ON THE BRAUER GROUP – NOTES FOR A TRIVIAL NOTIONS SEMINAR

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1. Classifying projective bundles modulo line bundles

This talk is about the Brauer group, an object of startling use and ubiquity – it exists equally well in the realms of pure algebra, pure geometry and at all points in between. So perhaps a good introduction to it is by way of another object which shares the same honor: the Picard group of X, which classifies structures which are equally deserving of the name "line bundle" and "rank 1 projective module". I assume that these objects as well as the objects called "vector bundles" or "finite rank projective modules" are also our old friends. Recall that $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_{X^*}) = H^1(X, \mathbb{G}_m)$ classifies line bundles on X, where here either X/S is an S-scheme (including the case $X = R$ a commutative ring and $\mathbb{G}_m$ gives the units in the structure sheaf and X/k is a scheme over a field k); or X is a topological space, and $\mathbb{G}_m$ is the the sheaf which, given any open $U \subset X$, returns the abelian group of continuous functions from U to $\mathbb{C}^\times$. (Or maybe X is a real or complex manifold...) Finally, recall that we can view the objects classified either geometrically (i.e. as morphisms $E \to X$ whose fibres are $\mathbb{A}^1/S$) or algebraically (as $\mathcal{O}_X$-modules).

So...what about projective bundles? Let X be either a topological space or an S-scheme. What happens if we try to do for projective bundles what we did with line bundles (and could have done with vector bundles, getting $K_0(X)$)? Just for fun, let's start with the case of X a topological space. Then by a $\mathbb{P}^n = \mathbb{CP}^n$-bundle over X we mean what the topologists mean: a locally trivial fibration $E \to X$ with $\mathbb{P}^n$-fibres and structure group $G := \mathrm{Aut}(\mathbb{CP}^n) = PGL_{n+1}(\mathbb{C})$-bundle. Then by a principle so general that everyone here must know it either topologically or algebraically (I would call it "the first principle of descent") we see that the (nonabelian!) cohomology *set* classifying $\mathbb{P}^{n-1}$-bundles on X is $H^1(X, PGL_n)$.

Notice that already things are more interesting than before: this is nonabelian cohomology, so there is no group structure in sight: that is, given two projective bundles of the same dimension, we do not have a natural (in any sense of the word!) way to combine them into another projective bundle of the same dimension. (Perhaps now you think we should work out the direct sum of two projective bundles and get a "Grothendieck group" K_0 as for vector bundles. We would get something this way, but it would not be the Brauer group. Honestly we are implicitly motivated by the Brauer group of a field, which we will dicuss later as an important special case.) Instead, we proceed not by *analogy* to the case of vector bundles, but rather by *reducing* to the case of vector bundles. Namely, observe that one way to construct an $n-1$-dimensional projective bundle over a space X is to take a rank n

vector bundle $E \to X$ and just projectivize: $\mathbb{P}E \to X$. Cohomologically speaking we have identified a map

$$H^1(X, GL_n) \to H^1(X, PGL_n)$$

Since we already "understand" the vector bundles on X, it makes some sense to ask to regard projectivizations of vector bundles as "banal" and ask for a classification of projective bundles modulo projectivizations of vector bundles. So suppose $B(X)$ is this set; we have constructed for all n injections

$$H^1(X, PGL_n)/H^1(X, GL_n) \hookrightarrow B(X)$$

such that $B(X)$ is the union of the images. I claim that there is a natural operation on $B(X)$ which makes it into a group, for which the inverse of a projective bundle is just the dual projective bundle. The group law is best viewed on the algebraic side, which we have so far not discussed. Consider that $H^1(X, PGL_n)$ classifies our projective bundles. But this set equally well classifies any fibre bundle on X with $PGL_n(\mathbb{C})$ as the structure group. Namely, the matrix algebra $M_n(\mathbb{C})$ has this automorphism group – this is the Noether-Skolem theorem, which asserts that the only automorphisms of a central simple algebra over a field are inner (i.e., given by conjugations). It follows from this that $H^1(X, PGL_n)$ also classifies the following structure: A is an $\mathcal{O}_X$-algebra, finite and locally free as an $\mathcal{O}_X$-module and such that its fibre at every point $x \in X$ is $A \otimes \mathcal{O}_{X,x} \cong M_n(\mathbb{C})$. In other words, given the same 1-cocycle we used to construct a projective bundle over X, we could equally well have constructed a matrix algebra bundle. Similarly, we can algebraically interpret the map $H^1(X, GL_n) \to H^1(X, PGL_n)$ as the map which, to any projective $\mathcal{O}_X$-module M associates the algebra $\mathrm{End}(M)$. In this light we want to classify matrix algebra bundles modulo bundles which are globally "End's" of projective modules.

