8320: Riemann Roch: a proof for non singular plane curves

We have given a proof of Riemann Roch along the exact lines of the original argument by Riemann and Roch. It thus suffers from the same omissions as their proof, namely a rigorous existence argument for the elementary differential forms of first and second kind. Riemann attempted to prove that for an arbitrary compact complex manifold of dimension one, and his analytic arguments have been supplemented by Hilbert and others. If we restrict to the case of Riemann surfaces that arise from plane curves, it is easier to produce these forms, as Riemann himself says, on pages 108-109 of the Kendrick Press translation of his works. On p. 108 he explicitly shows how to write down g linearly independent holomorphic one forms on a plane curve, and on page 109, lines 5-7 he states he could do so in the same way for the forms of second and third kinds, but that he “will not dwell on this now”.

We begin now to supplement our account of his argument, using the expressions of Riemann to write down the forms of first kind, i.e. the holomorphic forms, at least on a non singular plane curve. We could also try to give a similar approach to writing down forms of second kind, as partially explained in the book Plane algebraic curves, by Brieskorn and Knorrer, where an example is given on pages 656-657. But it is easier to finesse the existence question for these forms of second kind, and give instead an algebraic proof of the RRT based on playing back and forth with the Riemann inequality and the known forms of first kind, to deduce that the Riemann Roch formula must hold, again in the case of smooth plane curves. This argument may be due to Brill and Noether, who in their paper made an explicit attempt to render the proof of Riemann's result for plane curves, entirely algebraic. This argument appears in the book Introduction to algebraic curves, by Phillip Griffiths, and is also outlined in the exercises of Appendix A to chapter I of the book Geometry of algebraic curves, by Arbarello, Cornalba, Griffiths, and Harris.

Thus we will use topology and algebra together to establish the existence and the dimension of the space of holomorphic forms, and then use algebra to show this already suffices to prove the full Riemann Roch formula. I.e. for plane curves, the existence of g independent holomorphic differentials, plus the residue theorem, is enough to deduce the full RRT. We explain this next. In both cases we give an upper bound for the number of sections using analysis, and then give a lower bound using polynomials to represent them. We start with the canonical line bundle, which is the essential case.

Starting from a divisor D, we have shown how to construct a line bundle O(D) and a section of O(D) whose divisor equals D. The divisors linearly equivalent to D are the divisors of other sections of O(D). The vector space of holomorphic section of O(D) will be denoted either H^0(O(D)) or just H^0(D). If we want to remind that our line bundle is given on the curve X we may write H^0(X;O(D)), or H^0(X;D). That way we can distinguish between the space of sections of a line bundle on P^2 and the sections of the restriction of the same bundle to a plane curve X. If n is a non negative integer, we will denote by H^0(P^2;O(n)) the vector space of homogeneous polynomials of degree n on P^2. These are the holomorphic sections of a line
bundle denoted $O(n)$. Our first goal is to find an isomorphism between the space $H^0(X;K)$ of global holomorphic one forms on a non singular plane curve $X$ of degree $d$, and the space $H^0(O(d-3))$ of homogeneous polynomials on $P^2$ of degree $d-3$.

**An upper bound on the number of independent holomorphic forms on $X$**

On a compact connected Riemann surface $X$ the only global harmonic functions are constants, by the maximum modulus principle for harmonic functions. Thus the only exact harmonic differential is zero. If $H^0(K)$ denotes the space of global holomorphic one forms, and $H^0(Kbar)$ denotes the space of global conjugate-holomorphic one forms, this implies the map $H^0(K)+H^0(Kbar)\rightarrow H^1(X,k)$ taking a one form to integration over that one form, is injective. Since the cohomology space has complex dimension $2g$, and the map sending a holomorphic form $w$ to its complex conjugate $wbar$ is a real linear isomorphism between $H^0(K)$ and $H^0(Kbar)$, they both have complex dimension $\leq g$.

