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MATH 4600/6600
EXAM #2

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Name: Solutions

Give your reasoning (briefly and clearly) on all problems. Very little credit will be given for unadorned arithmetic.

1. (20 points) Aaron's computer needs to be rebooted on average once a day.

   a. What is a sensible random variable X giving the number of reboots in one day? A random variable Y giving the number in t days?

   Answer: We recognize this as the typical Poisson random variable (like wrong numbers, accidents, contracting a cold). Since the mean value of the Poisson random variable with parameter λ is λ, we infer that X is a Poisson random variable with parameter $\lambda = 1$ and Y is a Poisson random variable with parameter $\lambda = t$. The probability mass function for X is $p(i) = e^{-1}\dfrac{1}{i!}$ and that for Y is $q(i) = e^{-t}\dfrac{t^i}{i!}$.

   b. What is the probability that it needs to be rebooted twice on a given day?

   Answer: $p(2) = e^{-1}/2 \approx .184$

   c. What is the probability that it requires no reboots in a given two-day period?

   Answer: With $t = 2$, $q(0) = e^{-2}$ (which, we observe, is the same as $p(0)^2$).

   d. What is the probability that it requires at least one reboot each day, five days in a row?

   Answer: $P(X \geq 1) = 1 - P(X = 0) = 1 - 1/e$, so, assuming the computer's behavior on different days gives independent events, the probability of at least one reboot each of five days is $\left(1 - \frac{1}{e}\right)^5 \approx .101$.

2. (25 points) A student takes a multiple choice exam where each question has 5 possible answers. He works a question correctly if he knows the answer; otherwise he guesses randomly. Suppose he knows the answer to 70% of the questions.

   a. What is the probability that on a question chosen at random the student gets the correct answer?

   Answer: Let E be the event that the student gets the correct answer. Let F be the event that the student knows the answer. Note that when he guesses, he has a probability of $1/5$ of getting the right answer. Then, by Bayes's formula,

$$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c) = (1)(.7) + (.2)(.3) = .76\,.$$

b. Given that the student gets the correct answer to this question chosen at random, what is the probability that he actually knew the answer?

Answer: $P(F|E) = \dfrac{P(EF)}{P(E)} = \dfrac{P(E|F)P(F)}{P(E)} = \dfrac{.7}{.76} \approx .921.$

Suppose now that there are 20 questions on the exam. Let X denote the number of questions that the student gets correct.

c. Find $E[X]$. State explicitly any formula(s) you are using.

Answer: X is a binomial random variable with $n = 20$ and $p = .76$, so $E[X] = np = 15.2$.

d. Find Var(X). State explicitly any formula(s) you are using.

Answer: By our formula, $\text{Var}(X) = np(1-p) = (15.2)(.24) \approx 3.648$.

3. (15 points) Let E be the event that at least 56 of 100 random children are girls. Assuming that the children have equal probability of being either a boy or a girl, use an appropriate normal random variable to estimate $P(E)$. Give details.

Answer: We wish to use a normal random variable X to approximate the binomial random variable with parameters $n = 100$, $p = 1/2$, so it should have $\mu = np = 50$ and $\sigma = \sqrt{np(1-p)} = \sqrt{25} = 5$. Using the continuity correction,

$$P(E) \approx P(X > 55.5) = P(X - 50 > 5.5) = P\left(\frac{X-50}{5} > 1.1\right).$$

Now, $Z = \dfrac{X-50}{5}$ is the standard normal random variable with $\mu = 0$ and $\sigma = 1$, and $P(Z > 1.1) = 1 - P(Z \leq 1.1) = 1 - \Phi(1.1)$. Looking at our table, $\Phi(1.1) = .8643$, so our answer is $\approx .136$.

4. (20 points) Consider the continuous random variable X with probability density function $f(x) = c\dfrac{1}{(1+x)^2}$, $x \in [0,1]$.

a. Determine c.

Answer: In order for $f(x)$ to be a probability density function, we need

$$1 = \int_0^1 f(x)\,dx = \int_0^1 \frac{c}{(1+x)^2}dx = -c\left(\frac{1}{1+x}\right)\Bigg|_0^1 = \frac{c}{2},$$

so $c = 2$.

b. Find $P(X < 1/2)$.