Now we can give the group law: $A_1 \times A_2 := A_1 \otimes A_2$; with this interpretation the inverse of (the Brauer group element represented by) a matrix algebra bundle A is given by the opposite algebra A^{opp}. The map $A \otimes A^{opp} \to \mathrm{End}(A)$ induced by $(a \otimes b)(c) := acb$ is well-defined and clearly an isomorphism on the fibres (by this I mean that we can check whether a homomorphism of finite-dimensional simple algebras is an isomorphism by counting dimensions), so it's an isomorphism. Thus we have defined the Brauer group $Br(X)$ of a topological space, and we can see that it has interpretations both in terms of geometry and non-commutative algebra.

Very well: how can we compute it?

Proposition 1. *(Brauer group of a topological space*
a) $Br(X)$ is an abelian torsion group naturally embedded in $H^2(X, \mathbb{G}_m)$.
b) If we assume that X is paracompact, then via the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathbb{G}_m \to 0$$

we get

$$H^2(X, \mathbb{G}_m) \xrightarrow{\sim} H^3(X, \mathbb{Z}).$$

c) (Serre, thanks to Bott) If X is a finite CW-complex, then

$$Br(X) \xrightarrow{\sim} H^2(X, \mathbb{G}_m)[tors] \xrightarrow{\sim} H^3(X, \mathbb{Z})[tors].$$

Some words in the way of proof: For part a), consider the ladder of short exact sequences

$$1 \to \mu_n \to SL_n \to PGL_n \to 1$$

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$$

and applying H^1, we see that the map $H^1(X, PGL_n) \to H^2(X, \mathbb{G}_m)$ factors through $H^2(X, \mu_n)$, an n-torsion (abelian) group. For part b), just take cohomology of the exponential sequence and use the fact that the sheaf of continuous $\mathbb{C}$-valued functions on a paracompact space is soft (or somesuch), hence acyclic for sheaf cohomology. On the other hand, the proof of Serre's theorem uses Bott periodicity; it is very cool, but we do not digress to include it here.

In summary, the Brauer group of e.g. a compact manifold is always finite, and can be read off from the most basic invariants of the space.

2. The Brauer group of a scheme, a variety, a field

Say X is a scheme. Does the construction of the previous section define the Brauer group of X? No, there is one more issue to settle: what do we mean by a "locally trivial" projective bundle over X? It turns out that while it is acceptable to define *vector bundles* as being locally trivial for the Zariski topology, it is *not* acceptable to require *projective bundles* to be Zariski-locally trivial. Let me try to explain: one learns in commutative algebra that "fibrewise free" is the same as "locally free" for modules. On the other hand, there is no reason to think that an algebra over a ring which becomes a matrix algebra modulo every prime should itself be a matrix algebra, and indeed in a certain sense this is almost never true: namely, say $X/\mathbb{C}$ is a smooth proper variety, and $P \to X$ is a projective bundle over X which becomes trivial over a Zariski-open subset U. Then P lies in the kernel of $Br(X) \to Br(U)$. But letting $K = \mathbb{C}(X)$ the generic point, it follows then that P lies in the kernel of $Br(X) \to Br(U) \to Br(K)$, and by a result of Auslander-Goldman to be discussed shortly, the composite map is an injection.