**A lower bound on the number of independent holomorphic forms**

Now we do as Riemann did (p.108, Kendrick Press translation of Riemann’s works) and write down holomorphic differentials on a plane curve $f(s,t) = 0$, in the form $dt/\partial f/\partial s$. Recall that we can pull a line bundle back by a holomorphic map, and in particular we can restrict a line bundle from the plane to a plane curve. The fundamental line bundles on the plane are the ones $O(n)$ whose global sections for $n \geq 0$ are the polynomials of degree $n$. When we restrict a line bundle from the plane to a curve, its sections also restrict. The restricted bundle may have more sections that are not restrictions of sections of the bundle on the plane, but we get a lower bound this way on the number of sections of the restricted bundle. Recall, if we knew the RRT we would know that the only line bundle on $X$ of degree $2g-2$ having $g$ independent sections is the canonical bundle. I.e. if $g \geq 1$, and if there are $g$ sections, we can choose a non negative divisor $D$ representing the line bundle. Then the RRT theorem says if degree $D = 2g-2$ and $\dim L(D) = g$, that $\dim L(K-D)=1$. That says there is a holomorphic differential vanishing on the divisor $D$. But a holomorphic differential only has $2g-2$ zeroes, so it must vanish exactly on $D$. I.e. then $D$ is a divisor of a holomorphic differential, so $D$ is a canonical divisor.

We know a smooth plane curve $X$ of degree $d$ has genus $g = (1/2)(d-1)(d-2)$. Look at the sections of $O(d-3)$ restricted to $X$. Since $d-3 < d$, no non zero polynomial of degree $d-3$ can vanish identically on $X$, so the restriction map is injective on sections. Thus the restriction of $O(d-3)$ to $X$ has at least as many sections as there are polynomials of degree $d-3$ on $P^2$. That space of polynomials is “well known” to have dimension $(1/2)(d-1)(d-2)$. In fact the space of polynomials of degree $r$ on $P^2$ has dimension $(1/2)(r+2)(r+1)$. One way to see this is to write out $r+2$ dots in a sequence, and select two of these dots. This separates the remaining $d$ dots into three ordered subsets containing say $a,b,c$ dots. This corresponds to the monomial
In this way the number of distinct monomials in $X,Y,Z$ equals the binomial coefficient “$r+2$ choose 2”.

For example when $r = 5$ the sequence of 7 dots: . . . . . . . where we have bolded the 3rd and 6th dots, leaves subsets of (2,2,1) unbolded dots, hence defines the monomial $X^2.Y^2.Z$.

Thus the restricted line bundle $O(d-3)$ does have at least $(1/2)(d-1)(d-2) = g$ sections on a smooth plane curve $X$ of degree $d$. Moreover by Bezout, the divisors cut out on $X$ by these curves have degree $d(d-3)$. A short calculation shows $2g-2 = (d-1)(d-2)-2 = d^2 - 3d + 2 - 2 = d(d-3)$. So the restricted line bundle $O(d-3)$ on $X$ does have degree $2g-2$ and at least $g$ sections. We claim this line bundle is in fact the canonical bundle. [The reason we don't know this yet is now we are not assuming the RRT.]

To show the restriction of $O(d-3)$ to $X$ is isomorphic to the canonical bundle on $X$, we claim it suffices to find one holomorphic differential on our plane curve $X$, whose divisor is the same as the divisor cut out on $X$ by a polynomial of degree $d-3$. Then the canonical bundle $K$ and the bundle $O(d-3)$ have sections with the same divisor. Then the quotient of those sections will be a section of the quotient line bundle $K.D^{(d-1)}$ with divisor zero, i.e. a never zero holomorphic section. Such a section will define a trivialization of the line bundle $K.D^{(-1)}$, i.e. an isomorphism with the trivial bundle $X \times k$. This implies the two bundles $K.D$, or $O(K)$, $O(D)$, are isomorphic as claimed, hence have the same number of sections. Next we write down a differential form on a degree $d$ plane curve $X$, whose divisor equals the divisor of a polynomial of degree $d-3$. [If you wonder where the -3 is coming from, recall that $P^1$ has a differential $dz$ with divisor of degree -2. Similarly, $P^2$ has canonical divisors of degree -3, $P^3$ has canonical divisors of degree -4, and so on. Thus a canonical divisor on a surface of degree $d$ in $P^3$ would be cut out by a polynomial of degree $d-4$.]

**Differential forms on the Fermat curve**

OK lets write down a differential one form on a Fermat curve \{x^d + y^d + z^d = 0\}. In \{x\neq 0\} put $s = y/x$, $t = z/x$; and in \{z\neq 0\} put $u = x/z$, $v = y/z$. Then $s = v/u$, and $t = 1/u$. In the affine $(s,t)$ plane $x\neq 0$, our curve has equation \{1 + s^d + t^d = 0\}. To get a relation between differentials which holds on this curve, take $d$ of this equation: \(s^{(d-1)} ds + t^{(d-1)} dt = 0\). Thus $w = dt/s^{(d-1)} = -ds/t^{(d-1)}$ on our curve. Since $t$ and $s$ have no common zeroes on the curve \{1 + s^d + t^d = 0\} in the $(s,t)$ plane, this rational differential form is holomorphic on that affine part of our curve.

