RAMANUJAN GRAPHS AND SHIMURA CURVES

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What follows are some long, rambling notes of mine on Ramanujan graphs. For a period of about two months in 2006 I thought very intensely on this subject and had thoughts running in several different directions. In paricular this document contains the most complete exposition so far of my construction of expander graphs using Hecke operators on Shimura curves. I am posting this document now (March 2009) by request of John Voight.

Introduction

Let q be a positive integer. A finite, connected graph G in which each vertex has degree $q+1$ and for which every eigenvalue λ of the adjacency matrix $A(G)$ satisfies $|\lambda| = q+1$ (such eigenvalues are said to be *trivial*) or $|\lambda| \leq 2\sqrt{q}$ is a *Ramanujan graph*, and an infinite sequence G_i of pairwise nonisomorphic Ramanujan graphs of common vertex degree $q+1$ is called a *Ramanujan family*.

These definitions may not in themselves arouse immediate excitement, but in fact the search for Ramanujan families of vertex degree $q+1$ has made for some of the most intriguing and beautiful mathematics of recent times. In this paper we offer a survey of the theory of Ramanujan graphs and families, together with a new result and a "new" perspective in terms of relations to (Drinfeld-)Shimura curves.

We must mention straightaway that the literature already contains many fine surveys on Ramanujan graphs: especially recommended are the introductory treatment of Murty [**?**], the short book of Davidoff, Sarnak and Vallette [**?**] (which can be appreciated by undergraduates and research mathematicians alike), and the beautiful books of Sarnak [**?**] and Lubotzsky [**?**], which are especially deft at exposing connections to many different areas of mathematics. (Not to mention several works of W. Li, Terras, Stark-Terras,...) Originally I was leery adding another survey-type paper on Ramanujan graphs to the stack, until I realized: it is much more dubious when the stack is empty (perhaps no one else cares about the topic) or contains a single paper (perhaps the extant paper is definitive). Here, the fact that there are so many survey papers on Ramanujan graphs indicates that many are interested in the subject and that this is a theme which fruitfully admits many different variations.

The main result on Ramanujan families is that they are known to exist when $q+1$ is a prime power; all other cases are open. This has been the state of affairs for almost 15 years. Nevertheless more than 100 papers on Ramanujan graphs have appeared in the past decade. Some of these papers – notably, those of Li, Terras and their students – deal with families of Ramanujan graphs of variable vertex degree, a topic that we almost entirely omit here for lack of anything new to say. The remaining papers deal with generalizations of Ramanujan graphs, of which there are at least three: (i) a notion of "Ramanujancy" for graphs with unequal vertex degrees; (ii)

Morgenstern's "Ramanujan diagrams" – a special kind of infinite weighted graph closely related to the more exotic behavior of congruence subgroups acting on the Bruhat-Tits tree in the equicharacteristic case; and (iii) various higher-dimensional analogues: Ramanujan hypergraphs/complexes.

Here we discuss the case of graphs of unequal vertex degrees: it is not any more cumbersome technically, and leads to an array of interesting new problems: for instance, which trees cover at least one finite Ramanujan graph? It must be mentioned that the definition of irregular Ramanujan graphs is not universally agreed upon: a somewhat weaker notion, which we unimaginatively call *quasi-Ramanujan* may turn out to be more interesting. On the other hand, the notion of Ramanujancy for (r, s)-*semi-regular* graphs – i.e., graphs admitting a bipartition for which the vertices in the first set all have degree r and vertices in the second set all have degree s – looks quite promising, and in itself justifies the more general definition. We believe (following Lubotzky) that every (p^a, p^b)-semiregular Bruhat-Tits tree should lead to the construction of (p^a, p^b)-semiregular Ramanujan families.

The most exciting recent work has been on on Ramanujan hypergraphs/complexes. This has been pursued independently by several research groups, leading to slightly different setups and results. I would dearly like to compare and unify these higher-dimensional constructions, but they all use aspects of the theory of automorphic forms, p-adic representations, and Bruhat-Tits buildings (and, implicitly, Shimura varieties and p-adic uniformization) for a reductive group $G_{/\mathbb{Q}}$ of higher rank: these matters are well outside the scope of my current technical expertise. What I can hope is that clarifying the underlying geometry of the one-dimensional situation will give some clues as to the form a definitive higher-dimensional theory ought to take.

Finally we come to the main point of this paper: to present the construction of families of $q + 1$-regular Ramanujan graphs for all prime powers q in terms of a Hecke-equivariant correspondence between two quaternionic Shimura curves over a totally real global field F. When $F = \mathbb{Q}$ we are essentially just adding level N structure (i.e., passing to a congruence subgroup) to a 40 year-old construction of Ihara, which treats the case in which q is prime. We also attain as a special case the graphs constructed in the seminal paper of Lubotzky-Philips-Sarnak [**?**]. We note that the fact that the Ramanujan property for these graphs follows directly from the Riemann hypothesis for classical modular curves over finite fields seems to be missing from the literature: c.f. [**?**, Note 7.5.1]. When $F = \mathbb{F}_q(T)$ we are giving a more geometric description of a construction originally due to Morgenstern. Strangely, the construction does not seem to have been pursued for totally real *number* fields $F \neq \mathbb{Q}$ (except in passing in the work of Jordan and Livné), despite the fact when correctly set up, the construction goes through in the general case with only minor technical complications.[1]

[1]Except that when F has even degree over its prime subfield, we need to assume the existence of one more finite ramified prime than the general setup guarantees us. This shall be discussed in more detail later.

What is gained from these generalizations and recastings of work of Ihara, L-P-S and Morgenstern? First, there is a technical advantage: by exploiting the most basic difference between number fields and function fields – in particular, that for all integers $q+1 \geq 2$ there exists a totally real number field F and a squarefree ideal of F of norm $q+1$, we are able to prove the following

Theorem 1. *For $q \geq 2$, there is a family of $(q+1)$-regular graphs $\{G_n\}$ each of whose nontrivial eigenvalues satisfy*

$$\lambda(G_n) \leq 2^{\omega(q+1)}\sqrt{q},$$

where $\omega(n)$ is the number of distinct prime divisors of n. The graphs G_n are constructed deterministically.

Note that these graphs are Ramanujan graphs whenever $q+1$ is a prime power; for arbitrary q this is to my knowledge the best known bound on the nontrivial eigenvalues in a deterministically constructed family of $q+1$-regular graphs. In particular, it improves upon a prior bound of Pizer [**?**], who used only the case of quaternion algebras over $\mathbb{Q}$.

There is another conclusion to be drawn. As we shall recall, Ihara attached to every $q+1$-regular graph G a zeta function $Z_G(s)$, a rational function of s of the following special form:

$$Z_G(s) = (1-u^2)^{-q|G|/2} \det(I - A(G)u + qu^2)^{-1}, \ u = q^{-s},$$

where $|G|$ is the number of vertices of G and $A(G)$ is the adjacency matrix. It follows immediately that G is Ramanujan iff the reciprocal roots of the polynomial $P_G(u) = \det(I - A(G)u + qu^2)$ all have absolute value $\sqrt{q}$, i.e., are Weil q-numbers. No arithmetic geometer could see this without being reminded of the Riemann hypothesis for varieties over a finite field: for any smooth, connected projective variety $V_{/\mathbb{F}_q}$, all the roots of the characteristic polynomial $P_V(u)$ of the q-power Frobenius Fr_q acting on $H^1(V, \mathbb{Q}_\ell)$ are Weil q-numbers. Thus one way to show that a graph is Ramanujan is to identify an algebraic variety V such that $P_G(u) = P_V(U)$.

This is in fact what the above construction accomplishes: we take V to be the (good) reduction modulo a prime $\mathfrak{p}$ of F of a suitable Shimura curve $X_{/F}$. In other words: for every prime power q, all but finitely many known $q+1$-regular Ramanujan graphs G are obtained by identifying – up to trivial factors – the Ihara zeta function of G with the Hasse-Weil zeta function of an algebraic curve!

I find this result to be remarkable in that it is simultaneously very exciting and quite deflating: it indicates how completely dependent the constructions of Ramanujan families are on arithmetic geometry. Since Theorem 22 is surely the best bound we can wring from arithmetic-geometric methods, what then can we possibly do to improve upon it?

1. Spectra of finite graphs

1.1. **Graph-theoretic terminology.** We shall assume that the reader has a prior acquaintance with graph-theoretical notions and terminology. In view of the many minor variations in the definitions and terminology, we shall begin with a few basics.

