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MATH 4600/6600
EXAM #1

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Name: Solutions

Give your reasoning (briefly and clearly) on all problems. Very little credit will be given for unadorned arithmetic.

1. (15 points) Recall that a *void* in a bridge hand is the absence of some suit.
   a. How many different card distributions of a bridge hand (e.g., 5-3-2-3 denotes 5 spades, 3 hearts, 2 diamonds, and 3 clubs) can we have with no voids?

   Answer: We imagine 13 slots and put carets to separate the suits (first come the spades, then a caret, then the hearts, then a caret, then the diamonds, then a caret, and finally the clubs). There are 12 choices of locations for the 3 carets, so there are a total of $\binom{12}{3}$ possible different distributions of suits.

   b. How many different card distributions of a bridge hand are there if we allow voids?

   Answer: Since now we allow voids, we add three more slots and put the carets in (any) three of the slots. There are a total of $\binom{16}{3}$ possible different distributions of suits, allowing voids.

   c. How many different bridge hands are there with a 5-3-2-3 distribution?

   Answer: There are $\binom{13}{5}$ possible choices of 5 spades, $\binom{13}{3}$ possible choices of 3 hearts, $\binom{13}{2}$ possible choices of 2 diamonds, and $\binom{13}{3}$ choices of 3 clubs. Thus, there are $\binom{13}{5}\binom{13}{3}\binom{13}{2}\binom{13}{3}$ possible bridge hands with a 5-3-2-3 distribution.

2. (20 points) An urn contains 6 black balls and 5 red balls. Give explicitly an appropriate sample space and compute the probability that three balls of the same color are drawn consecutively from the urn
   a. if balls are *replaced* after each draw

   Answer: Let S be the set of all ordered triples consisting of $R_1, \dots, R_5$ (any of the 5 red balls) and $B_1, \dots, B_6$ (any of the 6 black balls). Thus, there are 11^3 elements in S. If we let E be the event that we draw three red balls and F be the event that we draw three black balls, there are 5^3 elements in E and 6^3 elements in F. Thus, $P(E \cup F) = \dfrac{5^3 + 6^3}{11^3}$.

b. if balls are *not replaced* after each draw

Answer: Now we let S be the set of all 3-element subsets of $\{R_1, \dots, R_5, B_1, \dots, B_6\}$. There are $\binom{11}{3}$ elements of S. With the same events (thought of as subsets of our new S) we have $\binom{5}{3}$ elements of E and $\binom{6}{3}$ elements of F. Thus, $P(E \cup F) = \dfrac{\binom{5}{3} + \binom{6}{3}}{\binom{11}{3}}$.

3. (10 points) Bart has a probability of 60% of getting the first test question correct and he has a probability of 40% of getting the second test question correct. Moreover, if he gets the first question correct, he has a probability of 50% of getting the second question correct. What is the probability that he'll get the first question correct if he gets the second one correct?

Answer: Let B_1 be the event that Bart gets the first question correct and B_2 be the event that he gets the second question correct. We are told that

$$P(B_1) = 0.6, \quad P(B_2) = 0.4, \quad \text{and} \quad P(B_2|B_1) = 0.5\,.$$

It follows that

$$0.5 = P(B_2|B_1) = \frac{P(B_1B_2)}{P(B_1)} \implies P(B_1B_2) = 0.3\,.$$

Thus,

$$P(B_1|B_2) = \frac{P(B_1B_2)}{P(B_2)} = \frac{0.3}{0.4} = 75\%\,.$$

4. (20 points) Suppose that Will arrives at work on time 70% of the time when he takes the first exit on the freeway, but is always late when he takes the second exit. However, he forgets to take the first exit 40% of the time. If he arrives late one day, what is the probability that he remembered to take the first exit?

Answer: Let E be the event that Will is late and F_i, $i = 1, 2$, be the events that Will takes the i^{th} exit. We are given that

$$P(F_1) = 0.6, \quad P(F_2) = 0.4, \quad P(E|F_1) = 0.3, \quad \text{and} \quad P(E|F_2) = 1\,.$$

From Bayes's Formula it follows that

$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) = (0.3)(0.6) + (1)(0.4) = 0.58\,.$$

Finally, the probability that he took the first exit given that he was late is

$$P(F_1|E) = \frac{P(F_1E)}{P(E)} = \frac{P(E|F_1)P(F_1)}{P(E)} = \frac{(0.3)(0.6)}{0.58} = \frac{18}{58} = \frac{9}{29} \approx 31\%\,.$$

5. (20 points) Eight people choose (randomly and independently) one of the letters A, B, C, or D.

a. What is the probability that at least one letter is chosen by no one?

