INTRODUCTION TO THE REAL SPECTRUM

PETE L. CLARK

1. Introduction

About eighteen months ago I was trying to learn a little bit about model theory for a student conference (the 2003 Arizona Winter School) explaining the connections between model theory and arithmetic geometry. There are indeed deep and important connections here, and I was quickly convinced that model theory is an area of mathematics that it is good for everyone to know at least a little bit about, but that is not my point here.

Rather, while reading a basic text, they cited as examples of basic algebraic structures: group, ring, module, field, ordered field, ..., and here I stopped: ordered field?!? Not since the tedious axiomatic description of $\mathbb{R}$ in my first undergraduate analysis class had I even encountered the term; what kind of loon would regard an ordered field as an object as basic as the others in the list?

This talk (given in 12/04 at Colorado College) is a meditation on how wrong I was.[1]

After a brief review of the notion of an ordered field (it's what you think it is!), we will aim to engage – and to some extent, to answer – the following questions:

- Which fields can be ordered?
- If a field can be ordered, in how many different ways can it be ordered?
- What information about a field is encoded in the set of all of its orderings?
- Can we regard the set of orderings of a field as some kind of space?
- When the field is the field of functions of an algebraic variety over a real-closed field R, how is this space related to the set of R-points of the variety?

2. Ordered fields and formally real fields

By an ordering $<$ on a field F we mean a total ordering on the field which is compatible with the field axioms in the following sense:

$$x_1 < x_2,\ y_1 < y_2 \implies x_1 + y_1 < x_2 + y_2. \tag{1}$$

$$x > 0,\ y > 0 \implies xy > 0. \tag{2}$$

Let us repackage these axioms slightly in a more convenient form. Namely, for any ordering, define the *positive cone* $P := \{x \in F \mid x \geq 0\}$[2]. Then P satisfies the

[1] §3 was omitted in the talk itself. As of late December 2006, I have made some minor changes and additions. In the last two years I have not made any serious attempts to deepen my understanding of this material or incorporate it into my own research, but I still find the subject striking and elegant, and I hope the reader will learn enough to feel the same way.

[2] Note that, unfortunately, zero is in the positive cone. The more pedantic terminology "non-negative cone" has not caught on.

following properties (here we use that for a subset $S \subset F$, $-S = \{-s \mid s \in S\}$):

(a) $P + P \subset P$.
(b) $P \cdot P \subset P$.
(c) $P \cup (-P) = F$.
(d) $P \cap (-P) = 0$.

Remark: In the presence of (a) and (b), (d) is equivalent to $P \cap (-P) \neq F$ and also to $-1 \not\in P \cap (-P)$.

Conversely, it is immediate to check that starting with a subset $P \subset F$ satisfying the given properties, one gets an ordering by decreeing $x < y$ iff $y - x \in P$. In the sequel, we will allow ourselves to do something that the analytic philosophers would not like, namely we will call P itself an ordering.

Note that $1 \in P$: indeed exactly one of 1 and -1 is in P, and if -1 were in P then $(-1) \cdot (-1) = 1$ would be in P, so it's definitely 1 that's in P. Similarly, for any $a \in F$, either $a \in P$ or $-a \in P$, so either way $(\pm a)^2 = a^2 \in P$. From this it follows that any element of the form $a_1^2 + \cdots + a_n^2$ lies in P. Denoting the set of all sums of squares in F by $\Sigma_\square$, we get that elements of $\Sigma_\square$ must be positive in any ordering.

Example 0: Of course the real numbers $\mathbb{R}$ with the usual $<$ forms an ordered field. In fact, since for any nonzero real number a, exactly one of a and $-a$ is a sum of squares, $\Sigma_\square$ itself forms an ordering in $\mathbb{R}$, so the usual ordering on $\mathbb{R}$ is the *unique ordering.*

Example 1: If $F_0 \subset (F, P)$ is a subfield of an ordered field, then $P_0 := P \cap F_0$ gives an ordering, i.e., a subfield of an ordered field gets an induced ordering. Since $\mathbb{R}$ has plenty of subfields, this gives lots of examples of ordered fields. For instance, the rational field $\mathbb{Q} \subset (\mathbb{R}, <)$ gets a canonical ordering. Using the platitude $\frac{1}{a} = \frac{1}{a^2} \cdot a$, one checks that if $a \in \sum_\square$, so is $\frac{1}{a}$. Thus, since every positive integer is a sum of squares,[3] also in $\mathbb{Q}$ one has the property that either x or $-x$ is a sum of squares, so the usual ordering on $\mathbb{Q}$ is again the unique one.