A graph G is a set V (of *vertices*), a set E (of *oriented edges*) and a map $E \to V \times V$, which we shall write as $e \mapsto (o(e), t(e))$; here $o(e)$ is the *origin* of e and $t(e)$ is the *terminus*. We shall require the existence of a fixed-point free involution $i : E \to E$, such that $o(i(e)) = t(e)$, $t(i(e)) = o(e)$; the i-equivalence classes will be referred to as *geometric edges*. A choice of a set of representatives $\mathcal{O}$ for E/i is called an *orientation* of G,[2] and a pair $(G, \mathcal{O})$ is called an *oriented graph*. However, in practice we will suppress the $\mathcal{O}$ from our notation.

To an oriented graph G one associates a one-dimensional CW-complex, with 0-skeleton V, and whose 1-skeleton is $\mathcal{O}$, with the obvious attaching maps. Conversely, the CW-complex determines the oriented graph. The underlying topological space (which, by abuse of notation, we will also denote by G) is independent of the choice of $\mathcal{O}$. The fundamental group of G (at any basepoint) is a free group, and the homology groups $H^i(G, \mathbb{Z})$ are free abelian groups: $h^i(G, \mathbb{Z}) = 0$ for all $i \geq 2$; $h^0(G, \mathbb{Z})$ is equal to the number of connected components; and we put $r = h^i(G, \mathbb{Z})$, the *rank* of G (which is the rank of $\pi_1(G)$ if G is connected). Suppose that G is connected: then, as it is locally simply connected it admits a universal cover $\mathcal{T}(G)$, a tree. Every covering space of G has in a canonical way the structure of a CW-complex, hence it makes sense to speak of the coverings of a graph G. It follows that there is a purely graph-theoretic notion of covering; we leave it to the reader to formalize this.

For $(i, j) \in V(G)^2$, let $a(i, j)$ denote the cardinality of the set of edges e with $o(e) = i$, $t(e) = j$. We say that i and j are *adjacent* if $a(i, j) > 0$. For any vertex i, the *degree* of i is $\sum_{j \in V(G)} a(i, j)$. Note well that we allow more than one edge to join a pair of vertices, and we also allow edges with $o(e) = t(e)$, i.e., *loops*. Moreover, according to our conventions, each geometric loop based at a vertex i contributes 2 to its degree. G is *locally finite* if each vertex has finite degree, and has *bounded degrees* if $\sup_{i \in V(G)} d(i) = D < \infty$.

1.2. **Spectra of graphs.**

To a graph G, we may consider the Hilbert space $H(G) = \ell^2(V(G))$ for which the vertices of G form an orthonormal basis. If G is locally finite, we can define the *adjacency operator* A, by

$$A(i) = \sum_{j \in V(G)} a(i, j) j.$$

Evidently A is a self-adjoint linear operator on $H(G)$; it is bounded iff G has bounded degrees, in which case a rich and fruitful spectral theory applies. From now on it shall go without saying that all our graphs have bounded degrees. Moreover, in the remainder of this introductory section we shall assume that G is finite: in this case, after identifying $V(G)$ with $\{1, \ldots, n\}$, we may represent the adjacency operator by a matrix A whose (i, j)th entry is simply $a(i, j)$. We define the *spectrum* $\operatorname{Spec} G$ to be the set of (real, since A is symmetric) eigenvalues of A, taken with (geometric = algebraic) multiplicities: we may write

$$\operatorname{Spec} G = \{\{\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_r\}\},$$

where the double braces are there to remind us that we have a set with multiplicities (or a "multiset").

[2]Notice that orientations always exist.

The spectrum of a finite graph is an interesting invariant, to be viewed in analogy with the spectrum of the Laplacian on a compact Riemannian manifold. In fact, one can define a combinatorial Laplacian ("on functions") on an arbitrary finite CW-complex; in the case of a finite graph, this turns out to be the linear operator whose matrix representative with respect to our canonical basis is $D - A$, where D is the diagonal matrix with (i, i) entry equal to $d(i)$.

The following facts are easy to prove and left to the reader.

Proposition 2. *For a finite graph G, let $\overline{d} = \frac{1}{|G|}\sum_{i \in V(G)} d(i)$ be the average vertex degree. Then* $\operatorname{Spec} G \subset [-\overline{d}, \overline{d}]$.

A graph is d-regular if every vertex has degree d; these graphs have an especially clean spectral theory and will be of most interest to us here (although the case of irregular graphs leads to many interesting open problems).

Proposition 3. *If G is d-regular, then the d-eigenspace consists of the locally constant functions.*

In particular, for a d-regular graph we have $\lambda_0 = d$, and the multiplicity of d is equal to the number of connected components (or, equivalently, to $h^0(G, \mathbb{Z})$.) If G is connected, then in $H(G)$, which is now just the set of all $\mathbb{C}$-valued functions on the vertices of G, the orthogonal complement of the d-eigenspace is the set of all functions f with $\sum_{i \in V} f(i) = 0$.

In general the top eigenvalue λ_0 depends only on the *local* geometry:

Theorem 4. *(Greenberg) Suppose G_1, G_2 are two connected finite graphs with isomorphic universal coverings. Then $\lambda_0(G_1) = \lambda_0(G_2)$.*

Example: The universal cover of any d-regular graph is the d-regular tree $\mathcal{T}_d$.

Recall that a graph is *bipartite* if we can paritition $V(G)$ into $V_1 \coprod V_2$ such that two vertices are adjacent only if one is in V_1 and the other is in V_2.

Example X: A graph is (d_1, d_2)-semiregular if it admits a bipartition $V_1 \coprod V_2$ such that each vertex in V_i has degree d_i. The universal cover of such a graph is the (d_1, d_2)-semiregular tree.[3] We claim that for a finite (d_1, d_2)-semiregular graph, $\lambda_0 = \sqrt{d_1 d_2}$. Indeed, let A be its adjacency matrix. Then $B := A^2$ is the matrix whose (i, j) entry counts the number of paths of length 2 from i to j. For any i, $\sum_j b(i, j)$ is equal to the number of length 2 paths starting at i, which is evidently $d_1 d_2$. Thus (as in Proposition 3), the constant function is an eigenfunction for B of eigenvalue $d_1 d_2$, which means that at least one of $\pm\sqrt{d_1 d_2}$ is an eigenvalue of A. The following proposition shows that both occur.

Proposition 5. *For a finite graph G, the following are equivalent:*
a) $\operatorname{Spec} G = -\operatorname{Spec} G$ *(with multiplicities).*
b) G is bipartite.
c) If G is d-regular, $-d \in \operatorname{Spec} G$.

[3] In other words, we claim that there is, up to isomorphism, a *unique* (d_1, d_2)-semiregular tree, a fact we leave to the reader to verify.

For a d-regular graph G, we define the *cuspidal spectrum* $\operatorname{Spec}_c G = \operatorname{Spec} G \cap (-d, d)$, and

$$\lambda = \lambda(G) = \max\{|\lambda| \mid \lambda \in \operatorname{Spec}_c G\}.$$

Example X: Let K_r be the complete graph on r vertices (i.e., $a(i,j) = 1 - \delta(i,j)$). Then

$$\operatorname{Spec} K_r = \{\{r-1, -1, \ldots, -1\}\},$$

where -1 occurs with multiplicity $r-1$. Indeed, the case of $r = 2$ is immediate, so assume $r \geq 3$. As K_r is not bipartite, $\operatorname{Spec}_c K_r = \{f : V \to \mathbb{C} \mid \sum_i f(v_i) = 0\}$. For any such function,

$$(Af)(v_i) = \sum_{j \neq i} f(v_j) = \sum_i f(v_i) - f(v_i) = -f(v_i).$$

Example 4: Let C_r be the r-cycle, i.e., the graph with vertex set $\mathbb{Z}/r\mathbb{Z}$ and with i adjacent to $i-1$ and $i+1 \pmod r$. It turns out that the eigenfunctions are the characters: for $0 \leq a < r$, let $\chi_a : V \to \mathbb{C}$ by $\chi_a(i) = e^{2\pi\sqrt{-1}\frac{a}{r}i}$. Then

$$A\chi_a(i) = \chi_a(i+1) + \chi_a(i-1) = \chi_a(i)\left(\chi_a(1) + \chi_a(-1)\right),$$

so χ_a is an eigenfunction with eigenvalue $\chi_a(1) + \chi_a(-1) = 2\cos(\frac{2\pi a}{r})$. So

$$\operatorname{Spec} C_r = \{2\cos(\frac{2\pi i}{r})\}_{0 \leq a < r}\}.$$

Semiloops: From a purely algebraic standpoint, it seems artificial to proscribe adjacency matrices whose diagonal elements can be odd positive integers. We can give such matrices a geometric interpretation (of a sort) in terms of *semiloops*: in other words, an edge e with $o(e) = t(e)$ and which is equal to its own inverse. Such a thing would contribute 1 to the degree of its incident vertex. In fact graphs with semiloops arise naturally, namely as the quotient of an honest graph under the action of a group which fixes a geometric edge but reverses its orientation. There is, in general, no reason why a group action *should not* do this, and by restricting to group actions *without inversions* one misses out on some fundamental phenomena (see [**?**]). In this note we will use graphs with semiloops as intermediate steps in constructions (so the reader who finds them distasteful is free to make other slightly more complicated, constructions to avoid them). One annoying feature is that a graph with (possible) semiloops is no longer uniquely determined by its adjacency matrix when $a(i,i) > 1$. To build graphs from adjacency matrices, we will make the convention here of minimal semiloops: i.e., given an adjacency matrix, we build the corresponding graph with zero (resp. 1) semiloops at a vertex i if $a(i,i)$ is even (resp. odd).