What about its divisor of zeroes? When $s\neq 0$, the partial derivative of the equation defining the curve is $s^{(d-1)} \neq 0$, so by the implicit function $t$ is a local coordinate there. That means $dt$ is non zero at such points, so $dt/s^{(d-1)}$ is holomorphic with no zeroes at points of the curve where $s\neq 0$. Where $t\neq 0$, similarly $s$
is a local parameter, and thus $ds$ is $\neq 0$, so $ds/t^{(d-1)}$ is non zero, and $w$ has no zeroes or poles in the affine $(s,t)$ plane \{$x\neq 0$\}.

It follows the divisor of $w$ is supported in the intersection of our curve with the line \{$x=0$\}. There are no points of our curve with $x = z = 0$, so all points of the divisor $x = 0$ on our curve are in the set $z \neq 0$. To compute that divisor we use coordinates $u = x/z$, $v = y/z$. Since $s = v/u$, and $t = 1/u$, thus $w = dt/s^{(d-1)} = d(1/u)/(v/u)^{(d-1)} = -u^{(d-3)}du/[u^{(d-1)}v^{(d-1)}] = -u^{(d-3)}du/v^{(d-1)}$. Since $u = x/z$, in the affine $(u,v)$ plane \{$z\neq 0$\} this divisor equals the one cut by \{$x^{(d-3)}=0$\}. Thus the divisor cut by $x^{(d-3)} = 0$ is the same as the one cut in \{$z\neq 0$\} by $u^{(d-3)} = 0$. I.e. the divisor $\text{div}(w)$ of our one form, is exactly the same as the divisor $x^{(d-3)}$ defining the line bundle $O(d-3)$ on $X$, as claimed. It follows that all $g$ linearly independent polynomials of degree $d-3$ define divisors of holomorphic differentials on the Fermat curve $X$, which thus has at least $g$ independent forms of first kind. By the upper bound already found there are thus exactly $g$ independent forms of first kind on $X$. The same argument works on every smooth plane curve as we show next.

\textbf{Differential forms on a smooth plane curve of degree $d$}

Let $X$: \{$F(x,y,z) = 0$\} be a smooth curve in $P^2$ of degree $d$. Then the partial derivatives of $F$ have no common zeroes, and the same holds for the partials of the affine equations. In $U$:\{$x\neq 0$\} we have affine coordinates: $s = y/x$, and $t = z/x$. In these affine coordinates the curve $X$ has equation $F(x,y,z)/x^d = F(1,s,t) = G(s,t) = 0$. Taking $d$ of the equation gives as above: $(\partial G/\partial s) \ ds + (\partial G/\partial t) \ dt = 0$. Since $X$ is non singular, at every point of $X$ in $U$ we have either $\partial G/\partial s \neq 0$ or $\partial G/\partial t \neq 0$. Thus the form $w = ds/(\partial G/\partial t) = -dt/(\partial G/\partial s)$, is holomorphic on the part of $X$ in the affine plane \{$x\neq 0$\}. Moreover, again by the implicit function theorem, $s$ is a coordinate on $X$ where $(\partial G/\partial t) \neq 0$, and $t$ is a coordinate where $\partial G/\partial s \neq 0$, so these representations show that $w$ has no zeroes as well as no poles in \{$x\neq 0$\}. Consequently its divisor is supported in the line \{$x=0$\}. A calculation like the one above shows that in the set $V$:\{$y\neq 0$\}, with coordinates $u = x/y$, $v = z/y$, the form $w = -u^{(d-3)}du/(\partial H/\partial v)$, where $s = 1/u$, $t = v/u$ and $H(u,v) = u^d.G(1/u, v/u)$. It follows again that $\text{div}(w)$ is the divisor of \{$x^{(d-3)} = 0$\}. Thus $O(K) \approx$ the restriction to $X$ of $O(d-3)$ and hence $O(K)$ has at least as many sections as there are polynomials of degree $d-3$ on $P^2$, i.e. a $g$ dimensional space of them. This plus the upper bound implies $\dim H^0(K) = g$.

