SECOND TALK ON RAMANUJAN GRAPHS

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1. Recall...

Last time we considered certain problems in the spectral theory of finite graphs. Namely, for an finite graph G (undirected, but possibly with loops and/or multiple edges) we may of course associate its adjacency matrix A, and by definition the **spectrum** of G is the multiset $\operatorname{Spec} G$ of eigenvalues

$$\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_r.$$

We say that G is d-**regular** if each vertex has degree d, and in this case $\lambda_0 = d$, with multiplicity 1 iff G is connected.

We are especially interested in graphs – or better, in sequences of graphs – whose first eigenvalues λ_1 are relatively small. In particular, for any finite graph, define the **spectral gap** $\omega(G) = \lambda_0(G) - \lambda_1(G)$. Define also the isoperimetric constant $h(G)$ to be the infimum

$$\frac{\#E(V_1, V_2)}{\min\{\#V_1,\ \#V_2\}}$$

over all partitions of the vertex set into two subsets V_1, V_2; here $E(V_1, V_2)$ is the set of edges connecting a vertex in V_1 to a vertex in V_2. The larger h is the "better connected" G is as a network. An **expander graph** is a sequence of finite graphs G_n with $\#G_n \to \infty$ (equivalently, any fixed isomorphism type occurs only finitely many times in the sequence) and $\inf h(G_n) > 0$. It turns out to be the case that this condition is equivalent to $\inf \omega(G_n) > 0$ (e.g. [2]).

As mentioned last time, this is easy to do: we could take G_n to be the complete graph on n vertices, and then $\omega(G_n) = n$ for all n. However, such a network has prohibitively many edges. Rather, we want the number of edges of G_n to grow linearly with n (one easily sees that one can ask for no more than this for a sequence of connected graphs). Last time we used this as an excuse to consider only d-regular graphs, but this turns out not to be necessary. Rather – and this is the crux of the more sophisticated perspective we are peddling in this talk – it suffices to consider graphs with a common universal covering tree.

2. Coverings of graphs

Let G be a finite graph, and $\rho : H \to G$ a finite, unramified, Galois covering graph of G, say with covering group π.

Lemma 1. *We have* $\operatorname{Spec} G \subset \operatorname{Spec} H$.

Proof: Indeed, a λ-eigenfunction $f \in L^2(G)$ is just a π-invariant λ-eigenfunction on $L^2(H)$. In other words, pulling back eigenfunctions visibly gives an inclusion $\operatorname{Spec} G \hookrightarrow \operatorname{Spec} H$ (with multiplicities).

Note that an equivalent formulation is the divisibility of characteristic polynomials: $P_G(t)|P_H(t)$.

More interesting is the following:

Proposition 2. *We have $\lambda_0(G) = \lambda_0(H)$.*

Sketch proof: This follows from the Perron-Frobenius theory of *irreducible* matrices with non-negative integer entries, which includes the case of adjacency matrices of connected finite graphs. In particular the theory says that the largest eigenvalue λ_0 is also maximal in modulus (and strictly maximal iff the graph is not bipartite), and is characterized by having an eigenvector with all non-negative entries. In our language there is a unique non-negative "Perron-Frobenius" eigenfunction f on G, which when pulled back gives the (unique up to scaling) Perron-Frobenius eigenfunction, say $\tilde{f}$ on H. Thus the eigenvalue $\lambda_0(G)$ of $\tilde{f}$ must be the largest eigenvalue of H.

Next we quote the following result:

Theorem 3. *(Leighton [5]) Let G_1 and G_2 be two finite graphs with isomorphic universal covering tree $\mathcal{T}$. Then there exist finite Galois coverings H_1 of G_1 and H_2 of G_2 such that $H_1 \cong H_2$.*

This is to be contrasted against the situation for compact Riemannian manifolds: e.g., two compact hyperbolic surfaces need not have a common universal cover!

We can deduce:

Corollary 4. *Let $\mathcal{T}$ be an infinite tree, and let $G(\mathcal{T})$ be the set of finite graphs covered by $\mathcal{T}$. Then the quantities*
(i) $\#E(G)/\#V(G)$ and
(ii) $\lambda_0(G)$
are independent of the choice of $G \in G(\mathcal{T})$.

In particular, the number λ_0 is really an invariant of $\mathcal{T}$, so it is tempting to denote it as $\lambda_0(\mathcal{T})$, but in view of later developments this would be quite confusing. So let us use the clunkier notation $PF(\mathcal{T})$ instead.