Anyway, the upshot is that "to do the right thing" we must require our projective bundles to be locally trivial (merely) in the sense of etale topology: namely a projective bundle on X is a morphism $P \to X$ such that for all $x \in X$ there exists an etale neighborhood U of x such that the basechange of P to U is isomorphic to the constant bundle $\mathbb{P}^n/U$. Again, it is useful to express this in algebraic language, and there is a new piece of terminology: an Azumaya algebra $A/\mathcal{O}_X$ is an $\mathcal{O}_X$-algebra which as a module is locally free of finite rank and such that in an etale neighborhood U of every point x, the basechange to U is isomorphic to a matrix algebra over U. There are many equivalent formulations of this: especially it is enough to require A to be the right kind of module (locally free of finite rank) and then just to require that the geometric fibres be matrix algebras.

Let us mention that using nonabelian etale cohology one can represent $Br(X)$ as in the previous section: namely, as the union of $H^1(X_{\acute{e}t}, PGL_n)/H^1(X_{'t}, GL_n)$, which lives inside $H^2(X_{\acute{e}t}, \mathbb{G}_m)[tors]$ but is *not* a priori equal to the whole thing (although it is in many cases). Finally, we should say that $H^2(X_{\acute{e}t}, \mathbb{G}_m)$ itself need not be a torsion group; the discrepancy between

$$Br(X) \subset H^2(X_{\acute{e}t}, \mathbb{G}_m)[tors] \subset H^2(X_{\acute{e}t}, \mathbb{G}_m)$$

was studied intensely by Grothendieck in his *Dix Exposes*; for some purposes it is more convenient to deal with the largest group than to worry about which of its elements are actually represented by Azumaya algebras (again, there is lots of work on this!), and the latter is often called the Grothendieck-Brauer group or the cohomological Brauer group (or – beware! – sometimes just the Brauer group.)

The Brauer group has all the nice functorial properties we have a right to expect from pulling back vector bundles, as well as being a sheaf for the Zariski topology on X (notice that this fact also implies that Zariksi-locally trivial vector bundles represent the trivial element of the Brauer group).

Here are two big theorems about the Brauer group that will come in handy later:

Theorem 2. *(Auslander-Goldman) Let R be a regular domain with quotient field K. Then the natural map "restriction to the generic point" $Br(R) \hookrightarrow Br(K)$ is an injection.*

Theorem 3. *(Auslander-Brumer) Let R be a discrete valuation ring with quotient field K and residue field k. Then we have*

$$0 \to Br(R) \to Br(K) \to X(\mathrm{Gal}_k) = \mathrm{Hom}(\mathrm{Gal}_k, \mathbb{Q}/\mathbb{Z}) \to 0$$

where the last term is the Pontrjagin dual ("character group") of the Galois group of k.

3. Brauer group of a field

This motivates us to at last consider the Brauer group of a field K (I should mention that this is an aggressively anachronistic treatment of the Brauer group; the Brauer group of a field came first of all). From our previous geometric perspective this may seem unlikely: how can we have a nontrivial bundle over a one-point space? But here the etale topology recovers the classical definition: by definition, a "projective bundle" over k is a k-scheme (it is its own fibre) which locally for the etale topology is a projective space. In other words, we have a variety X/k which after a finite separable field extension k'/k becomes isomorphic to projective space $\mathbb{P}^n/k'$. Such a guy is called a Severi-Brauer variety; it will represent the trivial element of $Br(k)$ precisely if it isomorphic to a projective space over k itself, and in turn this happens exactly when $X(k)$ is nonempty. (The condition is obviously necessary, and conversely if we have $P \in X(k)$ the map $\varphi : X/\overline{k} \to \mathbb{P}^n/k$ by $Q \mapsto$ the line joining P to Q is defined over k and an isomorphism over $\overline{k}$, hence is an isomorphism over k.)

Example 4. *The conic*

$$C/\mathbb{R} : X^2 + Y^2 + Z^2 = 0$$

represents a nontrivial element of $Br(\mathbb{R})$, indeed, the unique nontrivial element, as we are about to recall.