Example 3 (L operator): Consider the operator L on finite graphs (with semiloops) which takes the adjacency matrix A to $I + A$: geometrically speaking, L adds a semiloop to each vertex. Clearly $\operatorname{Spec} LG = 1 + \operatorname{Spec} G$.

Example 4 (bipartite cover): Consider the operator $G \mapsto \tilde{G}$, which takes A to $\begin{bmatrix} 0 & A \\ A^t & 0 \end{bmatrix}$; we call $\tilde{G}$ the *bipartite cover* of G.

Proposition 6. *a) For any finite graph G, $\tilde{G}$ is a bipartite graph.*
G itself is bipartite iff $\tilde{G}$ is not *connected.*
c) $\operatorname{Spec} \tilde{G} = \operatorname{Spec} G \coprod (-\operatorname{Spec} G)$.

Note that the construction is strongly reminiscent of the orientation covering of a manifold.[4] It also gives a sense in which nonbipartite graphs are more interesting than bipartite graphs: given any nonbipartite graph, we can always produce a bipartite graph with the same absolute values of eigenvalues (in particular, given a graph with semiloops we can always build an graph with the same positive eigenvalues) but the converse is not necessarily true.

Example 5: Let $K_{r,r} := \widetilde{LK_r}$. It follows from the above that its spectrum is $\{\{\pm r, 0\}\}$, where 0 occurs with multiplicity $2r-2$.

1.3. **Automorphisms and spectra.**

Clearly the spectrum of a given finite graph is a computable invariant, either exactly – as the characteristic polynomial of the adjacency matrix – or approximately – i.e., one can find the multiplicities of all the elements of the spectrum and computing them as decimal numbers to any prescribed degree of accuracy – , as a built-in feature of almost any mathematical software package.[5] (How much time and memory are required to do these computations as a function of the size of G is an issue that my ignorance forces me to omit completely.) However, given a family of graphs $\{G_i\}$ it is in general quite challenging to say something about the limiting behavior of their spectra. Necessarily then the exploration of graphs with interesting spectral properties involves finding certain special kinds of graphs whose spectra are easy (or at least easier) to compute.

The most useful idea is to consider graphs with symmetry: let $\mathcal{G}$ be a finite group acting effectively on $V(G)$ (on the right, let us say) by graph automorphisms. Then the elements g of $\mathcal{G}$ act isometrically on $H(G) = \{f : V(G) \to \mathbb{C}\}$ – indeed, on $H_{\mathbb{Q}}(G) = \{f : V(G) \to \mathbb{Q}\}$ – and commute with the self-adjoint operator A:

$$(gAf)(i) = g\sum_j a(i,j)f(j) = \sum_j a(i,j)f(g^{-1}j) = \sum_j a(g^{-1}i, g^{-1}j)f(g^{-1}j)$$

$$= \sum_{g(j)} a(g^{-1}i, j)f(j) = (Agf)(i).$$

Thus the λ-eigenspaces of A are $\mathcal{G}$-stable, from which we may deduce relations between $\mathrm{Aut}(\mathcal{G})$ and $\mathrm{Spec}\, G$: for instance, if all the eigenspaces are one-dimensional, then $\mathcal{G}$ embeds into the group of $|G| \times |G|$ diagonal matrices with real entries, so that $\mathcal{G}$ is abelian of exponent 2.

The following is a key result:

Theorem 7. *(Babai, Diaconis-Shashahani) Let G be a finite loopless graph, and $\mathcal{G}$ a finite group acting simply transitively on $V(G)$, so that after fixing a vertex 1 we may identify $V(G)$ with $\mathcal{G}$. Let S be the multiset in which $g \in V(G)$ occurs with multiplicity $a(1,g)$. Assume that S is stable under inner automorphisms of $\mathcal{G}$. Let $\rho_1, \ldots, \rho_r$ be the irreducible complex characters of $\mathcal{G}$, and $d_i = \dim \rho_i = \rho_i(e)$. Then the eigenvalues of G are $\lambda_i = \frac{1}{d_i}\sum_{s\in S}\chi_i(s)$, occurring with multiplicity d_i^2.*

[4] Indeed, I presume that both are special cases of some more general construction.

[5] Probably MATLAB is the best at these sort of computations

For the proof (which will obviously require some notions from representation theory) see [**?**, p. 106-107]. In the case that $\mathcal{G}$ is abelian, the result much simplifies: there are no nontrivial inner automorphisms, and the irreducible characters χ_i of $\mathcal{G}$ are all one-dimensional. Indeed, we need only observe that χ_i, viewed as an element of $H(G)$, is an eigenfunction with eigenvalue $\sum_{s \in S} \chi_i(s)$, i.e., the argument of Example X applies.

Example X: Let G be the one-skeleton of the n-dimensional hypercube. We have an evident simply transitive action by $\mathcal{G} = \mathbb{F}_2^n$ with $S = \{e_i\}_{i=1}^n$, where $e_i(j) = \delta(i,j)$ is the ith standard basis element. After identifying the groups $\mathbb{F}_2$ and ± 1, the $\mathbb{F}_2$-linear functionals $\varphi : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ on $\mathcal{G}$ are precisely the irreducible characters $\chi : \mathbb{F}_2^n \rightarrow \mathbb{C}^\times$ of $\mathcal{G}$. Thus the eigenvalues are $\sum_{i=1}^n \chi(e_i)$. Since the 2^n linear functionals are obtained by linear extension of the 2^n maps $f : S \rightarrow \mathbb{F}_2$, we find that the eigenvalues are precisely the sums of all possible sign sequences $(\epsilon_1, \ldots, \epsilon_n)$ of length n. So for all $k, 0 \leq k \leq n$, we have an eigenvalue $n-2k$ with multiplicity $\binom{n}{k}$.

Example X: Given graphs $G_1, \ G_2$, we can form a graph $G_1 \oplus G_2$ with vertex set $V = V(G_1) \times V(G_2)$ and edge set $E = E(G_1) \times V(E_2) \coprod V(G_1) \times E(G_2)$, with the obvious projection map $E \rightarrow V$. (This is indeed the direct sum in the category of graphs.) Then (when $G_1, \ G_2$ are finite) [**?**]

$$\operatorname{Spec}(G_1 \oplus G_2) = \operatorname{Spec} G_1 + \operatorname{Spec} G_2 = \{\{\lambda_i + \mu_j \mid \lambda_i \in \operatorname{Spec} G_1, \ \mu_j \in \operatorname{Spec} G_2\}.$$

Since $G = \bigoplus_{i=1}^n \bullet - \bullet$, and $\operatorname{Spec}(\bullet - \bullet) = \{\pm 1\}$, this gives another computation of its spectrum.

Conversely, given a (not necessarily finite) group G, and a multisubset S of $G \setminus 1$, we can construct a *directed* graph with a simply transitive (right) G-action: namely we take $V(G) = G$ and for every $g \in G$ and $s \in S$ insert an oriented edge e with $o(e) = g, \ t(e) = gs$. Assume moreover that $S = S^{-1}$; then $(g, gs) \mapsto (gs, g) = (gs, gss^{-1})$ defines a fixed-point free involution on the oriented edges giving us the structure of a graph. This construction is called the *Cayley graph* of G with respect to S and written $\operatorname{Cay}(G, S)$.