Example 1: Let $\mathcal{T} = \mathcal{T}_d$, the d-regular tree. This is the universal covering tree of finite d-regular graphs. This is, of course, the example which occupied our attention in the previous lecture (and has occupied the attention of the majority of those who study spectral/expanding/Ramanujan graphs); for instance, the complete graph K_{d+1} is an element of $G(\mathcal{T})$. Here we can see easily that $\#E(G) = \frac{d}{2}\#V(G)$ and that $\lambda_0(G) = d$: indeed the Perron-Frobenius eigenfunction is the constant function.

Example 2: For $r, s \geq 2$, let $\mathcal{T} = \mathcal{T}_{r,s}$, the (r, s)-semiregular tree. Meaning, we start at a red vertex, and draw r "outward" edges (this is just poetic license; we don't consider the orientation as part of the data), coloring each of the endpoints green. Then each green vertex gets s outward edges whose endpoints are painted red, and we repeat. By definition, a finite graph which is covered by $\mathcal{T}$ is called (r, s)-semiregular bipartite.

In particular such graphs exist! In analogue to the complete graph as above, the most natural-looking example is $K_{r,s}$, the complete bipartite graph connecting a set of r vertices to a set of s vertices. We can compute the Perron-Frobenius eigenvalue using a dirty trick: let A be the adjacency matrix of this graph. It would suffice, of course, to know the largest eigenvalue of A^2, but A^2 counts the number of length 2 paths from a vertex i to a vertex j, and is itself the adjacency matrix of some other graph on the same vertex set. But this new graph is visibly rs-regular, so that $PF(A^2) = rs$ and hence $PF(A) = \sqrt{rs}$.[1]

Example 3: For any tree $\mathcal{T}$, we can define a tree $\mathcal{T}'$, the tree obtained from $\mathcal{T}$ by barycentric subdivision. More plainly, we insert a new vertex in the middle of each of the edges of $\mathcal{T}$. For instance $\mathcal{T}_d' = \mathcal{T}_{d,2}$. Note that there is a natural injection $G(\mathcal{T}) \hookrightarrow G(\mathcal{T}')$; in particular, if $G(\mathcal{T}) \neq \emptyset$, then so is $G(\mathcal{T}')$. One's first instinct is that this is a rather trivial operation, but in fact the spectral theory is certainly changing. For instance, if we start with an n-cycle C_n and repeatedly barycentrically subdivide, we will get C_{2n}, C_{4n},..., $C_{2^k n}$: the spectral gaps ar approaching zero. This example is anomalous since in fact $\mathcal{T}_2 = \mathcal{T}_2' = \mathcal{T}_{2,2}$: in all other cases the subdivision changes the isomorphism type of the universal cover. Note also that G' is canonically bipartite for any G. This little observation is the trick lying at the root of a very important paper of Hashimoto [3].

These examples will do for us today, but in fact I do not know of any others that are similarly tractable, and there is relatively little about interesting irregular trees in the literature. Note well that not every tree covers a finite graph. Shortly we will see some rather obvious necessary conditions, but in fact a satisfactory characterization of such trees (called "uniform") is known and is a jumping off point for the book *Tree Lattices* of Bass and Lubotzky. Note that it is however clear on topological grounds that $G(\mathcal{T})$ is either empty or infinite.

3. $\mathcal{T}$-families and $\mathcal{T}$-expanders

Definition: A $\mathcal{T}$-**family** (or just a family, if $\mathcal{T}$ is understood) of graphs is an infinite sequence of connected finite graphs $\{G_n\}$ with common universal covering tree $\mathcal{T}$, and with $\#G_n \to \infty$.

An **expander** is a $\mathcal{T}$-family in which the infimum of the spectral gaps $PF(G_n) - \lambda_1(G_n)$ is positive.

Twenty-five years ago or so the explicit construction of even d-regular expanders was an open problem, but we have come much farther in recent years:

Theorem 5. *(Lubotzky [8]) Every uniform tree admits an expander.*

[1]This trick can be used to compute the entire spectrum of $K_{r,s}$: A^2 has distinct non-negative eigenvalues, and since $K_{r,s}$ is bipartite, both squareroots of any given eigenvalue of A^2 occur as eigenvalues of A.

Concerning the proof we can only say: "property τ". As usual, the Ramanujan property holds a special allure. Recall that a d-regular Ramanujan graph is a graph all of whose eigenvalues except $PF(G)$ – and, if G is bipartite, its negative – are at most $2\sqrt{d-1}$ in absolute value. This definition comes from the following result:

Theorem 6. *(Alon-Boppana) Let $\{G_n\}$ be any d-regular family. Then*

$$\liminf_n \lambda_1(G_n) \geq 2\sqrt{d-1}.$$

Note that last time we said precisely nothing about the proof. Here we want to view the proof as a conjunction of two different theorems in spectral theory and see how this could be generalized to irregular graphs.

