a stretching stress along the bottom, while to the right of the dotted line just the opposite is true. The stress is thus 0



at some intermediate surface, the *neutral plane*, whose profile, shown as a heavy line in the figure, is the graph of some function f. The figure also shows a cross section of the "fibre" through $(x, f(x))$, that is, the surface into which a vertical section of the plank is deformed.

The figure below is a greatly enlarged view of a small portion of the figure near the point $(x, f(x))$, bounded by two fibres. While the heavy line is the

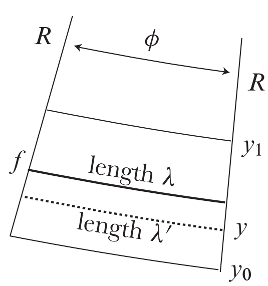


graph of f, the cross-section of the neutral plane, the dotted line indicates the cross-section of a surface where the stress has the constant value y. The whole region is filled up by such surfaces having values in some interval (y_0, y_1).

Let the curvature of the graph of f at the point $(x, f(x))$ be $\kappa(x)$, where we have

$$\kappa(x) = \frac{f''(x)}{1 + (f'(x))^2}.$$

Near the point $(x, f(x))$, the graph is very close to a segment of a circle subtending an angle ϕ with radius $R = 1/(-\kappa)$, where the minus sign is necessary because $f'' \leq 0$ at this point. So the solid line and the dotted line have lengths λ



and λ' given, to first order, by

$$\begin{aligned} \lambda &= R\phi \\ \lambda' &= (R + y)\phi. \end{aligned}$$

Consequently, the strain ε along the surface indicated by the dotted line is given by

$$\varepsilon = \frac{\lambda' - \lambda}{\lambda} = \frac{y\phi}{R\phi} = \frac{y}{R} = -y\kappa,$$

and the stress along this surface is

$$\sigma = \varepsilon E = -y\kappa E.$$

It is easy to check that for points where $f'' \geq 0$ we get exactly the same formula.

For the fibre A through the point $(x, f(x))$, let $\boldsymbol{\tau}(x)$ be the total torsion on A, with respect to the point $(x, f(x))$, from all the external forces to the left of the point (gravity acting down on the portion of the board to the left, together with the upward force of any supports to the left). Since A isn't rotating, we must have the following, where b is the width of the plank:

$$\begin{aligned}\boldsymbol{\tau}(x) &= \int_A \sigma \\ &= b\int_{y_0}^{y_1} \sigma(x, y) \cdot y\, dy \\ &= -E\kappa(x)\int_{y_0}^{y_1} by^2\, dy \\ &= -EI\kappa(x),\end{aligned}$$

where

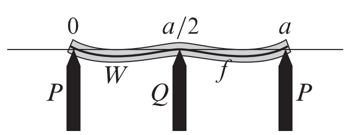
$$I = \int_{y_0}^{y_1} by^2\, dy$$

is, by definition, the moment of inertia of A, which we will assume can be taken to be a constant.

This gives us the equation $\boldsymbol{\tau}(x) = -EI\kappa(x)$. Finally, since f' is usually going to be extremely small, we simply throw away the $f'(x)$ term in the expression for $\kappa(x)$, leading to the *Euler-Bernoulli equation* for plank bending,

$$(*) \qquad -EIf''(x) = \boldsymbol{\tau}(x).$$

As a simple application of the Euler-Bernoulli equation, consider a plank of length a resting on three knife blade supports, two at the ends, and one in the



middle. The plank, of weight W, is assumed to have uniform density $w = W/a$; the outside supports each exert an upward force of P and the middle support exerts an upward force of Q, with $2P + Q = W$. For convenience, we choose the position of the x-axis so that our function f is 0 at 0, $a/2$, and a.

For $0 \leq x < a/2$ we have

$$\boldsymbol{\tau}(x) = -Px + \tfrac{1}{2}wx^2,$$

where the first term is the moment of the upward force P at distance x from our point, and the second term comes from the uniformly distributed force of w along the plank of length x to the left of our point. Thus

$$EIf''(x) = Px - \tfrac{1}{2}wx^2$$

and

$$EIf'(x) = \frac{Px^2}{2} - \frac{wx^3}{6} + C_1.$$

There is another equation for $a/2 < x \leq a$, involving another constant C_2, but in this case we can use symmetry to dispense with the second expression. Since we clearly have $f'(a/2) = 0$, we can immediately solve for C_1, to get

$$EIf'(x) = \frac{Px^2}{2} - \frac{wx^3}{6} + \frac{wa^3}{48} - \frac{Pa^2}{8},$$

and since $f(0) = 0$ this gives

$$EIf(x) = \frac{Px^3}{6} - \frac{wx^4}{24} + \left(\frac{wa^3}{48} - \frac{Pa^2}{8}\right)x.$$