\textbf{Remark: Curve with nodes; “adjoint polynomials” of degree $d-3$}

We know a general Riemann surface does not arise from a smooth plane curve, but in fact all Riemann surfaces do arise from nodal plane curves. I.e. we have proved that every Riemann surface embeds in some projective space. Then since the union of the secant lines to an embedded curve has dimension $\leq 3$, and the union of the tangent lines has dimension $\leq 2$, we can always project down to $P^3$ from a
point that misses all secants and tangents, and to $P^2$ from a point that at least misses all tangents. Thus every Riemann surface embeds in $P^3$ and immerses in $P^2$. One can also show that we can project to $P^2$ from a point of $P^3$ that avoids all trisecants, meets only a finite set of secants, and meets no secant lying in a bitangent plane. Hence every Riemann surface $X$ immerses in $P^2$ by a map of degree one, as a curve $C$ of some degree $d$ with at worst nodes. Then the same argument given above shows that holomorphic differentials on $X$ arise from polynomials of degree $d-3$ which pass through all the nodes of $C$. Since each node of $C$ reduces the genus of $X$ by one, and imposes at most one condition on polynomials of degree $d-3$, there are again at least $g$ holomorphic forms from this construction. This proof is detailed in Phillip Griffiths’ book mentioned above.

**Restricting polynomials gives lower bounds for sections of all line bundles**

We have computed the sections of the canonical bundle $K$ for plane curves in two steps. We used topology and analysis to give upper bounds on the number of sections, and then we used polynomials to simply write down enough sections to give a sharp lower bound. We will show next that the same approach computes the precise number of sections of every line bundle $O(D)$ on a plane curve. Now that we know the number of sections of $K$, we will also get an upper bound on the number of sections of any line bundle from the residue theorem, and then again polynomials will provide a sharp lower bound for the full space of sections. For plane curves of degree three we already have enough for RRT.

**RRT for plane cubics**

Recall that on a curve of genus one, the RRT is very simple: if $D$ is a non-negative divisor, then $\dim L(D) = \deg(D)$. Let’s try this on a plane cubic $C$. Take a non-negative divisor $D$ of degree $d$. Pass a curve $G$ of degree $m$ through all $d$ points of $D$. By Bezout, it will meet the curve $C$ further in a divisor $E$ of $3m-d$ points. Now consider all curves of degree $m$ in $P^2$. They have dimension $(1/2)(m+2)(m+1)$. Now consider the subspace that contain the excess divisor $E$ where $G$ meets $C$ outside $D$. This subspace has dimension at least $(1/2)(m+2)(m+1) - (3m-d)$. We want to compute the linear systems swept out by these curves. That subspace has a smaller subspace of curves that contain the entire curve $C$. These consist of products of the polynomial defining $C$ times a polynomial of degree $m-3$. That subspace thus has the dimension $(1/2)(m-1)(m-2)$ of all polynomials of degree $m-3$. Two polynomials of degree $m$ that differ by a polynomial containing $C$ cut out the same divisor on $C$. Thus the linear system swept out on $C$ by curves of degree $m$ containing the divisor $E$, is defined by a space of dimension $\geq (1/2)(m+2)(m+1) - (3m-d) - (1/2)(m-1)(m-2)$. Since these divisors are linearly equivalent to $D$, I hope this number equals $d$. Yes! It does! I.e. multiplying by 2, we hope to get $2d$, and in fact we do get:
This actually proves the RRT for non negative divisors on plane cubics, since it is easy to prove that \( \dim L(D) \leq \deg(D) \) for all \( D \geq 0 \). i.e., we showed earlier the upper bound \( \dim L(D) \leq 1 + \deg(D) \) in all cases. However we also know that if the upper bound is achieved, then there is a non negative divisor \( D \) of degree \( d \) with \( \dim L(D) = d+1 \). That defines a degree one map of \( X \) to a spanning curve of degree \( \leq d \) in \( \mathbb{P}^d \). We also know a spanning curve in \( \mathbb{P}^d \) has degree \( \geq d \), so the degree of the embedded curve is \( d \) and that curve is smooth. Then projecting further to \( \mathbb{P}^2 \) from points on our embedded curve shows \( X \) would be isomorphic to \( \mathbb{P}^1 \). So \( g(X) = 1 \) implies that \( \deg(D) \) is not just a lower bound but also an upper bound for \( \dim L(D) \).

### Riemann’s inequality for non singular plane curves

I claim the same lower bound computation works for all non singular curves of degree \( d \). Let’s try it, again by calculating dimensions of spaces of polynomials. We follow closely the outline in [ACGH], exercises.