What is going on algebraically? An Azumaya algebra A over a field k is a finite-rank k-algebra such that there exists a finite separable field extension k'/k such that $A \otimes_k k' \cong M_n(k)$. The smallest bit of non-commutative algebra shows that this condition – being a "twisted form" of a matrix algebra – is equivalent to A/k being a *central simple algebra* – that is, a finite-dimensional k-algebra with center k and with $0, A$ as the only two-sided ideals. Thus, one can view the representatives of $Br(k)$ as either Severi-Brauer varieties or central simple algebras over k. On the other hand, remember we regard matrix algebras as being trivial; recall Wedderburn's classification of central simple algebras: $A \cong M_n(D)$, where D/k is a division algebra. Since $M_n(D) \cong M_n(k) \otimes_k D$, the class of $A \cong M_n(D)$ in $Br(k)$ coincides with the class of the division algebra D (which is uniquely determined). Thus we can say that the elements of $Br(k)$ correspond bijectively to finite-dimensional division algebras over k with center k. This at last is what the early 20th century algebraists regarded as the Brauer group.

Some facts about $Br(k)$: There is no distinction between the Brauer group and the cohomological Brauer group: $Br(k) = H^2(Gal_k, \overline{k}^\times)$. (Again we are going backwards in time; algebraists realized they could construct central simple algebras by writing down certain two-variable functions in the Galois group of k; the fact that these functions satisfy a certain identity (the cocycle condition!) was one of the three phenomena that led them to study group cohomology, the other two being group extensions and the cohomology of Eilenberg-MacLane spaces.)

Example 5. *(Brauer group of a field which is "almost algebraically closed")*

The cohomological interpetation implies that the Brauer group of any algebraically closed field is trivial (exercise: use the minimal polynomial to give a pithy direct proof that every finite division algebra over an algebraically closed field is commutative). Also, since the absolute Galois group of $\mathbb{R}$ is $\mathbb{Z}/2\mathbb{Z}$, and one knows the cohomology of cyclic groups, we can compute the Brauer group of $\mathbb{R}$:

$$Br(\mathbb{R}) = H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^\times) = \mathbb{R}^\times / N_{\mathbb{R}}^{\mathbb{C}}(\mathbb{C}^\times) = \mathbb{R}^\times/\mathbb{R}^{>0} \cong \mathbb{Z}/2\mathbb{Z}.$$

So above we exhibited the only nontrivial element of the Brauer group; on the division algebra side it is represented by Hamilton's quaternions.

Exercise: Let k be a field whose absolute Galois group G_k is finite. Compute $Br(k)$ as above. (Hint: By a theorem of Artin-Schreier, $\#G_k \leq 2$, and if the order is 2 then the algebraic closure is obtained by adjoining a root of $X^2 + 1 = 0$.)

Example 6. *(Brauer group of a finite field)*

More than a hundred years ago Wedderburn proved directly that a finite division algebra is a commutative field, i.e., that the Brauer group of a finite field is zero. Actually the Brauer group of a finite field is zero "for many reasons" – that is, many of the proofs of the vanishing of the Brauer group of a finite field single out some more general class of fields with vanishing Brauer group of which the finite fields are members.

I. The norm map between any two finite extensions is surjective. (It's a little tricky to see this suffices. If the Brauer group is nonzero, there exists an element of exact order p for some prime. That is, $0 \neq \eta \in H^2(G, \mathbb{G}_m)[p] = H^2(G_p, \mathbb{G}_m)[p]$, where $G_p \leq G$ is a p-Sylow subgroup of $G = \mathrm{Gal}_k$. It follows that there exists an extension l/k of p-power order such that restriction to l kills η. By dévissage, at some field k' between k and l, η restricted to k' is nonzero and killed by restriction to an order p extension. But the hypothesis made prohibits the existence of a nonzero element of any finite extension of k from being killed by a cyclic extension, contradiction.)

II. The Galois group is procyclic and torsionfree – that is, if there exists a surjective map $\hat{\mathbb{Z}} \to G_k$ such that the restriction to G_k is injective. More generally:

III. The Galois group has cohomological dimension 1.
(See [CG].)

Example 7. *(Brauer group of a complete, discretely valued field)*

Let K be a field complete with respect to a discrete valuation v, with valuation ring R and residue field k. The following (split) exact sequence determines $Br(K)$:

$$0 \to Br(k) \to Br(K) \to X(\mathrm{Gal}_k) \to 0$$

where the last term is the Pontrjagin dual (= character group) of the Galois group of the residue field. One finds a very readable proof of this in [CL]; for later use, however, it is convenient to regard this result as obtained by combining the Auslander-Brumer theorem with the following