Finally, using $f(a/2) = 0$, and remembering that $aw = W$, this gives

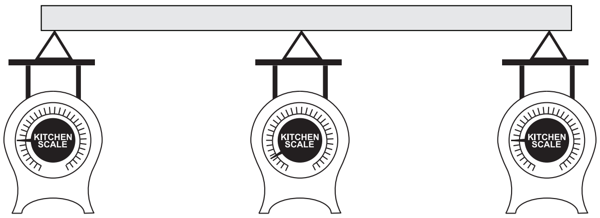
$$P = \frac{3}{16}W.$$

So each end provides an upward force of $\frac{3}{16}W$, while the middle support bears most of the weight, providing an upward force of $\frac{10}{16}W$.

I realized that I had the opportunity to end these lectures with a little experiment, balancing those provided in the first lecture, when I recalled that I had purchased a cute little kitchen scale at one of Tokyo's ubiquitous 100¥ shops.

Wandering through Shibuya until I found the store again, I was gratified to see that they still had an ample supply, so for another couple hundred ¥, I was able to set up a test of our result.

Supplying three cardboard pieces to provide knife edge supports above each scale, I set a board upon them, and eagerly looked at the readings of three scales,



confidently expecting the middle scale to register over 3 times the readings of the end scales. But, as I discovered, and showed at the lecture, the end scales actually registered much larger weights than the middle scale!

This, of course, was somewhat disconcerting! To be sure, a 100¥ scale probably isn't all that accurate, but only Descartes would try to explain away such a large discrepancy as experimental error!

I then reflected that this particular arrangement did not actually correspond to the problem as originally posed, because we assumed that compressions of the supports were of negligible magnitude, so that we had $f(0) = f(a/2)$. But now the compression of the supports (the scales) is certainly significant, since this compression is what causes the scale to register.

So we might ask what happens when the end scales are compressed down by the amount δ_1, while the middle scale is compressed down by the amount δ_2. We still have the equation

$$EIf'(x) = \frac{Px^2}{2} - \frac{wx^3}{6} + \frac{wa^3}{48} - \frac{Pa^2}{8},$$

since this depended only on the fact that $f'(a/2) = 0$. Using $f(0) = -\delta_1$ we then have

$$EIf(x) = \frac{Px^3}{6} - \frac{wx^4}{24} + \left(\frac{wa^3}{48} - \frac{Pa^2E}{8}\right)x - EI\delta_1,$$

so that $f(a/2) = -\delta_2$ then gives

$$EI(\delta_1 - \delta_2) = \frac{Pa^3}{48} = \frac{wa^4}{24 \cdot 16} + \left(\frac{wa^3}{48} - \frac{Pa^2}{8}\right) \cdot \frac{a}{2},$$

leading to

$$24EI(\delta_1 - \delta_2) = a^3\left(\frac{3}{16}W - P\right).$$

If we let λ be the appropriate constant from Hooke's law for the identical springs on each of the scales, this can be written as

$$\frac{24EI}{\lambda}(P - Q) = a^3\left(\frac{3}{16}W - P\right)$$

or

$$\frac{24EI}{\lambda}(3P - W) = a^2\left(\frac{3}{16}W - P\right).$$

Of course, it's the ratio P/W that interest us, so we write

$$\frac{24EI}{\lambda}\left(\frac{P}{W} - 1\right) = a^3\left(\frac{3}{16} - \frac{P}{W}\right)$$

which we can solve as

$$\frac{P}{W} = \frac{\dfrac{3a^3}{16} + \dfrac{24EI}{\lambda}}{a^3 + \dfrac{72EI}{\lambda}}.$$

Now this result at least seems reasonable at the extreme values for λ: If λ is very large (so that the springs on the scale are extremely strong, and consequently their compression is very tiny), then we get the answer

$$\frac{P}{W} \sim \frac{3}{16}$$

obtained previously, when we assumed no compression at all. On the other hand, if λ is very small, we get

$$\frac{P}{W} \sim \frac{1}{3},$$

i.e., the weight is distributed nearly equally among the supports.

However it seemed very unlikely that I would get an answer anywhere close to the results of the experiment if I inserted the proper value of λ (determined easily enough by seeing how far down a scale was depressed when it registered a particular weight), because even for a piece of wood, E is enormous, on the order of 10^6, so the formula would just give $\frac{P}{W} \sim \frac{1}{3}$.

So I ended this series of lectures confessing my inability to provide a good answer for this final problem. Perhaps some day I'll be able to persuade a mechanical engineer to sit down and explain to a mathematician just how this problem really should be handled.

~ THE END ~

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