Proposition 8. *a)* $\operatorname{Cay}(G,S)$ *is connected iff* S *generates* G.
b) $\operatorname{Cay}(G,S)$ *is simple iff elements of* S *occur with multiplicity* 1.

An order two element $x \in S$ plays a special role: as an automorphism of G x *inverts* the edge corresponding to s, whereas elements of order $n \in [3, \infty]$ act without inversions. Moreover, the condition $S = S^{-1}$ means that given a set $S' = \{\gamma_1, \gamma_{r+s}\}$ of nontrivial generators for a group G, if $\gamma_1, \ldots, \gamma_r$ have order 2 and $\gamma_{r+1}, \ldots, \gamma_{r+s}$ do not, then the corresponding symmetrized generating set $S = S' \cup S'^{-1}$ leads to a Cayley graph of degree $r + 2s$.

For an arbitrary multisubset $S' \subset G \setminus 1$, let us define $\operatorname{Cay}(G, S') = \operatorname{Cay}(G, S' \cup S'^{-1})$.

Example X.X: Let $F_n = \operatorname{Free}(x_1, \ldots, x_n)$ be the free group on the set $S' = \{x_1, \ldots, x_n\}$. Then $\operatorname{Cay}(F, S')$ is the regular tree of degree $2n$. Conversely, a group acting freely *without inversions* on a tree is necessarily a free group (CITE).

Example X.X: Consider the free product $G = \bigstar_{i=1}^r \mathbb{Z}/2\mathbb{Z} \star \bigstar_{j=1}^s \mathbb{Z}$, or if you like, the group with presentation:

$$\langle x_1, \ldots, x_r, y_1, \ldots, y_s \mid x_1^2 = \ldots = x_r^2 = 1 \rangle.$$

Then $\mathrm{Cay}(G, \{x_1, \ldots, x_r, y_1, \ldots, y_s\})$ is a regular tree of degree $r + 2s$ – indeed, the only relations $x_i^2 = 1$ correspond to paths with backtracking on the Cayley graph. Conversely, a group acting freely on a tree is necessarily the free product of some (possibly infinite) number of copies of $\mathbb{Z}/2\mathbb{Z}$ with some number of copies of $\mathbb{Z}$.

The Cayley graph construction generalizes to merely transitive group actions: indeed, given a group $\mathcal{G}$, a subgroup $\mathcal{H}$, and a symmetric multisubset S of $\mathcal{G}$, we may build a graph with vertex set given by the left coset space $\mathcal{H}\backslash\mathcal{G}$, and with edges $(g\mathcal{H}, sg\mathcal{H})$. This called a Schreier graph, and will be (a bit unfairly) denoted by $\mathrm{Cay}(\mathcal{G}/\mathcal{H}, S)$. There is a similar result to Theorem XX for the computation of the spectrum of a Schreier graph (when S is stable under conjugation): the only difference is that $H(G)$, as a $\mathbb{C}[\mathcal{G}]$-module, is now isomorphic to $\mathbb{C}[G/H]$ (or, if you like, the induction from H to G of the trivial module). Thus, for each irreducible character χ_i the eigenvalue $\frac{1}{d_i} \sum_{s \in S} \chi_i(s)$ occurs with multiplicity equal to d_i times the multiplicity of χ_i in $\mathbb{C}[G/H]$.

2. Expanders and Ramanujan graphs

2.1. **Definitions.** For a finite graph G, the *spectral gap* is the quantity $\omega(G) = \lambda_0(G) - \lambda_1(G)$.

We will be interested in studying infinite sequences of finite graphs. To avoid trivialities, it is convenient to define a *family* $\mathcal{F}$ to be a sequence $\{G_i\}$ of finite graphs satisfying:

(F1) $|V(G_i)| \to \infty$ with i.
(F2) The number of connected components $h_0(G_i, \mathbb{Z})$ is a bounded function of i.
(F3) There is a fixed tree $\mathcal{T}$ such that the universal cover of every connected component of every G_i is isomorphic to $\mathcal{T}$.

Definition: A *q-family* is a family of $(q+1)$-regular graphs.[6]

Remark: The ordering of the elements in a family is not of any importance. When one is comparing two families $\mathcal{F}_1$ and $\mathcal{F}_2$, it is convenient to choose some universal enumeration $\{G_i\}$ of all finite graphs and then put the elements of the family in order according to this enumeration (or, rather, to keep in mind that this can be done!).

Remark: By (F1), a given graph may occur up to isomorphism in a family $\mathcal{F}$ only finitely many times. Thus any family has a subfamily of pairwise nonisomorphic graphs; the maximal such family – i.e., in which we just take each isomorphism type occurring in the family exactly once. We call this $\mathcal{F}_s$, the *reduced subfamily* of $\mathcal{F}$.

Definition: We will say that a family $\mathcal{F}_1$ is *essentially contained* in another family

[6] The reasons for this strange normalization will become clear later.

$\mathcal{F}_2$ if all but finitely many graphs in $\mathcal{F}_1$ occur (up to isomorphism, of course) i $\mathcal{F}_2$; we denote this by $\mathcal{F}_1 \leq \mathcal{F}_2$. If $\mathcal{F}_1 \leq \mathcal{F}_2$ and $\mathcal{F}_2 \leq \mathcal{F}_1$ we will say that the two families are *essentially equal* and write $\mathcal{F}_1 \cong \mathcal{F}_2$.

Definition: An *expander* is a family $\{G_i\}_{i=1}^{\infty}$ of finite graphs such that $\inf \omega(G_i) > 0$.

Remark: Note that (E) implies that G_i is connected for all i, since $\omega(G) = 0$ for disconnected graphs.

An adequate explanation of the importance of expanding graphs in (e.g.) computer science and network design is beyond the scope of our expertise: we refer the reader to XXX. Roughly speaking, a family of expanders gives us a way of building a sequence of networks which are *sparse* – (F3) implies that the number of edges grows linearly with the number of vertices – but nevertheless propagate information very efficiently. To get at least an idea of what the latter statement might mean, we define the *isoperimetric constant* h of a connected finite graph G: namely, for all ways of partitioning $V(G)$ into two subsets $V_1 \coprod V_2$, consider the ratio of the number of edges running between V_1 and V_2 to the smaller of $|V_1|$ and $|V_2|$. The least number obtained in this way is, by definition, $h(G)$. Define $h(G) = 0$ if G is not connected.

Theorem 9. *Let $\{G_i\}$ be a family of finite graphs satisfying (E1) and (E2). Then $\inf \omega(G_i) > 0$ iff $\inf h(G_i) > 0$.*

This is a corollary of a more precise double inequality, due to Alon, Dodziuk, Milman and Tanner: see e.g. [**?**], [**?**].

It turns out that in fact, "most" families of graphs satisfying (E1) and (E2) also satisfy (E3). In other words, there exists $\epsilon > 0$, depending upon $\tilde{G} = \tilde{G_i}$, such that the probability that a graph G with n vertices and universal cover $\tilde{G}$ has $\omega(G) \geq \epsilon$ tends to 1 as $n \to \infty$. (We will not enter into the details of the probabilistic model; the result holds for any reasonable way of counting.) The proof is not at all difficult: see [**?**, §1.2] for the case when $\tilde{G}$ is the k-regular tree.

The idea of showing existence of a discrete structure with certain properties by showing that the probability that a random structure has the property is positive ("the probabilistic method") is due to Erdos and Renyi; their original paper in fact addresses this kind of result. The probabilistic method has had a revolutionary effect on discrete mathematics: for most problems, it is much easier to prove the existence of a structure with the desired properties probabilistically than to give an explicit construction.[7] Of course, if you are building a fiber-optic network, then the *existence*, or even the probabilistic ubiquity, of networks with good expansion properties, is not of much use to you: you want an explicit construction, and ideally a method for producing explicit constructions robust enough to incorporate other properties you want or need your network to have.

[7]Gian-Carlo Rota has written that the problem conversion of probabilistic methods into explicit constructions is one that should be of interest to philosophers of mathematics (rather than, say, the problem of the existence of the number 2.

We will view the above discussion as some motivation for being interested in the spectral gap ω. But in fact, we will be more interested in families $\{G_i\}$ not only with $\omega = \inf \omega(G_i) > 0$, but in fact with ω as large as possible. In other words, despite the fact that maximizing ω is not necessarily the same thing as maximizing $h = \inf h(G_i)$, we will be interested in the former problem. Our reasons for this will become more clear presently. In any event, we can ask: how large can the asymptotic spectral gap ω possibly be?