Example 2: $F = \mathbb{Q}(\sqrt{2})$. There are two embeddings $F \hookrightarrow \mathbb{R}$ which differ from each other by the nontrivial automorphism of F, which carries $\sqrt{2} \mapsto -\sqrt{2}$. (Algebraically speaking, $\sqrt{2}$ just denotes one of the two roots of the polynomial $X^2 - 2$; except by choosing an embedding into $\mathbb{R}$, there is no preference given to one or the other root.) In one of these embeddings, $\sqrt{2}$ goes to the positive real number whose square is 2, and in the other one it goes to the negative real number whose square is 2. Thus the two embeddings give distinct orderings, and it is easy to check that these are the only two orderings of F. In general, the orderings on any number field correspond K correspond bijectively to the embeddings from K into $\mathbb{R}$ ("the real places"); in particular, they are finite in number.

Non-example 1: $F = \mathbb{Q}(\sqrt{-1})$. Since $-1 = (\sqrt{-1})^2 \in \sum_\square$, any ordering on F

[3] It is a famous theorem of Lagrange that every positive integer is a sum of at most four squares, but here it suffices to note that the positive integer n can be represented as $1^2 + \ldots + 1^2$ (n times).

would have -1 in its positive cone, a contradiction.

Definition: A field F is *formally real* if -1 is *not* a sum of squares of elements of F. Now, we may apply the same reasoning as in Non-example 1:

Observation: A field F which admits an ordering is necessarily formally real.

Non-example 2: No field of characteristic $p > 0$ is formally real, since $-1 = 1 + \ldots + 1$ ($(p-1)$ times). So only fields of characteristic zero can admit orderings.

Non-example 3: No p-adic field is formally real. Indeed, it suffices to look at the fields $\mathbb{Q}_p$. If $p \equiv 1 \pmod 4$ then -1 is a square in $\mathbb{Q}_p$. If p is an odd prime, then it is well-known that $X^2 + Y^2 = -1$ has a solution over $\mathbb{F}_p$, and Hensel's Lemma lifts it to a solution in $\mathbb{Z}_p$, so -1 is a sum of two squares in $\mathbb{Q}_p$. A slightly more elaborate application of Hensel's Lemma shows that -1 is a sum of 4 squares over $\mathbb{Q}_2$.

Remarkably, the converse of the above observation holds.

Theorem 1. *(Artin-Schreier) Any formally real field admits at least one ordering.*

We will sketch the proof, if only to emphasize that it does not in any sense *construct* an ordering on a formally real field. Rather, given a formally real field, what we can produce is a canonical *preordering.* A preordering is a subset $T \subset F$ satisfying axioms (a), (b) and (d) for orderings, but in place of (c) the weaker:

(c') $\sum_\square \subset T$.

Now what is clear is that a field is formally real iff $\sum_\square$ is a preordering on F. Moreover, this is the unique minimal preordering in any formally real field. The union of a chain of preorderings is itself a preordering, so Zorn's Lemma[4] entitles us to a preordering which is not properly contained in any other preordering, and the point is to show that any maximal preordering does in fact satisfy the property $T \cup (-T) = F$, so gives an ordering. For this we need:

Lemma 2. *For a preordering $T \subset F$ and an element $a \in F$, the following are equivalent:*
(i) the set $T[a] = \{x + ya \mid x,\ y \in T\}$ is a preordering;
(ii) a is not *in $-T$.*

Proof: Since no preordering can contain both a and $-a$, (i) $\implies$ (ii) is clear. Convsersely, suppose a is not in $-T$. We claim that -1 is not in $T[a]$, which is enough, since if $T[-a]$ contained any nonzero element t and its additive inverse $-t$, it would contain $-t \cdot t \cdot \frac{1}{t^2} = -1$. Now, if $-1 = x + ya$, then $-ya = 1 + x$ is a nonzero element of T, so $a = -y^{-2}(1 + x) \in -T$, a contradiction.

Thus a maximal preordering $T \subset F$ must satisfy $T \cup -T = F$, i.e., T is an ordering. This completes the proof of the Artin-Schreier theorem. In fact, we showed more:

[4]Added on 12/30/06: I presume that the Artin Schreier theorem requires the Axiom of Choice in the formal sense but have not been able to track down a reference.

