Spring, 2015 | MATH 3510(H) | T. Shifrin

MATH 3510(H)
PROBLEM SET #2

DUE Wednesday, January 21, 2015.

Problems to work but not hand in:

§7.2: #1a,c, 2b,f, 8a, 11, 12a,d, 18.

§7.3: #1, 5, 9, 17, 20.

Problems to turn in:

WeBWork Homework 2.

Now log in at
`https://webwork.math.uga.edu/webwork2/MATH3510_Shifrin_S15`
using, as before, your UGAMyID as your username [*no caps*] and all but the last digit of your 81x number, formatted as 81xxxxxxx, as your password. You can then change your password.

A. (3) Let $\Omega \subset \mathbb{R}^3$ be that portion of the cube $0 \le x, y, z \le 1$ lying above the plane $y + z = 1$ and below the plane $x + y + z = 2$. Evaluate $\displaystyle\int_\Omega x\, dV$.

§7.2: #19 (3), 20 (5), 21 (2), 22* (4).

Challenge problems (Turn in separately):

§7.2: #23 and 24 (5).

B. (5) (an old Putnam problem) Evaluate (with proof) the integral

$$\int_0^{\pi/2} \frac{d\theta}{1 + (\tan\theta)^\pi}\,.$$

(Hint: Problem 7.2.20 or, rather, its extension to improper integrals may well be useful.)

—OVER—

*Be sure to justify limits, e.g., $\lim_{x\to\infty} g(x)$, with care. It is in general false—as those of you in MATH 3100(H) will find—that if $\phi_N(t) \to \phi(t)$ as $N \to \infty$, then $\int_0^1 \phi_N(t)\,dt \to \int_0^1 \phi(t)\,dt$. Hint: $e^{-x^2(t^2+1)} = e^{-x^2}e^{-x^2t^2}$.

C. (4) (inspired by a more recent Putnam problem) Suppose f is continuous on $\mathbb{R}^2$ and $\displaystyle\int_R f\,dA = 0$ for *every* rectangle R with area 1. Prove that the average values of f on each of the edges of each such R are all equal. For further extra credit: Deduce that $f = 0$.

D. (5) Suppose $f\colon [a,b]\times[c,\infty)\to\mathbb{R}$ is continuous, $\dfrac{\partial f}{\partial x}$ is continuous, $\displaystyle\int_c^\infty f\begin{pmatrix}x\\y\end{pmatrix} dy$ exists for all $x\in[a,b]$ and, moreover, $\displaystyle\int_N^\infty \frac{\partial f}{\partial x}\begin{pmatrix}x\\y\end{pmatrix} dy$ converges to 0 uniformly as $N\to\infty$ for $x\in[a,b]$.[†] Extend the result of Problem 7.2.20 to prove that, with these hypotheses, if we set $F(x) = \displaystyle\int_c^\infty f\begin{pmatrix}x\\y\end{pmatrix} dy$, then $F'(x) = \displaystyle\int_c^\infty \frac{\partial f}{\partial x}\begin{pmatrix}x\\y\end{pmatrix} dy$. You will probably want to use the result from MATH 2410/3100 that if ϕ_N and ϕ are integrable on $[a,b]$ and $\phi_N\to\phi$ uniformly, then $\displaystyle\int_a^b \phi(x)\,dx = \lim_{N\to\infty}\int_a^b \phi_N(x)\,dx$.

E. (3) Buried in problem **D** is an appropriate statement of Fubini's Theorem for improper integrals. How does this jibe with the function

$$f\begin{pmatrix}x\\y\end{pmatrix} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad\text{on}\quad [1,\infty)\times[1,\infty)?$$

§7.3: #24[‡] (5) (one of my all-time favorites).

[†]If $\phi_N,\phi\colon [a,b]\to\mathbb{R}$, we say ϕ_N converges to ϕ uniformly if, given $\varepsilon>0$, there is N_0 so that for all $N\ge N_0$ and for all $x\in[a,b]$ we have $|\phi_N(x)-\phi(x)|<\varepsilon$.

[‡]This problem may well have been done by Archimedes, using no calculus at all. Bonus points for that.