Let \( C \) be a non singular plane curve of degree \( n \), and let \( D \) be a divisor on \( C \) of degree \( d \). We cannot assume \( D \geq 0 \), if we want later to use just the degree of \( D \) to conclude from RRT that \( |D| \) is non empty. i.e. we are going to want to know that all divisors of degree \( \geq \deg \) are linearly equivalent to non negative divisors. So assume \( D = D' - D'' \) where \( D' \), \( D'' \) are non negative of degrees \( d' \), \( d'' \) and \( \deg(D) = d = d' - d'' \). Then choose any curve \( G \) of degree \( m \) passing through the points of the divisor \( D' \) on the curve \( C \), and meeting \( C \) further in a divisor \( E \) of degree \( (nm - d') \).

Now consider the space \( V \) of polynomials \( H \) of degree \( m \) in \( \mathbb{P}^2 \) containing \( E + D'' \). If \( H.C = E + D'' + B \), then since \( G,H \) have the same degree, the divisors \( G.C \) and \( H.C \) are linearly equivalent on \( X \), i.e. \( E + D'' \approx E + D'' + B \), so \( B \approx D' - D'' = D \). Consequently every divisor cut on \( C \) by a polynomial in \( V \) is linearly equivalent to \( D \). Now since to contain one point imposes at most one linear condition on polynomials, to contain \( E + D'' \) imposes at most \( (nm-d') + d'' = nm - d \) conditions. Hence \( V \) has dimension at least \( (1/2)(m+2)(m+1) - (nm-d) = (1/2)(m+2)(m+1) - nm + d \). However some of these polynomials may completely contain the curve \( C \) and hence do not define divisors, so restricting these polynomials to \( C \) has as kernel the subspace of \( V \) consisting of those polynomials of degree \( m \) that contain \( C \). Such a polynomial is the product of the equation for \( C \) by a polynomial of degree \( m-n \), hence the kernel of restriction is isomorphic to the space of polynomials of degree \( m-n \), which has dimension \( (1/2)(m-n+2)(m-n+1) \).

Consequently the space of sections of \( O(D) \) on \( X \) has dimension at least as great as the image of the restrictions to \( X \) of polynomials in \( V \), i.e. at least \( (1/2)(m+2)(m+1) - (nm-d) - (1/2)(m-n+2)(m-n+1) \). Computing twice this number gives: \( (m+2)(m+1) - 2(nm-d) - (m-n+2)(m-n+1) \)

\[
= m^2 + 3m + 2 - 2nm + 2d - (m-n)^2 - 3(m-n) - 2
\]
\[ m^2 + 3m + 2 - 2nm + 2d - m^2 + 2mn - n^2 - 3m + 3n - 2 = \]
\[ 2d - n^2 + 3n = 2d - n(n-3). \]

We want to show \( \dim L(D) \geq d + (1 - g) = d - (g-1) \), so we hope the number just above is twice that. We saw earlier that \( 2g - 2 = n(n-3) \), so indeed \( 2d - n(n-3) = 2d - 2(g-1) \), as claimed.

We have proved for every divisor \( D \), we have the lower bound \( d + 1 - g \leq \dim L(D) \). Next we want to check the upper bound \( \dim L(D) \leq d + 1 - g + \dim L(K-D) \) holds for non negative divisors \( D \geq 0 \). This uses analysis, i.e. the residue theorem.

**The residue obstruction for non singular plane curves**

If \( D = 0 \), then \( \dim L(0) = 1 \), and \( K-O = K \), so we already know the inequality \( \dim L(D) \leq d + 1 - g + \dim L(K-D) \) holds, since it reduces to \( 1 \leq 1 \). If \( D > 0 \) is a positive divisor of degree \( d > 0 \), we will define a residue pairing \( k^d \times H^0(K) \rightarrow k \), where \( k \) is the complex numbers. Let \( D = \sum pj \), and assume for simplicity the points \( pj \) of \( D \) are distinct. If \( c = (c_1, ..., cd) \) is a vector in \( k^d \) and \( w \) is a holomorphic one form, the value of the pairing at \( (c,w) \) is \( \sum \text{Res}(cjw/zj) \) where \( zj \) is a local coordinate centered at \( pj \). The product \( cjw/zj \) is only a locally defined meromorphic form, the residue at \( pj \) makes sense. Now map the space \( L(D) \) into \( k^d \) by sending \( f \) to the vector \( c = (c_1, ..., cd) \) of coefficients \( cj/zj \) of the term \( 1/zj \) in the local Laurent expansion of \( f \) at \( pj \). The kernel of this map as we have observed before, is the constants, so the image of \( L(D) \) in \( k^d \) is isomorphic to \( L(D)/k \). We claim this subspace is in the left kernel of the residue pairing.