Corollary 3. *Let F be a formally real field and x be any element of F which is* not *a sum of squares. Then there exists an ordering on F in which x is negative.*

Using Corollary 3 and a dollop of model theory, one gets an especially short and perspicuous affirmative solution to Hilbert's 17th Problem: a positive semidefinite polynomial $f \in \mathbb{R}[t_1, \ldots, t_n]$ is a sum of squares of rational functions.[5]

We give some examples of using the Artin-Schreier theorem to deduce that orderings must exist.

Proposition 4. *F can be ordered iff the rational function field $F(t)$ can be ordered.*

Proof: If $F(t)$ is formally real, so, of course, is its subfield F. Inversely, suppose $F(t)$ is *not* formally real:

$$-1 = \left(\frac{p_1(t)}{q_1(t)}\right)^2 + \ldots + \left(\frac{p_n(t)}{q_n(t)}\right)^2.$$

Now, seeking a contradiction, suppose F is formally real. Then F has characteristic zero, so in particular is infinite; hence there is an element a of F which is not a root of any of the finitely many polynomials $q_i(t)$. Plugging in $t = a$, we get that -1 is a sum of squares in F, so F is *not* formally real.

Exercise 1: Prove that F can be ordered if and only if $F((t))$ can be ordered.

Exercise 2: Let $F = \bigcup_{\alpha \in I} F_\alpha$ be an arbitrary union of subfields F_α. Show that F can be ordered if and only if each F_α can be ordered. Conclude that there exist orderable fields of all cardinalities, e.g., rational function fields over $\mathbb{Q}$ in lots of indeterminates.

Remark: Lang proved a vast generalization of Proposition 4: if $V_{/\mathbb{R}}$ is a (geometrically irreducible) algebraic variety and $F = \mathbb{R}(V)$ is its function field, then F is formally real iff V has a nonsingular real point. For instance, for $a \in \mathbb{R}$, the quotient field F_a of the ring $\mathbb{R}[x, y]/(x^2 + y^2 - a)$ is formally real iff $a > 0$.

3. Orderings on rational function fields

We now know that for any formally real field F, $F(t)$ admits orderings – but what are they? We can strengthen Proposition 4 by showing that for any ordered field (F, P), there are orderings on $F(t)$ which extend the ordering P on F.

There is one ordering P_∞ in which we decree a polynomial $a_n t^n + \ldots + a_0$ to be positive iff its leading coefficient a_n is a positive rational number.[6] This ordering has the property that for any rational number a, $t - a > 0$, i.e., $t > \mathbb{Q}$: t is said to be "infinitely large." An ordered field for which there exists an element which is larger than every rational number (or equivalently, than any positive integer) is called *non-Archimedean.* In fact P_∞ is the unique ordering on $\mathbb{Q}(t)$ in which t is infinitely large. There is also a unique ordering $P_{-\infty}$ in which $-t$ is infinitely large (we leave it as an exercise to describe it). Similarly there is a unique ordering P_{0+} in which

[5] Thus you see that model theory has its uses, and formally real fields are especially interesting to model theorists.

[6] Since in any ordered field, a nonzero element a is positive iff a^{-1} is positive, an ordering on an integral domain always extends uniquely to its quotient field.

$\frac{1}{t}$ is infinitely large: a polynomial is said to be positive if its smallest degree term is positive; similarly there exists an ordering P_{0-} for which $\frac{-1}{t}$ is infinitely large. There are lots more orderings: indeed if F is any field, the image $\varphi(P)$ of P under an automorphism φ of F is another ordering. The group of automorphisms of $F(t)$ inducing the identity on F is the group of Mobius transformations $t \mapsto \frac{at+b}{ct+d}$, where $ad - bc \neq 0$. Let x be any element of F. Under the automorphism $t \mapsto \frac{1}{t-a}$ P_∞ maps to an ordering P_{a+} in which $\frac{1}{t-a}$ is infinitely large: here t is bigger than a but is smaller than any element $b < a$: we think of t is being infinitely close to but to the right of a. And there is a similar ordering P_{a-} in which t is infinitely close but to the left of any given element a of F.

Does $P_{\infty+} \cup P_{\infty-} \cup \{P_{a+} \cup P_{a-} \mid a \in F\}$ give all orderings on $F(t)$ extending the ordering P on F? This is a surprisingly intricate question in general. Later we will come back to this point and see one choice of F for which the answer is "yes" and one for which the answer is "no."

Exercise 3: Show that a field admits an Archimedean ordering if and only if it can be embedded in $\mathbb{R}$. Deduce from Exercise 2 that "most" formally real fields have cardinality too large to admit Archimedean orders. (There are also countable fields admitting only non-Archimedean orders.)