The answer comes from the spectral theory of the adjacency operator A on the (infinite-dimensional!) Hilbert space $H(\tilde{G})$: like any bounded self-adjoint linear operator on Hilbert space, it has a spectrum $\operatorname{Spec} A$, the set of real numbers λ such that $\lambda I - A$ does not have a bounded inverse (i.e., either it is not invertible, or the inverse is unbounded), a compact subset of $\mathbb{R}$. In the case of the adjacency operator on a tree $\tilde{G}$, then as in the finite case, the bipartiteness implies that $\operatorname{Spec} A$ is symmetric about the origin. The maximum element of $\operatorname{Spec} A$ is called the *spectral radius* ρ of A; it can also be characterized as the norm of A and in many other ways. The following key result explains how the geometry of the universal cover gives an upper bound on the spectral gap.

Theorem 10. *(Greenberg) Let $\{G_i\}$ be a family of finite connected graphs with isomorphic universal covers, and let ρ be the spectral radius of the universal cover. Then $\lambda_1 := \inf \lambda_1(G_i) \geq \rho$.*

Since $\lambda_0(G_i)$ is constant in any family, the result is equivalent to an upper bound on the spectral gap: $\omega \leq \lambda_0(G) - \rho$.

The spectral radius of the universal cover of a finite graph is an algebraic number, and an algorithm for computing it is given in [**?**]. The most important case is the following:

Theorem 11. *(Li-Solé) For $r,\ s \geq 2$, let $\mathcal{T}_{r,s}$ be the (r,s)-semiregular tree. Then $\rho = \sqrt{r-1} + \sqrt{s-1}$.*

Taking $r = s$ and combining the previous two results we get the following important result (which predates Theorems 10 and 11 and was originally proved by more elementary methods):

Corollary 12. *(Alon-Boppana) Let $\{G_i\}$ be a sequence of q-regular graphs. Then $\liminf_i \lambda_1(G_i) \geq 2\sqrt{q}$.*

Definition (Lubotzky-Tagnibeda): A finite graph G is *Ramanujan* if $|\lambda_1(G)| \leq \rho(\tilde{G})$. A Ramanujan family is a family $\{G_i\}$ of Ramanujan graphs, i.e., a family with $|\lambda_1(G_i)| \leq \rho(\tilde{G})$ for all i. Note that Greenberg's theorem now implies that $|\lambda_1(G)| = \inf |\lambda_1(G_i)| = \rho(\tilde{G})$. Thus, although we have defined what it means for a single graph to be Ramanujan, the motivation for the definition comes from families: a Ramanujan family is precisely a family whose first eigenvalue $|\lambda_1| = \inf |\lambda_1(G_i)|$ attains the Alon-Boppana-Greenberg bound $\rho(\mathcal{T}(\mathcal{F}))$.

Definition: An infinite tree (all our trees will be infinite, from now on) $\mathcal{T}$ is Ramanujan if it is the universal cover of a Ramanujan family, and *weakly Ramanujan* if it is the universal cover of at least one Ramanujan graph.

Given a tree $\mathcal{T}$, define $\mathcal{R}(\mathcal{T})$ to be the set of all Ramanujan quotient graphs of $\mathcal{T}$. We abbreviate $\mathcal{R}_q = \mathcal{R}(\mathcal{T}_{q+1})$ and $\mathcal{R}_{r,s} = \mathcal{R}(\mathcal{T}_{r+1,s+1})$.

Example X ($q = 2$): The 2-regular tree $\mathcal{T}_2$ is a doubly infinite path, the Cayley graph of $G = \mathbb{Z}$. Its finite quotients are precisely the r-cycles C_r – corresponding to the "congruence subgroups" $r\mathbb{Z}$ of $\mathbb{Z}$ – and by Example X.X we have $\lambda_1(C_r) \leq 2 = 2\sqrt{2-1}$. Thus every finite quotient of $\mathcal{T}_2$ (and in fact there are no nontrivial infinite quotients) is Ramanujan. On the other hand, note that $\lambda_0(G_i) = 2$, so $\omega = \inf 2 - \lambda_1(G_i) = 0$, so this is not an expander. This is an exceptional situation:

Proposition 13. *For any $(r, s) \neq (1, 1)$, $\mathcal{T}$ (r, s)-Ramanujan families are expanders.*

Proof: By Example X.X, $\lambda_0(G_i) = \sqrt{(r+1)(s+1)} > \sqrt{r} + \sqrt{s} = \rho(\mathcal{T}_{r,s})$.

We are now ready to ask some very difficult questions:

Question 14. *For a bounded tree $\mathcal{T}$, describe $\mathcal{R}(\mathcal{T})$. Is it nonempty? Is it infinite?*

Clearly a necessary condition for $\mathcal{R}(\mathcal{T})$ to be nonempty is that $\mathcal{T}$ cover some finite graph – this is called *uniform* – and a necessary condition for this is that $\mathcal{T}$ be *almost regular*, i.e, that its automorphism group has finitely many orbits on $\mathcal{T}$. It is clear that "most" trees of bounded degree are not almost regular (e.g. the set of isomorphism classes of trees of bounded degree is uncountable but there are only countably many isomorphism classes of almost regular trees), so for most trees we have $\mathcal{R}(\mathcal{T}) = \emptyset$ for trivial reasons. On the other hand, a much more surprising result of Lubotzky-Nagnibeda constructs a large family of uniform trees which are not weakly Ramanujan.

Remark: More precisely, L-N construct trees $\mathcal{T}$ such that any finite quotient G of T has $\lambda_1(G) > \rho(T)$. In analogy with the fact that $\lambda_0(G) > \rho(T)$ for every regular graph, it is tempting to view $\lambda_1(G)$ as being another "trivial" eigenvalue. Indeed, if one can show of a tree $\mathcal{T}$ that it admits finite quotients and that for every finite quotient we have $k \geq 1$ eigenvalues exceeding, in absolute value, the spectral radius $\rho(\mathcal{T})$, then it is tempting to throw these out. In this way, one could define a tree $\mathcal{T}$ to be *quasi-Ramanujan* if there exists some $k \in \mathbb{Z}$ and an infinite sequence of finite quotients G_i such that for all but k of the eigenvalues λ of G_i we have $|\lambda| \leq \rho(\mathcal{T})$. It is then conceivable that every uniform tree is quasi-Ramanujan.

Although these speculations on the nature of Ramanujancy for very irregular graphs seem interesting, the case of semiregular graphs is much more interesting. Indeed, the following question gets to the heart of the matter:

Question 15. *Is every weakly Ramanujan tree Ramanujan?*

In other words, given one Ramanujan graph G, can we somehow "replicate" it to give infinitely many Ramanujan graphs with the same universal cover? In particular, can we find infinitely many covers of G which are Ramanujan? It is very tempting to believe that the answer is yes (in fact I do believe it), but there is no simple toplogical or group-theoretical construction. For instance, one can check that, unless G is 2-regular, not all of its finite coverings are Ramanujan graphs.

(EXPLAIN)

Nevertheless our best guess is that Question 15 has an affirmative answer. Indeed, there are no known counterexamples, and it is known to hold for infinitely many regular and semi-regular trees.

3. Examples of non/expanding and non/Ramanujan families

Example X: The graphs K_r, $K_{r,s}$ are all Ramanujan graphs. Therefore the trees $\mathcal{T}_{r,s}$ are weakly Ramanujan, and none of them are known *not* to be Ramanujan. As we shall see later, there are infinitely many pairs (r,s) for which $\mathcal{T}_{r,s}$ is known to be Ramanujan, but for most pairs – in particular, for all pairs with $2 < r < s$ – the problem remains wide open.

Example X.X: The hypercube graphs $(\mathrm{Cay}((\mathbb{Z}/2\mathbb{Z})^n, \{e_1, \ldots, e_n\})$ of Example X.X are Ramanujan iff $n \leq 6$.

Example X.X: Given a single non-Ramanujan graph G (which is not a tree), we can build a *non*-Ramanujan family by taking any family of finite coverings G_n of G. Indeed, if $G' \to G$ is a covering map of finite graphs then $\operatorname{Spec} G \subset \operatorname{Spec} G'$ (with multiplicities): indeed, each eigenfunction for A_G pulls back to an eigenfunction for $A_{G'}$ with the same eigenvalue. Combining with the previous example, we get non-Ramanujan q-families for $q \geq 6$.

