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MATH 4600/6600
EXAM #3

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Name: Solutions

Give your reasoning (briefly and clearly) on all problems. Very little credit will be given for unadorned arithmetic.

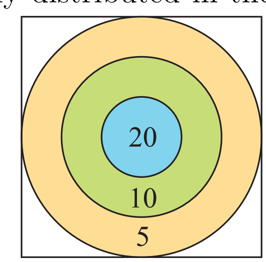
1. (20 points) A prisoner is trapped in a cell containing three doors. The first leads to a tunnel that returns him to his cell after 1 day of travel; the second leads to the outside after 2 days' travel and the third leads to the outside after 4 days' travel. If the prisoner selects doors 1, 2, and 3 with respective probabilities .3, .5, and .2, what is the expected number of days it takes the prisoner to reach freedom?

   Answer: Let X denote the number of days it takes for the prisoner to reach freedom and Y the door he takes. We have

$$\begin{aligned} E[X] &= E[E[X|Y]] \\ &= E[X|Y=1]P(Y=1) + E[X|Y=2]P(Y=2) + E[X|Y=1]P(Y=2) \\ &= (E[X]+1)(.3) + (2)(.5) + (4)(.2) = .3E[X] + 2.1\,, \text{ whence} \\ .7E[X] &= 2.1 \\ E[X] &= 3\,. \end{aligned}$$

   We expect it to take him 3 days to get to freedom.

2. (20 points) The accompanying dartboard is a square whose sides are of length 6. The three circles are centered at the center of the dartboard and have radii 1, 2, and 3, respectively. Point values are as indicated in the diagram. Darts that do not land within the circle of radius 3 do not score any points. Assume that each dart you throw will, independent of results of previous throws, land on a point uniformly distributed in the square.



   a. Find the probability that you score exactly 10 points on a throw of the dart.

      Answer: The total area of the square is 36, and the area of the 10-point ring is $\pi(2^2 - 1^2) = 3\pi$, so the probability that you score 10 points is $3\pi/36 = \pi/12 \approx 26.2\%$.

   b. Find the probability that your total score after 2 throws is 20 points.

      Answer: There are two ways you can score 20 points in 2 throws (call this event S). Case A is to hit the bullseye and then miss or vice versa; case B is to hit the 10-point ring twice. Since $S = A \cup B$ and $AB = \emptyset$,

$$P(S) = P(A) + P(B) = 2\cdot\frac{\pi}{36}\cdot\frac{36-9\pi}{36} + \left(\frac{\pi}{12}\right)^2 = \frac{\pi}{18}\left(1-\frac{\pi}{4}\right) + \left(\frac{\pi}{12}\right)^2 = \frac{\pi}{18} - \frac{\pi^2}{144} \approx 10.6\%.$$

c. Find the expected value of your score on a throw of the dart.

Answer: Let X denote the score on a throw of the dart.

$$\begin{aligned} E[X] &= 0 \cdot P(X=0) + 5 \cdot P(X=5) + 10 \cdot P(X=10) + 20 \cdot P(X=20) \\ &= 5 \cdot \frac{\pi(3^2-2^2)}{36} + 10 \cdot \frac{\pi(2^2-1^2)}{36} + 20 \cdot \frac{\pi(1^2)}{36} = \frac{75\pi}{36} = \frac{25\pi}{12} \approx 6.54\,. \end{aligned}$$

3. (20 points) Given the joint probability density $f(x,y) = 8xy$ on the region $R = \{0 \le x \le 1,\ 0 \le y \le x\}$, find

a. $E[X]$

Answer: $E[X] = \int_0^1 \int_0^x x \cdot 8xy\, dy\, dx = \int_0^1 4x^2y^2\Big]_0^x dx = \int_0^1 4x^4\, dx = \frac{4}{5}.$

b. $\mathrm{Var}(X)$

Answer: $\mathrm{Var}(X) = E[X^2] - E[X]^2$.
Well, $E[X^2] = \int_0^1 \int_0^x x^2 \cdot 8xy\, dy\, dx = \int_0^1 4x^3y^2\Big]_0^x dx = \int_0^1 4x^5\, dx = \frac{2}{3}$, so
$\mathrm{Var}(X) = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75}.$

c. $P(Y \le 1/2 \,|\, X = x)$

Answer: If $x \le 1/2$, we automatically have $Y \le X \le 1/2$, so the conditional probability is 1. If $1/2 < x \le 1$, we want

$$\frac{P(Y \le 1/2 \text{ and } X = x)}{P(X = x)} = \frac{\int_0^{1/2} 8xy\, dy}{\int_0^x 8xy\, dy} = \frac{8x \cdot \frac{1}{8}}{4x^3} = \frac{1}{4x^2}\,.$$

(Note that when $x = 1/2$, this formula does agree with our previous answer.)