Exercise 4: Let (F, P) be an ordered field. Show that there are precisely two orderings on $F((t))$ extending the ordering P on F.

4. Pfister's theorem

We now know that the collection of all orderings of a formally real field can be quite complicated. Is there a good reason to try figure out what they all are?

To show you that the answer is "yes," I want to present a result due to Pfister. This is a result about quadratic forms, so let me give a brief review. A rank n quadratic form $f(x_1, \ldots, x_n)$ over a field F – whose characteristic is *not* 2! – is just a homogeneous polynomial in which every term has degree two. Such a polynomial can be given by a matrix product $f(x_1, \ldots, x_n) = vAv^T$, where $v = [x_1 \ldots x_n]$ and A is symmetric. Two quadratic forms are *isomorphic* if we can get from one to the other by making an invertible linear change of variables – in terms of matrices, this means we pass from A to PAP^T, where P is invertible. Every quadratic form is isomorphic to a diagonal quadratic form $a_1x_1^2 + \ldots + a_nx_n^2$, which we denote by $\langle a_1, \ldots, a_n \rangle$. We also assume that our form is nondegenerate, which amounts to saying that none of the a_i's are zero.

J.J. Sylvester defined an invariant of a nondegenerate quadratic form $f = \langle a_1, \ldots, a_n \rangle$ over $\mathbb{R}$: its signature, which is the number of positive entries minus the number of negative entries. (One must show that this difference is well-defined independent of the choice of diagonalization – this is Sylvester's Law of Inertia.) It is easy to see that a nondegenerate rank n quadratic form is determined up to isomorphism by its signature.

This can be generalized as follows: if P is any ordering on F, the P-signature $\sigma_P(f)$ is the number of P-positive entries ($a_i \in P$) minus the number of P-negative entries ($-a_i \in P$) – again, this turns out to be well-defined. So for any field F (still not of characteristic 2), one can assemble a *total signature* invariant: let $X(F)$ be the set of all orderings on F (it will be empty iff F is not formally real, but this case is *not* excluded); for any quadratic form $f = \langle a_1, \ldots, a_n \rangle$, $\sigma(f)$ is the function $X(F) \mapsto \mathbb{Z}$ carrying $P \mapsto \sigma_P(f)$.

It is too much to ask that quadratic forms over an arbitrary field F are determined up to isomorphism by its rank and total signature (for instance this would say that all conics $aX^2 + bY^2 + cZ^2$ over $\mathbb{Q}$ with $a, b, c > 0$ are isomorphic, and this is certainly false). But it is amazingly close to being true: there is a natural addition law on quadratic forms: $\langle a_1, \ldots, a_n \rangle \oplus \langle b_1 \ldots, b_m \rangle := \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle$. In particular we have for any positive integer k, $k \cdot f = f \oplus f \oplus \ldots \oplus f$, k times.

Theorem 5. *(Pfister's local-global theorem) Suppose f and g are two nondegenerate quadratic forms of the same rank over a field F.*
a) If $\sigma(f) = \sigma(g)$, then there exists a positive integer ℓ such that $2^\ell \cdot f \cong 2^\ell \cdot g$.
b) If $S : X(F) \to \mathbb{Z}$ is any function, there exists a positive integer m such that $2^m S = \sigma(f)$ for some quadratic form f.

In other words, the problem of classifying all quadratic forms over *any field* F is solved by the total signature invariant "up to 2-primary torsion." One amazing thing about Pfister's result is that it is nontrivial even when F is not formally real: in this case $X(F)$ is empty, and since there is exactly one function from the empty set to $\mathbb{Z}$, we get that any two quadratic forms f and g of the same rank are such that $2^\ell f \cong 2^\ell g$ for sufficiently large ℓ.

5. A topology on $X(F)$

There is a natural topology on $X(F)$: namely the open sets are given by finite intersections of (subbasic) open sets of the form

$$H(a) = \{P \in X(F) \mid a \in P\}$$

as a ranges through nonzero elements of F: that is, $H(a)$ is the set of orderings which regard a as positive. Note that $H(-a) = X(F) \setminus H(a)$, so that the $H(a)$ and (and hence also all the basis elements) are closed as well open: this implies that $X(F)$ is totally disconnected and Hausdorff. It is also compact. To see this, note that an ordering P of F gives rise to an element of $Y = \{\pm 1\}^{F^\times}$, namely for each nonzero element a, we assign $+1$ if $a \in P$ and -1 if $-a \in P$. Giving $\{\pm 1\}$ the discrete topology and Y the product topology, by Tychonoff's theorem it is a *profinite* – i.e., compact Hausdorff totally disconnected – space. It remains to be shown first that the topology on $X(F)$ defined above is the same as the topology it gets as a subspace of Y,[7] and second that $X(F)$ is closed as a subspace of Y. Neither of these is especially difficult and we leave them to the interested reader.