Example X.X: Let M_n be the nth *Mobius ladder*, i.e., the Cayley graph for the group $Z_{2n} = \langle \gamma \mid \gamma^{2n} = 1 \rangle$ with respect to the symmetric generating set $S = \{\gamma, \gamma^{-1}, \gamma^n\}$. The eigenvector corresponding to the character $\chi_2 : \gamma^r \mapsto e^{\pi\sqrt{-1}2r/n}$ has eigenvalue $\chi_2(1) + \chi_2(-1) + \chi_2(n) = 2\cos(2\pi/n) + 1$. As $n \to \infty$, this eigenvalue approaches $3 = \lambda_0(M_n)$, so this gives a simple example of a cubic nonexpander (*a fortiori* a non-Ramanujan family). By taking the element γ^n with multiplicity $k \geq 1$, we get a family of nonexpanders of degree $2 + k$, i.e., all possible degrees.

Example X.X: More generally, for any $k \geq 2$, it is not too hard to see that if $\{G_i\}$ is any family of k-regular Cayley graphs on finite abelian groups, then $\lim_i \lambda_1(G_i) = k$. See [Murty et. al.] for a quantitative refinement.

3.1. **Planar Ramanujan graphs.** Non-example X: For fixed $q > 1$, there are only finitely many *planar* Ramanujan q-graphs. More generally: define the *embeddability genus* of a finite graph to be the least $g \geq 0$ such that G embeds in a compact orientable surface of genus g. (To see that g is finite, we may embed G in $\mathbb{R}^3$ and then take the boundary of a sufficiently small tubular neighborhood.) Note that "planar" is equivalent to "embeddability genus 0."

Theorem 16. *(Kelner) Let G_n be a sequence of graphs whose vertex degrees and embeddability genera are both uniformly bounded. Then $\lim_{n\to\infty} \omega(G_n) = 0$.*

Even if we consider all vertex degrees $d \geq 3$ simultaneously, it is an open problem whether there are infinitely many planar Ramanujan graphs; apparently the largest known example has 84 vertices.

Example: The one-skeleton of each of the five platonic solids – i.e., the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron – is a planar Ramanujan graph. What is to be made of this?

One can check that not all of the (one-skeleta of the) semiregular polytopes are Ramanujan graphs, so this is the wrong generalization. On the other hand, consider first the cube (degree 3) and the dodecahedron (degree 5). These are the first two graphs in an infinite sequence, as follows: for an odd prime let $X(p) = \Gamma(p)\backslash\overline{\mathcal{H}}$ be the compactification of the Riemann surface uniformized by the principal congruence subgroup $\Gamma(p) \subset PSL_2(\mathbb{Z}) = \Gamma(1)$. Since $\Gamma(p)$ is normal with quotient $PSL_2(\mathbb{F}_p)$, we get a Galois covering $X(p) \rightarrow X(1) = \Gamma(1)\backslash\overline{H} \cong \mathbb{P}^1(\mathbb{C})$, i.e., the Riemann sphere, the isomorphism being given by the modular function j. On $X(1)$ consider the three points with j-invariant $1728, 0, \infty$; these are the three ramification points of the cover, with ramification indices $(2, 3, p)$. There is a unique geodesic circle connecting these three points; this gives a triangulation of $X(1)$ into two triangles, which we intuitively think as having two different colors, say red and green. The preimage of this triangulation gives a triangulation on $X(p)$ with a natural biparition: there are $\#PSL_2(\mathbb{F}_p)$ red and $\#PSL_2(\mathbb{F}_p)$ green triangles, and the automorphism group acts simple transitively on triangles of a given color. Notice however that $PSL_2(\mathbb{F}_p)$ does not act transitively on the vertices, but rather has three orbits, corresponding to the three vertex degrees 4, 6 and $2p$. However, we can restrict attention to the cusps, i.e., the preimages of $j = \infty$, which have degree $2p$: namely, we remove all the edges which run between the vertices of degree 4 and 6. In other words, we remove preimage of the open geodesic segment $(0, 1728)$ on $X(1)$. On the resulting graph, the preimages of 0 and 1728 now have degree 2 and we cease to regard them as vertices. This defines a finite graph $G(p)$ which is immediately seen to be a p-regular Schreier graph of $PSL_2(\mathbb{F}_p)$ modulo an order p (unipotent) subgroup U.

Theorem 17. *(Gunnells) For all odd primes p, the graph $G(p)$ is a p-regular Ramanujan graph.*

For $p = 3$ and 5 (and no other primes) $X(p)$ has genus zero, so that the graphs $G(3)$ and $G(5)$ give triangulations on the Riemann sphere, with respective symmetry groups $PSL_2(\mathbb{F}_3) \cong A_4$ and $PSL_2(\mathbb{F}_5) \cong A_5$. These spherical triangulations are well-known: $G(3)$ is the tetrahedron and $G(5)$ is the icosahedron. What about the octahedron? Indeed $X(4)$ also has genus zero, and $PSL_2(\mathbb{Z}/4\mathbb{Z}) \cong S_4$, and the analogously defined triangulation $G(4)$ is the octahedron. (It follows from Gunnells' work that for most composite values of n, $G(n)$ is *not* a Ramanujan graph; the case of $n = 4$ is exceptional.)

There are many other interesting graphs related to this construction. For instance, consider instead the preimage of $[0, 1728]$ under the covering $\mathcal{H} \rightarrow \Gamma(1)\backslash\mathcal{H}$. We get the action of $PSL_2(\mathbb{Z})$ on the semiregular tree $\mathcal{T}_{2,3}$ with vertex stabilizers $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ which exhibits the classical isomorphism $\mathrm{PSL}_2(\mathbb{Z})\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ (cite Serre). For any congruence subgroup $\Gamma \subset \Gamma(1)$, we may therefore consider the finite graph $\Gamma\backslash\mathcal{T}_{2,3}$. Assuming that Γ is torsionfree – e.g. $\Gamma(p)$ for all p – we get a family of $(2, 3)$-semiregular quotients. These are the barycentric subdivisions of 3-regular graphs, indeed the Schreier graphs $G'(p) = X(PSL_2(\mathbb{F}_p)/\langle R\rangle, \{S, T\})$, where $R = [01-1-1]$, $S = [01-10$ and $T = RS = [1101]$. It follows from deep results of Selberg on the spectrum of the hyperbolic Riemannian surfaces $Y(p)$ that this family of graphs forms an expander. The Cayley graphs themselves also appear in the picture: we can pair each red triangle with a unique green triangle so as to make a triangle with angles $\pi/3$, $\pi/3$, $2\pi/p$: $PSL_2(\mathbb{F}_p)$ acts simply transitively on these paired triangles; regarding two paired triangles as adjacent if they share an edge, we get $\mathrm{Cay}(PSL_2(\mathbb{F}_p), \{S, T\})$. Might it be possible to relate the spectra of $G'(p)$ and $G(p)$ and therefore deduce this from Theorem 17?

More generally, given a hyperbolic triangle group $\Delta(a, b, c)$, one can define, following Beasley-Cohen and Wolfart, a family of congruence subgroups Γ of Δ; the principal congruence subgroups are normal, of finite index, with quotients PSL_2 or PGL_2 with coefficients in certain finite rings. The map $X(\Gamma) = \Gamma\backslash\overline{\mathcal{H}} \to \Delta(a, b, c)\backslash\overline{\mathcal{H}} = X(1) \cong \mathbb{P}^1$ is a Galois Belyi map, so as above we get an associated triangulation of the Riemann surface $X(\Gamma)$ (a "dessin d'enfant"). For all but 85 triples (a, b, c), these are (so-called!) non-arithmetic Fuchsian groups, so the methods of Selberg do not apply. However, it seems likely that we will nevertheless get families of expanders.
In particular, any q-family with bounded embeddability genus (e.g. planar graphs!) is not even an expander, let alone (for $q > 1$) a Ramanujan family.