Answer: Let E_i, $i = 1, 2, 3, 4$, be the probability that the i^{th} letter is chosen by no one. For any i we have $P(E_i) = \left(\frac{3}{4}\right)^8$. For $i \neq j$, we have $P(E_iE_j) = \left(\frac{2}{4}\right)^8$, and for i, j, k distinct, we have $P(E_iE_jE_k) = \left(\frac{1}{4}\right)^8$. Note that $E_1E_2E_3E_4 = \emptyset$.

By the inclusion/exclusion principle, we have

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\begin{aligned}
P(\bigcup_{i=1}^{4} E_i) &= \sum_i P(E_i) - \sum_{i \neq j} P(E_iE_j) + \sum P(E_iE_jE_k) \\
&= \binom{4}{1}\left(\frac{3}{4}\right)^8 - \binom{4}{2}\left(\frac{2}{4}\right)^8 + \binom{4}{3}\left(\frac{1}{4}\right)^8 \approx 37.71\%\,.
\end{aligned}
$$

b. What is the probability that every letter is chosen at least once?

Answer: Note that the event that every letter is chosen at least once is the complement of the event $\bigcup\limits_{i=1}^{4} E_i)$. Thus, the probability that every letter is chosen at least once is approximately $1 - 0.3771 = 62.29\%$.

6. (15 points) Lucy and Ricky play a game in which the first person to roll a 1 (using *one* fair, six-faced die) wins. If Lucy starts and they alternate turns, what is the probability that she will win the game? Let p_k denote the probability that she wins on *her* k^{th} turn.

a. What are p_1, p_2, p_k?

Answer: We have $p_1 = \frac{1}{6}$ and $p_2 = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \left(\frac{5}{6}\right)^2 \frac{1}{6}$ (remember that Ricky has to take his turn, too!). Following the pattern, the probability that Lucy wins on *her* k^{th} turn is $p_k = \left(\frac{5}{6}\right)^{2(k-1)} \frac{1}{6}$.

b. Find $\sum\limits_{k=1}^{\infty} p_k$.

Answer: Recall that $\sum\limits_{j=0}^{\infty} r^j = \dfrac{1}{1-r}$ when $|r| < 1$.

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\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{2(k-1)} = \frac{1}{6}\sum_{j=0}^{\infty}\left(\frac{25}{36}\right)^j = \frac{1}{6} \cdot \frac{1}{1-\frac{25}{36}} = \frac{6}{11}\,.
$$

c. Solve the problem an alternative way. If P is the probability that Lucy wins, give the probability that Ricky wins in terms of P in two different ways. Now finish.

Answer: Let P denote the probability that Lucy wins (eventually) and Q the probability that Ricky wins (eventually). Then $P+Q=1$. (Yes, there's always a remote possibility that the game goes on forever, but this eventuality has probability 0 and I did not expect you to worry about this.)
But let's think about the game from Ricky's perspective. It's identical to the game from Lucy's perspective, except for the fact that in order for Ricky to get a turn, Lucy must first roll something other than a 1. There's a probability of 5/6 that this happens, and then everything proceeds identically. That is, $Q = \frac{5}{6} \cdot P$. (Officially, if E is the event that Ricky wins and F is the event that Lucy rolls a 1 on her first turn, we have:

$$Q = P(E) = P(E|F)P(F) + P(E|F^c)P(F^c) = 0 \cdot \frac{1}{6} + P \cdot \frac{5}{6}.)$$

Thus, $1 = P + Q = P + \frac{5}{6}P = \frac{11}{6}P$, and so $P = 6/11$, as before.

FOR EXTRA CREDIT

7. (10 points) Suppose a bin contains 15 tennis balls, of which 8 have never been used. Aaron comes along, takes out two at random, hits with them for an hour, and then returns them to the bin. David comes along a while after and takes out two balls at random. What is the probability that both of these balls have been used before?