If $F_1 \hookrightarrow F_2$ is a field embedding, then the aforementioned process of restricting orders on F_2 to orders on F_1 gives a map $X(F_2) \to X(F_1)$ which is easily seen to be continuous. So X is a functor from fields to profinite topological spaces. Our second "converse theorem" identifies its image:

[7] One might wonder why we didn't save ourselves the trouble and define the topology on $X(F)$ in this latter way. It turns out that the sets $H(a)$, called the *Harrison subbasis*, are important in their own right.

Theorem 6. *(Craven) Any profinite space X is homeomorphic to $X(F)$ for some field F.*

In full generality the proof is rather involved. However, the following exercises outline a (new?) proof in case X has a countable basis (or equivalently, by Urysohn's theorem, is metrizable).

Exercise 5: Let $F = \varinjlim_\alpha F_\alpha$ be a direct limit (i.e., directed union) of fields. Show $X(F) = \varprojlim_\alpha X(F_\alpha)$ as topological spaces.

Exercise 6: Let $F/\mathbb{Q}$ be a (possibly infinite) formally real Galois extension. Show that $\mathrm{Aut}(F) = \mathrm{Gal}(F/\mathbb{Q})$ acts continuously and simply transitively on $X(F)$, and conclude that in this case $X(F)$ is homeomorphic to the underlying topological space of a profinite group. In particular, if $F/\mathbb{Q}$ is infinite, $X(F)$ is an infinite profinite space without isolated points and with a countable basis, so is homeomorphic to the Cantor set. (A good example is $F = \mathbb{Q}(\{\sqrt{p}\})$ as p ranges over all the prime numbers: here $\mathrm{Aut}(F) \cong (\mathbb{Z}/2\mathbb{Z})^{\aleph_0}$ really looks like the Cantor set.)

Exercise 7*: Use weak approximation of valuations to show that any inverse system

$$\ldots \to S_{n+1} \to S_n \to \ldots \to S_1$$

of finite sets can be realized as the system of $X(F_n)$'s where

$$F_1 \ldots \hookrightarrow F_n \hookrightarrow F_{n+1} \hookrightarrow \ldots$$

is a tower of number fields. Conclude that any profinite space with a countable basis arises as the space of orderings of an algebraic field extension of $\mathbb{Q}$.

6. Orderings on rational function fields, again

Let us revisit the case of the orderings on $F(T)$ extending a given ordering P on F. Recall that we found infinitely many orderings: indeed two for each point on the projective line $\mathbb{P}^1/F$.

Suppose that $F = \mathbb{Q}$ (with its unique ordering); have we found all the orderings? It cannot be so – the subspace $\mathbb{P}^1(\mathbb{Q}) \subset X(\mathbb{Q}(t))$ with its natural topology is without isolated points. But there is no such thing as a countably infinite compact Hausdorff space without isolated points – this contradicts the Baire category theorem. There must therefore be more orderings. They arise as follows: if $\alpha \in \mathbb{R}$ is any real number (rational or not), the two orderings $P_{\alpha+}$, $P_{\alpha-}$ on $\mathbb{R}(t)$ restrict to orderings on $\mathbb{Q}(t)$, and it is easy to see that all these orderings are distinct. Moreover, if $\alpha \in \mathbb{R}$ is transcendental, then we get an Archimedean ordering by embedding $\mathbb{Q}(t) \hookrightarrow \mathbb{R}$ as $t \mapsto \alpha$. We claim that these are all the orderings on $\mathbb{Q}(t)$ and give the main idea the proof: an ordering P on $F(t)$ for any t determines a *gap* in F, i.e., a partition of F into two subsets L and R (it is permissible for one of them to be empty) such that every element of L is less than every element of R. Indeed, we put $L := \{x \in F \mid x < t\}$ and $R := \{x \in F \mid x > t\}$. We leave it to the reader to figure out how to recover the ordering from the gap (except that in the case of orderings corresponding to irrational algebraic numbers, each gap corresponds to two orderings).

