Fall, 2014

MATH 3500(H)
PROBLEM SET #12

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DUE Wednesday, November 12, 2014.

Problems to work but not hand in:

§5.1: #4a, 9.

§5.2: #1a,i,g, 3.

Problems to turn in:

No WeBWork this week.

A. (2) Suppose $X \subset \mathbb{R}^n$ is compact and $\{\mathbf{x}_k\} \subset X$ is a sequence.

(i) Can it be the case that $\|\mathbf{x}_k - \mathbf{x}_{k+1}\| \geq 1$ for all $k \in \mathbf{N}$? Prove your answer.

(ii) Can it be the case that $\|\mathbf{x}_k - \mathbf{x}_{k+1}\| \geq k$ for all $k \in \mathbf{N}$? Prove your answer.

§5.1: #7 (3), 10 (3), 11* (3), 12 (3), 13 (4).

B. (3) Suppose $f\colon B(\mathbf{0}, 5/4) \to \mathbb{R}$ is a $\mathcal{C}^1$ function and $\boldsymbol{\nabla} f(\mathbf{x}) \cdot \mathbf{x} > 0$ for all $\mathbf{x}$ with $\|\mathbf{x}\| = 1$. Prove that there is a point $\mathbf{a} \in B(\mathbf{0}, 1)$ with $\boldsymbol{\nabla} f(\mathbf{a}) = \mathbf{0}$.

C. (4) Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable and $f\begin{pmatrix} x \\ y \end{pmatrix} = f\begin{pmatrix} y \\ x \end{pmatrix}$ for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

(i) Show that $\begin{bmatrix} x \\ y \end{bmatrix}$ is a critical point of f if and only if $\begin{bmatrix} y \\ x \end{bmatrix}$ is also. Hint: Use the chain rule.

(ii) Contrary to popular belief, there need not be critical points with $x = y$. Find *all* the critical points of

$$f\begin{pmatrix} x \\ y \end{pmatrix} = 2(x+y) + xy(1+x+y)\,.$$

(Hint: Be judicious with your algebra.)

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*Hint: Choose $\mathbf{x}_k \in S_k$ for each $k \in \mathbb{N}$.

D. (3) Suppose $A \subset \mathbb{R}^n$ is closed but unbounded, $f\colon A \to \mathbb{R}$ is continuous, and $f(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$ with $\mathbf{x} \in A$.† Prove that there is $\mathbf{y} \in A$ so that $f(\mathbf{x}) \geq f(\mathbf{y})$ for all $\mathbf{x} \in A$.

Hint: Pick an arbitrary $\mathbf{a} \in A$ and, referring to the footnote, take $N = f(\mathbf{a})$. Using this, define a compact set $X \subset A$ and apply Theorem 1.2. Last, argue that $f(\mathbf{x}) \geq f(\mathbf{y})$ for *all* $\mathbf{x} \in A$.

E. (3) If the sum of three nonnegative numbers is 12, what must these numbers be in order for the product of one of them, the square of the next, and the cube of the last to be as large as possible? (Be sure to set up a function of *independent variables* on an appropriate domain and argue why it has a maximum. Then examine the frontier of your domain.)

F. (4) (Be sure to set up a function of *independent variables* on an appropriate domain and argue why it has a maximum. Then examine the frontier of your domain.)

(i) Suppose x, y, z are the angles of a triangle. Prove that the function $\sin x + \sin y + \sin z$ has a maximum value and find it.

(ii) Among all triangles inscribed in the unit circle, prove that the equilateral triangle has the greatest area.

Challenge problems (Turn in separately):

§5.1: #5 (3), 6 (2).

G. (4) Here we explore the function $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $f\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \|\mathbf{x}-\mathbf{y}\|$.

(i) Prove that f is continuous by using the triangle inequality to show that if $\|\mathbf{x}-\mathbf{a}\| < \varepsilon/2$ and $\|\mathbf{y}-\mathbf{b}\| < \varepsilon/2$, then

$$\big|\|\mathbf{x}-\mathbf{y}\| - \|\mathbf{a}-\mathbf{b}\|\big| < \varepsilon\,.$$

(ii) If $X \subset \mathbb{R}^n$ is compact, show that $X \times X = \left\{\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} : \mathbf{x}, \mathbf{y} \in X\right\} \subset \mathbb{R}^n \times \mathbb{R}^n$ is compact.

(iii) Deduce that there are $\mathbf{x}_0, \mathbf{y}_0 \in X$ so that $f\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} \leq f\begin{pmatrix}\mathbf{x}_0\\ \mathbf{y}_0\end{pmatrix}$ for all $\mathbf{x}, \mathbf{y} \in X$. The latter value is called the *diameter* of X.

†More precisely, this means that given any $N > 0$, there is $R > 0$ so that whenever $\mathbf{x} \in A$ and $\|\mathbf{x}\| > R$, we have $f(\mathbf{x}) > N$.