There is a natural infinite sequence of cubic (i.e., 3-regular) graphs whose first three members are the tetrahedron, the octahedron and the icosahedron. Namely, for any prime p, consider the modular curve $X(p)$ (as, for now, a compact Riemann surface) and its canonical map $X(p) \to X(1) \cong \mathbb{P}^1$, the isomorphism being given by the modular function j. The standard (Dirichlet) fundamental domain for $\Gamma(1) = PSL_2(\mathbb{Z})$ acting on $\mathcal{H}$ gives rise to a tiling of the extended upper halfplane by pairs of $(2, 3, p)$ hyperbolic triangles joined along a common side; since $\Gamma(p)$ is normal of finite index in $\Gamma(1)$, the quotient $PSL_2(\mathbb{F}_p)$ acts on the tiling, and the orbit of a given double triangle gives a fundamental domain for $\Gamma(p)$. Taking the one-skeleton, we get a planar graph $\tilde{G}(p)$. Moreover, for every oriented exterior edge e, there exists a unique element $\gamma \in \Gamma(p)$ such that $\gamma(e)$ is another oriented exterior edge, and this defines a quotient graph $G(p)$ which admits a natural action of $PSL_2(\mathbb{F}_p)$ with vertex stabilizer a unipotent (cyclic) group of order p. In particular, for odd p, $G(p)$ is regular of degree $\frac{1/2(p+1)(p^2-p)}{p} = \frac{p^2-1}{2}$ (and degree 3 for $p = 2$). In particular $G(p)$ is a Schreier graph for $PSL_2(\mathbb{F}_p)/U$, where U is a unipotent group (with respect to a particular set of generators), and from this observation Gunnells shows, using nothing but the known representation theory of $PSL_2(\mathbb{F}_p)$, that all the graphs $G(p)$ are Ramanujan graphs.

(Explain how the Ramanujancy of the cube and the dodecahedron follows from "planar duality.")

Example X.X: For a graph G, we define its barycentric subdivision $G_{(2)}$ by placing a vertex in the middle of each edge. This replaces an r-regular graph by a $(r, 2)$-semiregular bipartite graph.

Proposition 18. *(Hashimoto) Let G be a d-regular finite graph. Then G is Ramanujan iff $G_{(2)}$ is Ramanujan.*

Proof: . . .

Remark: It seems likely that the result holds for all finite graphs, although I have not tried to do the computation. (Perhaps it is even true that the Ramanujan property is preserved by all edge subdivisions, i.e., is a homeomorphism invariant.) However, it is especially easy to check that starting with a (r, s)-biregular Ramanujan graph, repeated barycentric subdivision preserves the Ramanujan property. Thus there exist weakly Ramanujan trees $\mathcal{T}$ with arbitrarily many orbits of $\mathrm{Aut}(\mathcal{T})$.

Most bounded trees do not have enough symmetry to cover any finite graphs, let alone Ramanujan graphs. More surprisingly, Lubotzky and Tagnibeda have constructed classes of trees $\mathcal{T}$ covering finite graphs but with $\mathcal{R}(\mathcal{T}) = \emptyset$. Their trees have the property that there is a unique minimal finite G_0, with $\lambda = \lambda_1(G_0) > \rho$. This implies, as in Example X.X, that every finite quotient G of $\mathcal{T}$ has the eigenvalue λ: since $\rho < \lambda < \lambda_0(G) = \lambda_0(G_0)$, G is again not Ramanujan.

3.2. **Some non-Ramanujan graphs.**
G. Rota has written of the importance of illuminating a key definition with *non-examples* as well as examples. So let us give two ways of constructing families of non-Ramanujan graphs, and indeed, non-expanders.

Note that $X(p)$ has genus 0 only if $p \leq 5$ (indeed $X(7)$ is Klein's quartic surface, of genus 3), so we may not conclude that the other $G(p)$'s are planar (presumably only finitely many of them are planar). On the other hand, using the fact that $G(p)$ is the 1-skeleton of a 2-dimensional cell complex, we may consider the "dual" graph $G^{\vee}(p)$ whose vertices are the 2-cells and whose edges are the 1-cells; this is evidently a cubic graph. It follows from a celebrated theorem of Selberg (on the point spectrum of the Riemannian manifold $Y(p) = \Gamma(p)\backslash\mathcal{H}$) that the $G^{\vee}(p)$'s form an expander; we refer the interested reader to the discussion in [**?**]. From this result and Kelner's theorem we may conclude that the genus of $X(p)$ tends to infinity with p![8]

Example: Let $\Delta = \Delta(2, 3, 7)$ be the Fuchsian group generated by $r_a \circ r_b$, $r_b \circ r_c$, $r_c \circ r_a$, where r_a, r_b, r_c are reflections through the sides of a hyperbolic triangle whose angles are $\pi/2$, $\pi/3$, $\pi/7$. This Fuchsian group is well-known to be arithmetic; it is the group of norm 1 units in the quaternion algebra over $F = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ which is ramified at two infinite places and unramified at every finite place. For any prime ideal $\mathfrak{p}$ in the ring of integers of F, we have the notion of a principal congruence subgroup $\Gamma(\mathfrak{p})$ and hence a cover $X(\mathfrak{p}) \to X(1) \cong \mathbb{P}^1$. Similarly to the above example, we get a tesselation of a fundamental region for $\Gamma(\mathfrak{p})$ by pairs of

[8] Of course, this also follows from the above description of the branched cover $X(p) \to X(1)$ together with the Riemann-Hurwitz formula.

$(2,3,7)$ hyperbolic triangles as the orbit of a single paired triangle under the group $PSL_2(\mathbb{F}_q)$, where $\mathbb{F}_q = \mathfrak{o}_F/\mathfrak{p}$. This leads to graphs $\tilde{G}(\mathfrak{p})$and $G(\mathfrak{p})$, endowed with a transitive action of $PSL_2(\mathbb{F}_q)$ with vertex stabilizer an order 7 (cyclic!) subgroup H. When $\mathfrak{p}$ is the unique of F above 7, this is exactly the graph $G(7)$ constructed above; for all other primes H is not a unipotent subgroup, so this gives something new. One should be able to check whether these graphs are Ramanujan by using the same (well-known, but nontrivial) methods as [?] and [?]; I have not done the computation. On the other hand, the "dual" cubic graphs $G^\vee(\mathfrak{p})$ form a family of expanders for the same reason: Selberg's work extends to all congruence arithmetic Fuchsian groups.

More generally, for any hyperbolic triangle group $\Delta = \Delta(a,b,c)$, one can define congruence subgroups $\Gamma(\mathfrak{p})$ corresponding to prime ideals of $F = F(\ldots)$ and get a similar family of Schreier graphs $G(\mathfrak{p})$ for $PSL_2(\mathbb{F}_q)$ modulo a cyclic subgroup of order c. Whether or not these graphs are Ramanujan is again a purely algebraic question; more interesting is whether the dual cubic graphs $G^\vee(\mathfrak{p})$ form an expanding family in the general case, since only 85 of the hyperbolic triangle groups are arithmetic.

Other classes of Schreier graphs on $PSL_2(\mathbb{F}_q)$ having the Ramanujan property (or, in some cases, narrowly missing the Ramanujan bound) are constructed by Li and Meemark, also by representation-theoretic methods. Whereas our graphs are closely related to triangulations of certain "modular" Riemann surfaces, it is remarked in [?] that their constructions can be interpreted as triangulations (in a certain formal sense) on Drinfeld modular curves, i.e., on the geometry of analytically uniformized curves in positive characteristic (!).

Remark: One cannot help but wonder what to make of the "duality" between $G(p)$ and $G^\vee(p)$: can the (deep!) expander property of $G^\vee(p)$ be deduced directly from the (elementary!) Ramanujancy of $G^\vee(p)$?

Question: If the point of this entire dicussion was to "explain" why the platonic solids are Ramanujan graphs, then what is the corresponding explanation for the Peterson graph?

3.3. **Probabilistic constructions of expanders.** It may well be that *most* $q+1$-regular graphs are Ramanujan graphs, i.e., that the proportion of Ramanujan $q+1$-regular graphs on N vertices tends to 1 as $N \to \infty$. The following result shows that this at least "very nearly" the case (whether q is a prime power or not!):

Theorem 19. *(Friedman) For* any $q > 1$ *and* $\epsilon > 0$*, the probability that a simple* q*-graph on* n *vertices has* $\lambda_1 \geq 2\sqrt{q} + \epsilon$ *tends to* 0 *as* $n \to \infty$*.*

something to do with a duality between the spectra of two different combinatorial Laplacians on the same family of 2-dimensional cell complexes.

3.4. **The zig-zag product.** We must mention the recent beautiful work of Reingold, Vadhan and Wigderson, which attains a holy grail in the theory of expanding graphs: given two finite graphs G_1 and G_2, they define a *zig-zag product* G of G_1 and G_2 such that the spectral gap $\omega(G)$ can be bounded below in terms of the spectal gaps $\omega(G_1)$ and $\omega(G_2)$. However the zig-zag product does not preserve the degree, so while it gives what is certainly the most elementary explicit construction of d-regular expanders for infinitely many d (including $d = 3$ and $d = N^2$ for any $N > 1$), it does not, so far as I know, yield expanders of *all* vertex degrees.