Now if $F = \mathbb{R}$, things are simpler: when L and R are nonempty, then either L contains its least upper bound α, in which case the ordering is $P_{\alpha+}$, or R contains its greatest lower bound α, in which case the ordering is $P_{\alpha-}$. Thus:

Proposition 7. *Let $F = \mathbb{R}(t)$. Then $X(F) = \{P_+, P_- \mid P \in \mathbb{P}^1(\mathbb{R})\}$.*

Here is a topological description of $X(\mathbb{R}(t))$: $\mathbb{P}^1(\mathbb{R})$ itself would just be the one-point compactification of $\mathbb{R}$, i.e., a circle S^1. A good way to picture $X(\mathbb{R}(t))$ is as the set of "oriented points" on the circle: i.e., we get a point of $\mathbb{P}^1(\mathbb{R})$ together with a local orientation. It is in fact the case that the quotient space $X(\mathbb{R}(t))/\sim$ where $P_+ \sim P_-$ is just the usual circle S^1. In fact, if $F = \mathbb{R}(C)$ is the function field of any (geometrically irreducible) curve $C/\mathbb{R}$, then one can show that $X(F)$ is again the set of oriented real points of C. To be sure, this is a bit strange: $X(\mathbb{R}(C))$ is a totally disconnected space, but it has as a continuous image via a very simple quotient map the honest space of real points $C(\mathbb{R})$. It thus seems clear that the algebraic notion of a space of orderings "knows about" the topological space $C(\mathbb{R})$.

Question 8. *Is there some more direct order-theoretic construction that will give us the topological space $C(\mathbb{R})$, and that can be generalized to higher dimensional varieties and other fields?*

Lang's Theorem (end of §2) provides some reason to suspect an affirmative answer: using the correspondence between one variable function fields and smooth algebraic curves, we get at least that $C(\mathbb{R}) \neq \emptyset \iff X(\mathbb{R}(C)) \neq \emptyset$. For $d > 1$, there is no unique way to go from a function field F of transcendence degree d over $\mathbb{R}$ to a d-dimensional algebraic variety $V_{/\mathbb{R}}$: the function field is only a birational invariant. It is interesting to note that Lang's theorem implies that having a smooth rational point is a birational invariant of a real algebraic variety. Indeed this is true for varieties over any field and is due, independently, to Nishimura and Lang himself.[8] So those with experience with conventional algebraic geometry and the connections to valuation theory (especially, the "Zariski Riemann surface") might suspect that we should rather be looking at orderings on *rings* rather than just fields.

7. The real spectrum of a commutative ring, of an $\mathbb{R}$-variety, of a scheme

The answer to Question 8 is *yes*, and as hinted at just above, the key idea is to "cross-breed" the notions of space of orderings and the prime spectrum of a commutative ring. We can only outline it briefly, hoping that its appeal will nevertheless be evident.

Let A be any commutative ring. We will define the notion of an ordering on A. The set of all orderings on A, endowed with a topology as above, will be the real spectrum of A, denoted $\operatorname{Spec}_r(A)$.

An ordering on A can be given by a "cone" $T \subset A$ which is now required to satisfy the following properties: $T + T \subset T$, $T \cdot T \subset T$, $T \cup -T = A$, and: lastly but most importantly, the axiom $P \cap -P = 0$ for fields is replaced by: $T \cap -T = \mathfrak{p}$ is a prime ideal, called the *support* of T. Here $a >_T 0$ means that a is in T and is *not* in $-T$. Note the essential difference: elements in $\mathfrak{p} = \operatorname{supp} T$ are neither T-positive

[8] I wonder whether Lang was motivated by his work on the real case; it may well be so.

nor T-negative; thus, if $\mathfrak{p}$ is not the zero ideal, T is not a total ordering.

Another way of saying the same thing is that an ordering on A is given by a choice of prime ideal $\mathfrak{p}$ of A together with an ordering P on the quotient field $F_\mathfrak{p}$ of the integral domain $A/\mathfrak{p}$. (We recover T as the subset of elements a of A such that the image of a in $A/\mathfrak{p} \subset F_\mathfrak{p}$ is non-negative with respect to the ordering P.)

The set of all orderings, $\mathrm{Spec}_r(A)$, is topologized as follows: a subbasis of open sets is given by the sets

$$H(a) = \{T \in \mathrm{Spec}_r(A) \mid a >_T 0\}$$

for all $a \in A$. With this topology, $\mathrm{Spec}_r(A)$ becomes quasi-compact but not necessarily Hausdorff. Because orderings need not be total orderings, we will have $H(-a) \neq \mathrm{Spec}_r(A) \setminus H(a)$ in general. Indeed, $\mathrm{Spec}_r(A)$ is usually *not* totally disconnected. This is good news: the topology on $\mathrm{Spec}_r(A)$ is not just some formal thing but contains honest geometric information about A.