4. (20 points) Sean's scores in bowling are given by a normal random variable with mean 170 and standard deviation 8. Ted's scores are given by a normal random variable with mean 82 and standard deviation 3. What is the probability that Sean's score is greater than twice Ted's score?

Answer: Let X be Sean's score and Y Ted's. We want $P(X > 2Y) = P(2Y - X < 0)$. Now, $Z = 2Y - X$ is a normal random variable with mean $2(82) - 170 = -6$ and standard deviation $\sqrt{(2 \cdot 3)^2 + (-8)^2} = 10$. (Any reasonable person would infer that X and Y are independent.) Thus,

$$P(Z < 0) = P\left(\frac{Z - (-6)}{10} < \frac{6}{10}\right) = \Phi(0.6) = .7257\,.$$

There is a probability of 72.6% that Sean's score is greater than twice Ted's.

Strictly speaking, bowling scores, being discrete, can only be *approximated* by a normal random variable. So, if you want to use the continuity correction, you will have

$$P(Z < -0.5) = P\left(\frac{Z-(-6)}{10} < \frac{5.5}{10}\right) = \Phi(0.55) = .7088\,.$$

5. (20 points) We deal 13 cards at random from a deck of 52 cards. Let X be the number of spades and Y be the number of aces we obtain. Find $E[X]$, $E[Y]$, and $\mathrm{Cov}(X,Y)$. (Hint: Consider the indicator functions for each of the relevant cards.)

Answer: Deal the 13 cards in order, and for $i = 1,\ldots,13$, let $X_i = 1$, $Y_i = 1$ if the i^{th} card is a spade or an ace, respectively. We proceed as above, but note that now $E[X_i] = P(\text{the } i^{\text{th}} \text{ card is a spade}) = 1/4$ and $E[Y_j] = P(\text{the } j^{\text{th}} \text{ card is an ace}) = 1/13$. Then, once again, we have

$$\begin{aligned} E[X] &= E[X_1 + X_2 + \cdots + X_{13}] = 13 \cdot \frac{1}{4} = \frac{13}{4} \\ E[Y] &= E[Y_1 + Y_2 + \cdots + Y_{13}] = 13 \cdot \frac{1}{13} = 1\,. \end{aligned}$$

It follows from our discussion of random deals of cards early in the semester that when $i \neq j$, X_i and Y_j are independent random variables, so $\mathrm{Cov}(X_i, Y_j) = 0$. On the other hand, when $i = j$, $X_iY_i = 1$ precisely when the i^{th} card is the $A\spadesuit$, which occurs with probability 1/52. So $\mathrm{Cov}(X_i,Y_i) = E[X_iY_i] - E[X_i]E[Y_i] = \frac{1}{52} - \frac{1}{4}\cdot\frac{1}{13} = 0$. Thus,

$$\mathrm{Cov}(X,Y) = \mathrm{Cov}\left(\sum X_i, \sum Y_j\right) = \sum_{i,j} \mathrm{Cov}(X_i, Y_j) = 0\,.$$

(Without observing independence for $i \neq j$, note that when $i \neq j$, $X_iY_j = 1$ precisely when the i^{th} card is a spade and the j^{th} card is an ace, and this occurs with probability $\frac{1}{4}\cdot\frac{1}{13} = \frac{1}{52}$:

$$\begin{aligned} &P(i^{\text{th}} \text{ card is a spade} \,|\, j^{\text{th}} \text{ card is an ace but not } A\spadesuit)P(j^{\text{th}} \text{ card is an ace but not } A\spadesuit) \\ &\quad + P(i^{\text{th}} \text{ card is a spade} \,|\, j^{\text{th}} \text{ card is } A\spadesuit)P(j^{\text{th}} \text{ card is } A\spadesuit) = \\ &\qquad \frac{13}{51}\cdot\frac{3}{52} + \frac{12}{51}\cdot\frac{1}{52} = \frac{39+12}{52\cdot 51} = \frac{1}{52}\,. \end{aligned}$$

This means that $E[X_iY_j] = E[X_i]E[Y_j]$, so $\mathrm{Cov}(X_i,Y_j) = 0$.)