Theorem 8. *(Grothendieck) Let R be a Henselian (e.g. complete!) local ring with residue field k. Then the natural map $Br(R) \to Br(k)$ is an isomorphism.*

(If you care, a stronger result is true: namely that the base change from R to k gives a bijection from rank r^2 Azumaya algebras over R to rank r^2 central simple algebras over k. The proper generality is: if G/R is a smooth groupscheme, then the canonical map $H^1(R_{\acute{e}t}, G) \to H^1(k, G/k)$ is an isomorphism; you can find this in SGA if you dare.)

As an application of this theorem, let K be a "local field," i.e. a field complete with respect to a complete valuation with finite residue field k. We deduce:

Corollary 9. *The Brauer group of a local field is isomorphic to $X(\hat{Z}) = \mathbb{Q}/\mathbb{Z}$.*

Example 10. *(Brauer group of a global field)*

Let K be a number field or a the function field of an algebraic curve over a finite field. Then there is a "local-to-global principle" that computes $Br(K)$ (the Brauer group is where local-to-global principles live); let K_v be the completions of K with respect to all the places of K. Then

$$0 \to Br(K) \to \oplus_v Br(K_v) \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z} \to 0$$

Here, each $Br(K_v)$ is canonically isomorphic to $\mathbb{Q}/\mathbb{Z}$ (this is called the "invariant"), and the map Σ simply adds up these finitely many elements in $\mathbb{Q}/\mathbb{Z}$. For instance, take $K = \mathbb{Q}$. Then for any $n \geq 2$, the theorem says there are infinitely many nonisomorphic division algebras over $\mathbb{Q}$ of dimension n^2. (Nevertheless one knows exactly how to write them down, and this counts as a rather innocuous example of a Brauer group.)

4. The Brauer group of a variety as the unramified Brauer group of its function field

In this final section I want to show that there is an interplay between the geometry of the Brauer group of a variety and the algebra of the Brauer group of its function field. To fix ideas, let X/k be a proper variety over a field of characteristic zero; write $K := k(X)$ for its function field. A prime divisor D on X gives rise to a discrete valuation $v = v_D$ on K with valuation ring R_D and residue field is $k(D)$. The Auslander-Brumer theorem gives us

$$0 \to Br(R_D) \to Br(K) \to \mathrm{Hom}(G_{k(D)}, \mathbb{Q}/\mathbb{Z}) \to 0 \tag{1}$$

Compiling these maps (they are called "ramification maps") over all prime divisors D we get a single map

$$Br(K) \to \bigoplus_D \mathrm{Hom}(G_{k(D)}, \mathbb{Q}/\mathbb{Z}) \tag{2}$$

What is the kernel of this map? Regard an element of $Br(K)$ as a "generically defined" Azumaya algebra on X – if we write down structure constants for a central simple algebra over $K = k(X)$, these are actually functions on X, which will be regular and nonzero outside on the complement of a divisor on X and hence will define a Severi-Brauer variety over a certain (Zariski-)open subset U of X. The sequence (1) is thus telling us that a generically defined Azumaya algebra is actually defined on the divisor D iff its image in the Hom vanishes. Therefore the kernel of (2) is the set of generically defined Azumaya algebras which are regular at every prime divisor D of X. That is (using, technically, the fact that the Brauer group is a Zariski sheaf), the kernel is just the Brauer group of X:

$$0 \to Br(X) \to Br(K) \xrightarrow{L} \bigoplus_v \mathrm{Hom}(G_v, \mathbb{Q}/\mathbb{Z}) \tag{3}$$

This is the sequence that links geometry to algebra. Let us quickly draw out some consequences.

First, since the set of prime divisors of X correspond bijectively to the discrete valuations on the function field K which are trivial on k, this means that the map L can be defined purely algebraically in terms of K/k. Thus its, kernel, the Brauer group of X, is a "purely algebraic object," called *the unramified Brauer group of* X for hopefully evident reasons. To be sure, we are saying that the Brauer group of a variety is entirely determined by the Brauer group of its function field, so among smooth projective varieties the Brauer group is a birational invariant.