The second definition of orderings gives us a canonical "forgetful" map f from $\mathrm{Spec}_r(A)$ to the usual prime spectrum $\mathrm{Spec}\, A$: $(\mathfrak{p}, P \in X(F_\mathfrak{p})) \mapsto \mathfrak{p})$. This map is continuous, and its image is precisely the set of primes whose quotient fields $F_\mathfrak{p}$ are formally real ("real primes"). In fact the fibre $f^{-1}(\mathfrak{p})$ over a real prime is homeomorphic to $X(F_\mathfrak{p})$.

On the other hand, $\mathrm{Spec}_r(A)$ itself has the structure of a partially ordered set: we write $T_1 \leq T_2$ if $T_1 \subset T_2$. The closed points of $\mathrm{Spec}_r(A)$ are precisely the orderings which are maximal elements of this poset. Because $T_1 \subset T_2$ implies $\mathrm{supp}\, T_1 \subset \mathrm{supp}\, T_2$ and distinct orderings with the same support are incomparable, an ordering whose support is a maximal ideal is necessarily maximal. However the converse is not true, and we will soon see an example. Denote by $\mathrm{MaxSpec}_r(A)$ the subset of maximal orderings with the induced (subspace) topology. This is a compact Hausdorff space (but not totally disconnected!).

If you are familiar with the topology of $\mathrm{Spec}\, A$, you know that there is a similar but simpler partial ordering (just inclusion of ideals) and that all the above statements are true for $\mathrm{Spec}\, A$. However, the following fact is true for $\mathrm{Spec}_r\, A$ and almost never for $\mathrm{Spec}\, A$: namely, the collection of orderings $T \in \mathrm{Spec}_r\, A$ containing a given ordering T_0 is *linearly ordered*. It follows that every ordering is contained in a *unique* maximal ordering, which gives us a *retraction map*

$$r : \mathrm{Spec}_r(A) \to \mathrm{MaxSpec}_r(A)$$

sending T to the unique maximal ordering containing it. This map is continuous and is in fact a quotient map. In other words, this construction is just what we need in order to "depointilize" the space $X(\mathbb{R}(t))$.

Example: $A = \mathbb{R}[t]$. A is a principal ideal domain, and $\mathrm{Spec}\, A$ consists of the zero ideal (0) – the "generic point" – whose quotient field is $F := \mathbb{R}(t)$ that we studied above – and whose closed points correspond to maximal ideals, i.e., either to polynomials of the form $t - a$ for a in $\mathbb{R}$ or to irreducible quadratic polynomials. The quotient field of $A/(t - a)$ is $\mathbb{R}$, which admits a unique ordering, so to each $a \in \mathbb{R}$ we get a unique ordering whose prime ideal is $(t - a)$; in the case of a closed point corresponding to an irreducible quadratic, the quotient field is the complex numbers, which is *not* orderable, so these are the points of $\operatorname{Spec} A$ which are not in the image of $\operatorname{Spec}_r(A)$. But don't forget the generic point (0), each of whose orderings gives an ordering of A. In other words, we get an embedding $X(\mathbb{R}(t)) \hookrightarrow \operatorname{Spec}_r(A)$. In this context, our above "quotient map" $X(\mathbb{R}(t)) \to S^1$ is an instance of the above retraction map: the closed point $x - a$ is the unique maximal point lying over both P_{a-} and P_{a+}.

Now in the setting of algebraic geometry, the ring A determines an affine variety V (as does any finitely generated $\mathbb{R}$-algebra), whose corresponding set of "real point" $V(\mathbb{R})$ are precisely the maximal ideals whose quotient field is $\mathbb{R}$ (rather than $\mathbb{C}$). For $A = \mathbb{R}[t]$, $V(\mathbb{R}) \cong \mathbb{R}$ is "the affine line." The real spectrum "knows all of this" and more: we find $V(\mathbb{R})$ as a subspace of $\operatorname{Spec}_r(A)$, since each maximal ideal admits a unique ordering. You can check that the topology $V(\mathbb{R})$ gets as a subspace is the right one – it really is homeomorphic to the real line. The full space $\operatorname{Spec}_r(A)$ is an enriched version of $V(\mathbb{R})$. We have also $\operatorname{MaxSpec}_r(A)$ and the retraction map. Consider the composite:

$$V(\mathbb{R}) \subset \operatorname{Spec}_r(A) \xrightarrow{r} \operatorname{MaxSpec}_r(A).$$

$\operatorname{MaxSpec}_r(A)$ consists of all the points on the real line together with two orderings, $\pm\infty$, which are maximal despite the fact that their support is not a maximal ideal. In other words, the embedding gives the two-point compactification of the real line.

