Fall, 2014

MATH 4600/6600
PROBLEM SET #7

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DUE Friday, October 10, 2014.

Problems to work but not hand in:

Self-Test Problems and Exercises, Chapter 4: #5, 9, 22.

To hand in:

Chapter 4: #24, 27, 33, 42, 47, 48*, 51.

A. Here's another approach to the question of how many cards it should take to turn over our first ace. Number the non-aces 1, 2, ..., 48, and let X_i be the random variable that gives us 1 if we turn over card i before any ace, 0 otherwise. Explain why $E(X_i) = 1/5$ (Hint: Consider card i and the 4 aces) and finish the problem.

B. Back to the prizes we receive randomly at Burger King. Find the expected number of trips you will need to Burger King in order to receive a complete set of five prizes. (Hint: Let X_i, $i = 0, 1, 2, 3, 4$, be the number of additional trips you need after i distinct types of prizes have been collected in order to obtain a new type. What is $E[X_i]$?)

Theoretical exercises: Chapter 4: #7.

Graduate problem:

C. Here's yet another approach to the question of how many cards it should take to turn over our first ace. Along the way you'll prove a fascinating generalization of Theoretical Exercise 8 in Chapter 1.

(i) Let X denote the number of cards we must turn over to get our first ace. Check that

$$E[X] = 4\sum_{k=0}^{48} \frac{\binom{48}{k}}{\binom{52}{k+1}}.$$

(ii) Prove by induction that for positive integers m and n we have

$$\int_0^1 t^m(1-t)^n\,dt = \frac{1}{m+n+1}\cdot\frac{m!n!}{(m+n)!} = \frac{1}{m+n+1}\binom{m+n}{m}^{-1}.$$

(iii) Prove that $\sum\limits_{k=0}^{m}\binom{m}{k}\binom{m+n}{k+1}^{-1} = \dfrac{m+n+1}{n(n+1)}$. (Hint: Write $m+n = (k+1)+(m+n-k-1)$ and use part (ii) and then the binomial theorem for $\big(t+(1-t)\big)^m$.)

(iv) Finish the problem.

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*Compute this numerically using both binomial and Poisson random variables.