Having said that everything in sight is algebraic, let's do just the opposite and exploit the *geometric* interepretation of $Br(X)$ – if we can compute this than we have a very good handle on $Br(k(X))$.

Suppose k is algebraically closed. Then using a standard exact sequence in etale cohomology (the Kummer sequence), one can compute $Br(X)$ up to a finite group. Being an abelian torsion group, it is enough to work prime by prime and compute $Br(X)[\ell^\infty]$: for simplicity, we will not do the case when $l = p = \Gamma k > 0$. Taking etale cohomology of the Kummer sequence

$$0 \to (\mu_n)_X \to \mathbb{G}_m \stackrel{[n]}{\to} \mathbb{G}_m \to 0$$

and passing to the limit over n, one gets

$$0 \to \mathrm{Pic}(X) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^2(X, \mu_l^\infty) \to H^2(X, \mathbb{G}_m)[l^\infty]) \to 0. \tag{4}$$

The properness of X ensures that the groups $H^2(X, \mu_l^\infty)$ are of "cofinite type" – they are Cartier dual to the l-adic cohomology groups, so of the form $(\mathbb{Q}_l/\mathbb{Z}_l)^r \oplus M$, where M is finite, and here $r = h^2(X, \mathbb{Q}_l) = B_2$, the second Betti number. Finally, using the exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to NS(X) \to 0$$

and the fact that since $\mathrm{Pic}^0(X)$ is l-divisible, $\mathrm{Pic}^0(X) = 0$, we find that up to a finite group

$$Br'(X)[l^\infty] \sim (\mathbb{Q}_l/\mathbb{Z}_l)^{b_2 - \rho},$$

where ρ is the $\mathbb{Z}$-rank of (finitely generated free) abelian group $NS(X)$. Thus the size of the Brauer group is "almost independent of l".

Remark: Let $X/\mathbb{C}$ be a smooth projective variety such that the singular cohomology group $H^2(X(\mathbb{C}), \mathbb{Z})$ has nontrivial l-torsion for some l. Being a finitely generated abelian group, it will be p-torsion free for all sufficiently large p. Since the Picard number ρ does not depend on l or p, we see that we will have ${}_{\mathbb{Z}/l\mathbb{Z}}Br'(X)[l] <_{\mathbb{Z}/p\mathbb{Z}} Br'(X)[p]$, so the above is the strongest possible independence result.

Example 11. *(Brauer group of a curve over $k = \overline{k}$)*

Let C/k be a smooth projective curve over an algebraically closed field. Here the divisors are just points, so the Galois group $G_v = 0$, so the Brauer group of an algebraically closed curve is equal to the Brauer group of its function field. In the above formula $b_2 = \rho = 1$, so the divisible part of $Br(C)$ is zero. Looking a bit more carefully at the finite group we may have lost, it's not at all hard to see that it's zero independently of l (even if $l = p = \mathrm{char}(k)$.) We conclude that $Br(k(C) = 0$, a famous theorem of Tsen.

Example 12. *(Brauer group of projective space over $k = \overline{k}$)*

In this case, the geometric part of the above argument is the same: we get $\rho = b_2 = 1$ and indeed that the entire geometric part of the Brauer group is zero. So

$$Br(k(x_1, \ldots, x_n)) = \bigoplus_D \mathrm{Hom}(G_{k(D)}, \mathbb{Q}/\mathbb{Z})$$

which, if $n \geq 2$ is quite large. That is, an Azumaya algebra over projective space must have nontrivial ramfication and is determined by the data of a cyclic extension of the function field of each prime divisor.

Example 13. *(Brauer group of an abelian variety $A/\mathbb{C}$)*

Let $A/\mathbb{C}$ be a d-dimensional abelian variety, $d > 1$. Then for every l the Brauer group of A is l-divisible. If you know something about abelian varieties, you should check my work on the following estimate on the rank r such that $Br(A)[l^\infty] \sim (\mathbb{Q}_l/\mathbb{Z}_l)^r$:

$$\binom{2d}{2} - \frac{d(d+1)}{2} \leq r \leq \binom{2d}{2} - 1.$$