I.e. if \( f \) is a function in \( L(D) \), then \( fw \) is a global meromorphic form, so the residue theorem implies the sum of its residues is zero. Thus every vector \( c \) in the subspace \( (L(D)/k) \) of \( k^d \) pairs to zero with every holomorphic form \( w \). If we look for the right kernel, it is easy to see it equals those holomorphic forms \( w \) which vanish at every point \( pj \) if \( D \). I.e. these forms cancel the pole of every local expression \( cjw/zj \), but if \( w \) does not vanish say at \( p1 \), then \( w \) does not pair to zero with the vector \( (1, ..., 0) \). As we have observed before, the space of holomorphic forms vanishing on \( D \) is isomorphic to the space \( H^0(K-D) \). Thus we have an induced pairing \( [k^d/(L(D)/k)] \times [H^0(K)/H^0(K-D)] \rightarrow k \), whose “right kernel” is now zero. If the “left kernel” were also zero, the pairing would be non singular, or “perfect” and would define an isomorphism of the left hand space with the dual of the right hand space and vice versa. In particular the two spaces would be isomorphic. The left hand kernel is not necessarily zero, so we may need to mod out more on the left side before the spaces become isomorphic. That implies that now the dimension of the right hand space is less than or equal to the dimension of the left hand space. I.e. we have proved the inequality:
\[
\dim [H^0(K)/H^0(K-D)] \leq \dim [k^d/(L(D)/k)].
\]
Thus we have the inequality:
\[
\dim H^0(K) - \dim H^0(K-D) \leq d - \dim (L(D)/k) = d + 1 - \dim L(D).
\]
Since \(\dim H^0(K) = g\), this implies the following inequality:
\[\text{If } D \geq 0, \text{ or even if } \dim L(D) > 0, \text{ then } \dim L(D) \leq d + 1 - g + \dim L(K-D).\]

Now we claim the two inequalities we have suffice to conclude the RRT in all cases.

**Case one:** Both \(L(D) = \{0\}\) and \(L(K-D) = \{0\}\). The lower bound \(d+1-g \leq \dim L(D)\), which we proved in all cases, implies that \(\dim L(D) > 0\) if \(\deg(D) \geq g\). Thus both \(D\) and \(K-D\) have degree \(\leq g-1\). Since they add up to \(K\), which has degree \(2g-2\), thus both \(D\) and \(K-D\) have degree exactly \(g-1\). Then RRT says \(\dim L(D) = (g-1)+1-g = 0\), which is correct.

**Case two:** Assume \(\dim L(D) > 0\) and \(\dim L(K-D) = 0\). Then we have both inequalities: \(d + 1 - g \leq \dim L(D) \leq d + 1 - g\). So RRT holds again: \(\dim L(D) = d + 1 - g\).

**Case three:** Assume \(\dim L(D) = 0\) and \(\dim L(K-D) > 0\). Then as above we have the two inequalities:
\[
(2g-2-d) + 1 - g \leq \dim L(K-D) \leq (2g-2-d) + 1 - g,
\]
so \(\dim L(K-D) = g-1-d\). Thus \(\dim L(D) = d + 1 - g + \dim L(K-D) = 0\), so RRT holds again.

**Case four:** Both \(L(D)\) and \(L(K-D)\) have positive dimension. Then we have both sets of inequalities:
\[
d + 1 - g \leq \dim L(D) \leq d + 1 - g + \dim L(K-D),
\]
and
\[
(2g-2-d) + 1 - g \leq \dim L(K-D) \leq (2g-2-d) + 1 - g + \dim L(D).
\]
This simplifies as follows:
\[
g-1-d \leq \dim L(K-D) \leq g-1-d + \dim L(D),
\]
which implies the opposite inequality:
\[
d + 1 - g \geq - \dim L(K-D) \geq d+1-g-\dim L(D).
\]
Hence \(\dim L(D) \geq d+1-g+\dim L(K-D)\). Combining this with the upper bound \(\dim L(D) \leq d + 1 - g + \dim L(K-D)\), this implies RRT holds again.

This completes the proof of RRT for smooth plane curves, except for simplifying the computations at times by taking \(D\) to have distinct points. Allowing points to be multiple causes very little change in the argument.