Alternatively, let X_i, $i = 1, 2, \ldots, 13$, and Y_j, $j = 1, 2, 3, 4$, be the indicator functions for the 13 spades and 4 aces, respectively, in our random subset of 13 cards. Recall that $E[X_i] = P(\text{spade } i \text{ is in our subset}) = 13\cdot\frac{1}{52} = \frac{1}{4}$ and $E[Y_j] = P(\text{ace } j \text{ is in our subset}) = 13\cdot\frac{1}{52} = \frac{1}{4}$. Then we have

$$\begin{aligned} E[X] &= E[X_1 + X_2 + \cdots + X_{13}] = 13 \cdot \frac{1}{4} = \frac{13}{4} \\ E[Y] &= E[Y_1 + Y_2 + Y_3 + Y_4] = 4 \cdot \frac{1}{4} = 1\,. \end{aligned}$$

Let's now find $E[XY]$. Note that $X_iY_j = 1$ if and only if the i^{th} spade and j^{th} ace are both in our hand. Let's agree that $A\spadesuit$ is the first spade and the first ace, i.e., $X_1 = Y_1$. Now,

$$E[X_1Y_1] = P(A\spadesuit \text{ is in the hand}) = 13 \cdot \frac{1}{52} = \frac{1}{4};$$

and for $(i,j) \neq (1,1)$,

$$E[X_iY_j] = P(X_i = 1 \text{ and } Y_j = 1) = P(X_i = 1 \big| Y_j = 1)P(Y_j = 1) = \frac{12}{51} \cdot \frac{13}{52};$$

and so

$$E[XY] = E[X_1Y_1] + \sum_{(i,j)\neq(1,1)} E[X_iY_j] = \frac{1}{4} + 51 \cdot \frac{13}{52} \cdot \frac{12}{51} = \frac{13}{4}\,.$$

Therefore, as before, $\operatorname{Cov}(X,Y) = E[XY] - E[X]E[Y] = \dfrac{13}{4} - \dfrac{13}{4} \cdot 1 = 0$.

(One might ask whether, in *this* set up, X_i and Y_j are independent when $(i,j) \neq (1,1)$. Note that we saw that $P(X_i = 1 \big| Y_j = 1) = 12/51$, whereas $P(X_i = 1) = 13/52 = 1/4$. So they are not independent.)

N.B. Interestingly, we have another situation where the covariance is 0, but X and Y are *not* independent. For example $P(Y = 0 \big| X = 13) = 0$, but $P(Y = 0) = \binom{48}{13}/\binom{52}{13}$.

TABLE OF VALUES OF $\Phi(x)$

x	$\Phi(x)$
0.0	0.5000
0.05	0.5199
0.1	0.5398
0.15	0.5596
0.2	0.5793
0.25	0.5987
0.3	0.6179
0.35	0.6368
0.4	0.6554
0.45	0.6736
0.5	0.6915
0.55	0.7088
0.6	0.7257
0.65	0.7422
0.7	0.7580
0.75	0.7734
0.8	0.7881
0.85	0.8023
0.9	0.8159
0.95	0.8289
1.0	0.8413
1.05	0.8531
1.1	0.8643
1.15	0.8749
1.2	0.8849
1.25	0.8944
1.3	0.9032
1.35	0.9115
1.4	0.9192
1.45	0.9265
1.5	0.9332

x	$\Phi(x)$
1.5	0.9332
1.55	0.9394
1.6	0.9452
1.65	0.9505
1.7	0.9554
1.75	0.9599
1.8	0.9641
1.85	0.9678
1.9	0.9713
1.95	0.9744
2.0	0.9772
2.05	0.9798
2.1	0.9821
2.15	0.9842
2.2	0.9861
2.25	0.9878
2.3	0.9893
2.35	0.9906
2.4	0.9918
2.45	0.9929
2.5	0.9938
2.55	0.9946
2.6	0.9953
2.65	0.9960
2.7	0.9965
2.75	0.9970
2.8	0.9974
2.85	0.9978
2.9	0.9981
2.95	0.9984
3.0	0.9987

FOR EXTRA CREDIT

6. (10 points) Assume that the dartboard in problem 2 is centered at $(0,0)$ and the X- and Y-coordinates of the dart are independent normal random variables with mean 0 and standard deviation σ. If 2/3 of the darts score at least 10 points, find σ. (Hint: Use polar coordinates. Recall that the probability density function for a normal random variable with mean μ and variance σ^2 is $\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}$.)

7. (15 points) Passengers exiting an elevator do so independently of one another, and with the floors at which they exit uniformly distributed. The number of people Y who enter an elevator on the ground floor is a Poisson random variable with mean 15. If there are 30 floors above the ground floor at which the elevator can stop, what is the expected number of stops X that the elevator will make before all its passengers are discharged?