Answer: $P(X < 1/2) = \displaystyle\int_0^{1/2} f(x)\,dx = \int_0^{1/2} \frac{2}{(1+x)^2}dx = -2\left(\frac{1}{1+x}\right)\Bigg|_0^{1/2} = \frac{2}{3}.$

c. Find $E[X]$.

Answer: You can do the following integral either by parts or by substitution. I'll do the latter here, letting $u = 1 + x$ ($du = dx$):

$$E[X] = \int_0^1 xf(x)\,dx = 2\int_0^1 \frac{x}{(1+x)^2}\,dx = 2\int_1^2 \frac{u-1}{u^2}du = 2\int_1^2 \left(\frac{1}{u} - \frac{1}{u^2}\right)du$$
$$= 2\left(\ln u + \frac{1}{u}\right)\Big|_1^2 = 2\ln 2 - 1.$$

5. (20 points)

a. State and derive Markov's inequality.

Answer: For any nonnegative random variable X and $a > 0$, we have

$$P(X \geq a) \leq \frac{E[X]}{a}\,.$$

Proof: Let $f(x)$ be the probability density function of X. Since X takes on only nonnegative values,

$$E[X] = \int_0^\infty xf(x)\,dx \geq \int_a^\infty xf(x)\,dx \geq a\int_a^\infty f(x)\,dx = aP(X \geq a)\,,$$

as desired.

b. State Chebyshev's inequality.

Answer: For any random variable X and $k > 0$ (with finite mean μ and variance σ^2),

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

c. If X is an arbitrary random variable with mean 60 and variance 64, what can you say about $P(42 < X < 78)$?

Answer: First of all, $\sigma = 8$, so

$$P(42 < X < 78) = P(|X - 60| < 18) = P(|X-\mu| < \frac{9}{4}\sigma).$$

By Chebyshev's inequality,

$$P(|X-\mu| \geq \frac{9}{4}\sigma) \leq \frac{1}{(9/4)^2} = \frac{16}{81}\,;$$

thus, $P(|X-\mu| < \frac{9}{4}\sigma) \geq 1 - \dfrac{16}{81} = \dfrac{65}{81} \approx .802.$

d. Suppose X is a normal random variable. Now what can you say in part c? Show your work.

Answer: If X is a normal random variable,

$$P(|X-\mu| \leq \frac{9}{4}\sigma) = \Phi(2.25) - (1-\Phi(2.25)) = 2\Phi(2.25) - 1$$
$$= 2(.9878) - 1 \approx 97.6\%.$$

TABLE OF VALUES OF $\Phi(x)$

x	$\Phi(x)$
0.0	0.5000
0.05	0.5199
0.1	0.5398
0.15	0.5596
0.2	0.5793
0.25	0.5987
0.3	0.6179
0.35	0.6368
0.4	0.6554
0.45	0.6736
0.5	0.6915
0.55	0.7088
0.6	0.7257
0.65	0.7422
0.7	0.7580
0.75	0.7734
0.8	0.7881
0.85	0.8023
0.9	0.8159
0.95	0.8289
1.0	0.8413
1.05	0.8531
1.1	0.8643
1.15	0.8749
1.2	0.8849
1.25	0.8944
1.3	0.9032
1.35	0.9115
1.4	0.9192
1.45	0.9265
1.5	0.9332

x	$\Phi(x)$
1.5	0.9332
1.55	0.9394
1.6	0.9452
1.65	0.9505
1.7	0.9554
1.75	0.9599
1.8	0.9641
1.85	0.9678
1.9	0.9713
1.95	0.9744
2.0	0.9772
2.05	0.9798
2.1	0.9821
2.15	0.9842
2.2	0.9861
2.25	0.9878
2.3	0.9893
2.35	0.9906
2.4	0.9918
2.45	0.9929
2.5	0.9938
2.55	0.9946
2.6	0.9953
2.65	0.9960
2.7	0.9965
2.75	0.9970
2.8	0.9974
2.85	0.9978
2.9	0.9981
2.95	0.9984
3.0	0.9987