Spring, 2015 | T. Shifrin

MATH 3510(H)
PROBLEM SET #7

DUE Wednesday, February 25, 2015.

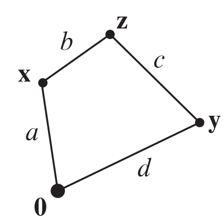
Problems to work but not hand in:

§8.2: #4a,c.

Problems to turn in:

§6.3: #4 (2), 7b,e (4), 9 (3), 12 (4).

A. (3) Prove that $X = \{\mathbf{x} \in \mathbb{R}^4 : \|\mathbf{x}\|^2 = 30 \text{ and } x_1^3 - x_2^3 + 2x_3^3 - 3x_4^3 = 0\}$ is a 2-dimensional manifold and give a basis for its tangent space at $\mathbf{a} = \begin{bmatrix} 4 \\ -1 \\ 2 \\ 3 \end{bmatrix}$.



B. (4) Consider the linkage in $\mathbb{R}^2$ pictured to the right. The lengths a, b, c, and d are fixed, positive numbers with $a + b \neq c + d$, $a \neq b + c + d$, and $d \neq a + b + c$. The origin $\mathbf{0}$ is fixed, and $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are allowed to move. Prove that the set of all "accessible" linkage positions $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} \in \mathbb{R}^6$ forms a manifold. What is its dimension? (Hint: It might be preferable to think about $DF^\top$ as three rows of vectors.)

C. (3) Let $M = \{\mathbf{x} \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4\,, x_1x_2 + x_3x_4 = 0\}$, $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

(i) Prove that M is manifold. Of what dimension?

(ii) Give a basis of the tangent space of M at $\mathbf{a}$.

(iii) Consider $\mathbf{g}\colon (-\pi,\pi)\times(-\pi,\pi) \to \mathbb{R}^4$, $\mathbf{g}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \cos u + \sin v \\ \cos u - \sin v \\ \sin u - \cos v \\ \sin u + \cos v \end{bmatrix}$.

Check that $\mathbf{g}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{a}$ and that $\mathbf{g}$ gives a parametric representation of M near $\mathbf{a}$. Use this representation to give a basis of the tangent space of M at $\mathbf{a}$.

§8.2: #2 (2), 3 (2), 5 (3).

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Challenge problems (Turn in separately):

§6.3: #13 (4), 14 (3).

C. (4) Using the notation of part **3** of the Definition on p. 262 of a k-dimensional manifold, perhaps shrinking W, prove that there is a smooth function $\mathbf{h}\colon W \to U$ whose restriction to $M \cap W$ is $\mathbf{g}^{-1}$. (Hint: Without loss of generality, assume $\mathbf{g}(\mathbf{u}_0) = \mathbf{p}$ and write $\mathbf{g}(\mathbf{u}) = \begin{bmatrix} \mathbf{g}_1(\mathbf{u}) \\ \mathbf{g}_2(\mathbf{u}) \end{bmatrix} \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, where $D\mathbf{g}_1(\mathbf{u}_0)$ is nonsingular.)

§8.1: #1 (3), 2 (2).