4. Spectral theory on $L^2(\mathcal{T})$

As above let $\mathcal{T}$ be an infinite tree which covers some finite graph. This necessitates that $\mathcal{T}$ have certain properties which make for a nice spectral theory. In particular $\mathcal{T}$ is locally finite, so the adjacency relation gives rise to a well-defined linear operator on the Hilbert space $L^2(\mathcal{T})$. In fact the vertex degrees of $\mathcal{T}$ must be uniformly bounded, and this implies that A is a bounded self-adjoint operator on $L^2(\mathcal{T})$. Whereas in the last lecture we amused ourself by interpreting the adjacency matrix of a finite graph as a self-adjoint operator on a finite-dimensional Hilbert space, the current situation is less quaint: we are interested in the spectrum of A, that is, the set of real numbers λ such that $\lambda I - A$ fails to have a bounded inverse: this is a compact subset of the real line, so the **spectral radius** ρ is well-defined.

Warning: $\rho \neq PF(\mathcal{T})$! E.g. for $\mathcal{T}_d$ it looks like the constant function should again be an eigenfunction with eigenvalue d, but a constant function is not in L^2!

In fact we have the following theorem:

Theorem 7. *The spectral radius of the d-regular tree $\mathcal{T}_d$ is $2\sqrt{d-1}$.*

Now one has to think that this explains the Alon-Boppana bound. And indeed it does, and more:

Theorem 8. *(Greenberg-Cioaba [1]) Fix a uniform tree $\mathcal{T}$ and $\epsilon > 0$. Then there exists $C = C(\epsilon, \mathcal{T})$ such that for any finite graph G covered by $\mathcal{T}$, we have*

$$\#\{\lambda \in \operatorname{Spec} G \mid \lambda \geq \rho(\mathcal{T}) - \epsilon\} \geq C\#V(G).\}$$

We remark that with $|\lambda|$ in place of λ, this is a theorem of Greenberg, which was unfortunately never published. Cioaba's recent (my copy is dated 12/05) preprint gives a nice writeup of the slightly stronger result. In the case of d-regular graphs the result is in fact due to Serre. Serre's beautiful 1997 JAMS paper places this result in the larger context of equidistribution of eigenvalues of graphs and of Frobenius operators on curves of finite fields (highly recommended!).

Putting these two theorems together we get the Alon-Boppana theorem. Note that this is not at all Alon-Boppana's proof: in fact their proof gives an explicit error bound in terms of the girth of the graph. There have been many other papers pursuing variations.

This of course makes us want to compute the spectral radius of the universal covering tree of other finite graphs. There are algorithms for doing this (which I do not understand as well as I would like); we state only one more case, which is quite striking:

Theorem 9. *(Li-Solé, [6]) The spectral radius of the semiregular tree* $\mathcal{T}_{r,s}$ *is* $\sqrt{r-1}+\sqrt{s-1}$.

Thus:

Corollary 10. *In any infinite family of* (r,s)*-semiregular bipartite graphs we have* $\inf \lambda_1 \geq \sqrt{r-1}+\sqrt{s-1}$.

In fact:

Proposition 11. $\lambda_1(K_{r,s}) = \sqrt{r-1}+\sqrt{s-1}$.

This very strongly suggests that an (r,s)-semiregular graph should be called Ramanujan if it satisfies $\lambda_1 \leq \sqrt{r-1}+\sqrt{s-1}$. This definition was in fact made about 20 years ago in a paper by Hasimoto, who as mentioned above, thought very deeply about bipartite graphs. There are in fact other reasons for believing that this is "the right definition" of a semiregular Ramanujan graph, involving the zeta function, which we cannot try to squeeze into this talk. One nice consequence of Hashimoto's theory is the following:

Corollary 12. *([3]) If* G *is a Ramanujan* d*-regular graph, then the barycentric subdivision* G' *is a Ramanujan* $(d,2)$*-semiregular graph.*

In particular, combining with the known results on regular Ramanujan graphs, this shows that the trees $\mathcal{T}_{q+1,2}$ cover infinitely many Ramanujan graphs for any prime power q. Because we have at least the one Ramanujan graph $K_{r,s}$ for any pair (r,s), however, it is natural to conjecture more:

Conjecture 13. *Every* $\mathcal{T}_{r,s}$ *covers infinitely many Ramanujan graphs.*

In fact there is a chance at using the strategy of the proof of the $\mathcal{T}_{q^d+1}$ case to prove that, e.g. when $(r,s) = (p+1, p^2+1)$, $\mathcal{T}_{r,s}$ covers infinitely many Ramanujan graphs. All we can say here is that the $q+1$-regular tree shows up naturally as the Bruhat-Tits tree for SL_2 of a local field $K/\mathbb{Q}_p$ whose residue field has cardinality q (a sort of p-adic symmetric space). There are certain more exotic K-adic Lie groups with K-rank 1 whose Bruhat-Tits building is a $(p+1, p^2+1)$-semiregular tree: the complete list of possibilities is found in a paper of Hashimoto and Hori [4]. It should be possible to use work of Helm on a Jacquet-Langlands correspondence for the associated Shimura varieties to derive the Ramanujancy of these semiregular graphs! This would make for a hell of a thesis project.