This is not quite what we expected (it would have been more reasonable to guess that the quotient map would have identified $\pm\infty$ to get the one-point compactification of $\mathbb{R}$, i.e., S^1), but it is at least as interesting. This construction can be applied to *any* affine $\mathbb{R}$-variety to get a natural compactification, which is very interesting on purely geometric grounds. If we really wanted to see S^1 in the picture, there are several ways to get it: we could start not with the affine ring $\mathbb{R}[t]$ but with the ring $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$, whose corresponding $V(\mathbb{R})$ is homeomorphic to S^1. (This is a profound difference between algebraic geometry over $\mathbb{R}$ and over $\mathbb{C}$: over the complex numbers, the topological space associated to an affine variety is never compact, and we must introduce the machinery of projective varieties. But real projective space $\mathbb{RP}^n$ is itself the space associated to an affine variety!) Or you can globalize the construction: to any scheme X (which is a global geometric object obtained by gluing together different prime spectra $\operatorname{Spec} A$ you can associate a real spectrum $\operatorname{Spec}_r X$, so in particular one can consider $\operatorname{Spec}_r(\mathbb{RP}^1)$, whose maximal spectrum is just S^1.

The real power of $\operatorname{Spec}_r V$ comes when V is a variety over a formally real field which is not $\mathbb{R}$ exactly – one can (e.g.) replace $\mathbb{R}$ by any *real-closed field*, a field which is formally real and for which no algebraic extension is formally real.[9] If $V/\mathbb{R}$ is a nonsingular projective variety, then $V(\mathbb{R})$ is a compact manifold: in particular it has finitely many connected components. If $\mathbb{R}$ is replaced by an arbitrary

[9] In a model-theoretic sense, the real-closed fields are non-standard models of the real numbers, but they arise naturally in many contexts and indeed are very closely related to the orderings of a field in a way we have not had time to discuss.

real-closed field, then one still wants a good notion of the connected components of $V(R)$ (even though $V(R)$ will, if R is a non-Archimedean field topologized in the naive way via open intervals, be a totally disconnected space). There are other ways to get at this – e.g. there is something called "semi-algebraic geometry" in which polynomial equations are replaced with inequalities – but the point is that $\mathrm{Spec}_r V$ just as a topological space is exactly the right thing: its connected components in the usual sense correspond to the semialgebraic connected components (which are a pain to define).

This is a jumping-off point for some very beautiful (if often difficult) recent research due to, among others, J.-L. Colliot-Thélène and Claus Scheiderer: the usual (singular) cohomology groups $H^i(\mathrm{Spec}_r V, \mathbb{Z}/2\mathbb{Z})$ are closely related to the étale cohomology groups $H^i(V_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z})$ (which were previously all but ignored because they can be nonzero for all i) with corresponding implications for the Galois cohomology of algebraic groups over fields of virtual cohomological dimension one. In particular, using the real spectrum, Scheiderer proved a local-global principle in Galois cohomology which is (loosely) analogous to Pfister's theorem for quadratic forms.

In summary, the orderings of a field contain vital information – algebraic, geometric, and cohomological – and give rise to algebro-geometric objects which are themselves quite beautiful and worthy of further study.

Some references: The basic reference for this material – and especially for some of the omitted proofs on the topology of $\mathrm{Spec}_r(A)$ – is T. Y. Lam's paper *An introduction to real algebra* (Rocky Mountain Journal of Mathematics 14 (1984)). There is much more in this paper that we did not have time to touch upon. The construction of the real spectrum of a commutative ring is due to M.-F. Coste-Roy (one person!), and its applications to real algebraic geometry are pursued in the book *Real algebraic geometry* by Bochnak, Coste and Coste-Roy. Here the emphasis is on the topological/analytic side of things. Connections between the cohomology of the real spectrum of a variety and its étale cohomology have been pursued by Claus Scheiderer, beginning in his book, *Real and étale cohomology.* This book is written at a very high level (familiarity with Grothendieck-style algebraic geometry is one of the many prerequisites), but it is also written very carefully and there are some wonderful results. In particular, Scheiderer proves some local-global theorems in Galois cohomology of algebraic groups (extended in his 1995 Inventiones paper) that were what piqued my interest in the subject.