5. What is an irregular Ramanujan graph?

Note that we have not quite said what it means for a finite graph to be Ramanujan. The preceding theory spells out a plausible definition, again propounded by Greenberg in his thesis. I am not sure whether it is the "correct" definition. Let me give it and then explain why I feel this way.

Our setup thus far involves starting with a tree $\mathcal{T}$ which covers at least one (and hence infinitely many) finite graphs G. Associated to $\mathcal{T}$ we have the spectral radius ρ, and as we saw earlier the finite graphs G all have a common Perron-Frobenius eigenvalue $\lambda_0(G) = PF(\mathcal{T})$, with the property that any eigenvalue $\lambda \neq \pm PF(\mathcal{T})$ is smaller in absolute value. This suggests the following:

A finite graph G is Ramanujan if for every eigenvalue λ with $|\lambda| < \lambda_0$, we have $\lambda < \rho(\mathcal{T})$, where $\mathcal{T}$ is the universal covering tree of G.

One's first reaction to the definition is that given an actual finite graph G, it is not completely clear how to compute $\rho(\mathcal{T}(G))$. In fact there is a paper of Nagnibeda [10] which gives an algorithm for this.

But it is a little strange to ask whether a given graph is Ramanujan according to this definition: Ramanujancy is a property which (as far as we have said) becomes significant only upon consideration of a family of graphs with a common universal covering tree. So more natural is to define a (uniform, i.e., covering some finite graphs) tree $\mathcal{T}$ to be Ramanujan if it covers infinitely many finite graphs G with $\lambda_1(G) \leq \rho(\mathcal{T})$. Again, with this definition we unforunately do not know whether $\mathcal{T}_7$ is Ramanujan.

On the other hand, we know that $\mathcal{T}_7$ covers at least one Ramanujan graph, namely K_8. (For the record, it is easy to modify K_8 to get a fair-sized handful of Ramanujan 7-regular graphs; what we cannot do as yet is produce infinitely many.) So let us say that a uniform tree $\mathcal{T}$ is **weakly Ramanujan** if it covers at least one Ramanujan graph. Now we have the following surprising result:

Theorem 14. *(Lubotzsky-Nagnibeda* [9]*) There exist uniform trees which are not even weakly Ramanujan.*

This is a very interesting result, but I confess that it makes me think that this definition of Ramanujancy is not the right one. Note for instance that the Greenberg-Cioaba theorem deals with much more than the second-largest eigenvalue: it says that asymptotically *lots* of eigenvalues will have to approach the spectral radius of the universal covering. In the simplest cases it happens that there is always at least one eigenvalue – the Perron-Frobenius eigenvalue – which lies above this bound. If we are willing to ignore one eigenvalue, why not try ignoring a finite number?

So here is another definition, which starts at the bottom rather than the top.

Suppose G is a finite graph. For a finite covering graph $H \to G$, define the relative spectrum $\operatorname{Spec}_{new}(H)$ to be $\operatorname{Spec} H \setminus \operatorname{Spec} G$.

Definition: We say that H is a **Ramanujan covering** of G if every $\lambda \in \operatorname{Spec}_{new}(H)$ satisfies $|\lambda| \leq \rho(G)$.

It is then tempting to make the following

Conjecture 15. *(Ramanujan covering conjecture) Every finite graph G admits infinitely many Ramanujan covering graphs.*

Note that an immediate consequence of this conjecture is that weakly Ramanujan implies strongly Ramanujan. Even applied to d-regular graphs the conjecture is *a priori* stronger than this: it is conceivable that the set of all d-regular Ramanujan graphs is infinite but "horizontal" in the sense that no fixed graph has infinitely many coverings. But we note that in all cases in which we know an infinite family of d-regular Ramanujan graphs, we actually have infinitely many coverings of any fixed graph. (In fact, much more is true: for a given graph in the family, the Ramanujan coverings are cofinal in the set of all finite covering graphs.) Let us note that Joel Friedman has some recent results which show – using probabilistic methods – that any finite graph has infinitely many coverings whose new eigenvalues are "close" to the Ramanujan bound.

Finally, the conjecture can be cleanly phrased in terms of the Ihara(-Hashimoto-Bass) zeta functions of the graphs, a topic which we unfortunately found no time to discuss.

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