4. Fundamental Theorems on Ramanujan graphs

For the last decade, the state of our knowledge on Ramanujan q-families has been the following:

Theorem 20. *(Ihara, Lubotzky-Phillips-Sarnak, Chiu, Morgenstern) For any prime power q, there exists a family of $(q + 1)$-regular Ramanujan graphs.*

Remark: In other words, there is a Ramanujan q-family for all prime power q, which begins to explain the terminology.

It is important to remark that this result is in all senses *constructive*. Indeed, in all of the constructions except Ihara's the graphs are ultimately gotten as Cayley graphs of $PSL_2(\mathbb{F}_q)$ or $PGL_2(\mathbb{F}_q)$ with respect to certain explicitly given sets of generators. These graphs have indeed been implemented by computer scientists.

Let us say a bit about the history of this important result. The case in which q is a prime was shown in 1966 by Ihara in the seminal work [**?**], which analyzes zeta functions of discrete cocompact subgroups the $q+1$-regular tree. His approach uses Eichler's Hecke-equivariant isomorphism of a certain Brandt module category with the Hecke module of modular forms on an appropriate modular curve, together with a "good reduction" result of Igusa and the Riemann hypothesis for the zeta function of a curve over a finite field, due of course to Weil.

Like much of Ihara's work, [**?**] was ahead of its time, and it appears that it began to be truly appreciated in the late 1980's. The case in which q is an odd prime is revisited from a different perspective in the 1988 paper [**?**]. Apart from rephrasing the construction in terms of quaternary quadratic forms (an aesthetically pleasing but inessential conceit), the paper introduces many innovations: first, the graphs themselves are presented in a truly elementary way, as Cayley graphs of either $PSL_2(\mathbb{F}_q)$ or $PGL_2(\mathbb{F}_q)$ with respect to an explicitly given set of generators. As we shall see later, the fact that the graphs can be taken to be Cayley graphs follows from the fact that the definite quaternion algebra intervening in the construction has class number one – L-P-S use the "Hamilton" algebra $H_{/\mathbb{Q}}$ of discriminant 2 (in which a maximal order has a left-Euclidean algorithm). Moreover, their graphs are shown to satisfy many other extremal properties, such as having simultaneously larger girth and chromatic number than any other deterministically constructed family of graphs! They also coin the term "Ramanujan graph," because in their quadratic form setup the needed spectral bounds come as a sort of *deus ex machina* from the estimates on the coefficients of a weight k cusp form on $\Gamma_1(N)$ conjectured by Ramanujan and Petersson: $|a_p| \leq 2p^{\frac{k-1}{2}}$ and proved in full generality by Deligne as a consequence of the Riemann hypothesis for higher-dimensional abelian varieties.

Let us pause for a crotchety remark. Ramanujan's original conjecture concerned the coefficients of the weight 12 cusp form Δ, and was generalized by Petersson to cusp forms of all weights (and all possible level structures). The *full* conjecture was proved by Deligne in 1973, but the modular forms which intervene in all the constructions of Ramanujan graphs have weight 2, so appeal only to the case of the Weil conjectures resolved by Weil himself, as well as work of Eichler, Igusa and – for a general congruence subgroup – Shimura. (Indeed, note that Ihara's paper predates the resolution of the Ramanujan-Petersson conjecture.) Thus the naming of these graphs "Ramanujan" is a rhetorical flourish: they would more accurately be called Ihara-Eichler-Igusa-Weil graphs. Of course the authors of [**?**] know this (indeed, it appears in their paper itself), but many more recent authors are not as conversant with these subtleties.

The L-P-S construction was extended to the case of $q = 2$ in a 1991 paper of P. Chiu [**?**]. In place of the quadratic form / order in H considered by L-P-S, Chiu needs to work with the quadratic form associated to a maximal order of a class number one definite quaternion algebra $D_{/\mathbb{Q}}$ which is split at 2. He uses the quaternion algebra of discriminant 13, with norm form $W^2 + 2X^2 + 13Y^2 + 26Z^2 = 0$. Why 13? Well, there are only finitely many definite rational quaternion algebras of class number one; among these, the ones that are split at 2 have discriminant 3, 5, 7 and 13. So there is not much choice here.[9]

In general, it seems tempting to work with totally definite quaternion algebras over totally real number fields. However, in this context one cannot expect to always be in a class number one situation – indeed, it seems likely that there exist only finitely many totally definite quaternion algebras over *any* totally real field of class number one.[10] So Morgenstern changes gears considerably and works with "definite" quaternion algebras over the function field $\mathbb{F}_q(x)$, and one can show that there are always Euclidean orders over such fields. Now Morgenstern uses Drinfeld's solution of the function field version of the Ramanujan-Petersson conjecture (again for automorphic forms of all integral weights) to carry through the analogue of the L-P-S construction, including the bounds on girth, chromatic number, and so forth.

When q is a prime power, we still know rather little about the family $\mathcal{R}_q$ of all $(q + 1)$-regular Ramanujan graphs. Computations suggests that it may be quite large indeed; Sarnak and XX present numerical evidence that about 5?% of all cubic graphs ($q = 2$) are Ramanujan. This seems quite far out of present reach.

Many mathematicians have sought to extend Theorem 20 to the case where q is not a prime power. From a hard-nosed perspective these efforts have come to precisely nothing: whether or not there exists a family of $q+1$-Regular Ramanujan graphs for any non-prime power q – e.g. $q = 6$ is wide open. Rather, many researchers have become intrigued by the method of proof of Theorem 20 and sought more general contexts in which this methods may be applied.

It turns out that the method of proof of Theorem 20 can nevertheless be used

[9]Still, why 13 and not 3, 5 or 7? The construction will still work with these other primes, but presumably the fact that 13 is 1 (mod 4) and 1 (mod 3) simplifies some of the explicit calculations.

[10]As far as I know, this is an open problem, although the finiteness is known over the class of all totally real fields in which the maximal number of roots of unity in a CM quadratic extension remains bounded, so in particular over all totally real fields of fixed degree.

to give an explicit construction of $q+1$-regular expanders for all $q>1$. This was done by Pizer, who proved the following result.

Theorem 21. *(Pizer) For $q \geq 2$, there is a family of $(q+1)$-regular graphs $\{G_n\}$ satisfying*

$$\lambda(G_n) \leq d_0(q+1)\sqrt{q}$$

for all n, where $d_0(n)$ is the number of divisors of n.

Here we will combine Pizer's idea with work of Weil, Eichler, Shimura, Jacquet and Langlands to prove the following result, a simultaneous generalization of Theorems 20 and 21.

Theorem 22. *For $q \geq 2$, there is a family of $(q+1)$-regular graphs $\{G_n\}$ satisfying*

$$\lambda(G_n) \leq 2^{\omega(q+1)}\sqrt{q},$$

where $\omega(n)$ is the number of distinct prime divisors of n. The graphs G_n are constructed deterministically.

Thus, like Pizer's theorem, Theorem 22 gives an explicit construction of $(q+1)$-regular expanding families for all q, but with a better bound on the spectral gap (although, when q is not a prime power, the bound is still not as good as Friedman's bound obtained using the probabilistic method).

The basic strategy of the proof is one which may have occurred to the reader in the course of our account of the various cases of Theorem 20: namely, we will exploit the arithmetic of totally definite quaternion algebras over totally real number fields. This is very much in the spirit of Ihara's original proof, except that we get a Hecke-module isomorphism from a Brandt module category to $S_2(X(\Gamma''))$ where $X(\Gamma)$ is a certain quaternionic Shimura curve over F. On this Shimura curve $X(\Gamma)$ we can apply the Eichler-Shimura congruence relation and then the Weil bounds to get the desired estimates for the eigenvalues of the adjacency matrix.

Finally we follow Pizer's idea of using not just the $T_{\mathfrak{p}}$-Hecke operator for some prime ideal $\mathfrak{p}$ but instead a product of such Hecke operators corresponding to a factorization of $q+1$ into $p_1^{a_1}\cdots p_r^{a_r}$. Pizer in his argument simply takes the $q+1$-Hecke operator which splits as a product $T(q+1) = \prod_{i=1}^r T(p_i^{a_i})$. He therefore gets the bound $\prod_i (a_i+1)\sqrt{p_i^{a_i}} = d_0(q+1)\sqrt{q}$. Instead we will choose a totally real field F having prime ideals $\mathfrak{p}_i$ of norm $p_i^{a_i}$, and then we lose nothing from the Ramanujan bound in taking a prime power instead of a prime.

It is amusing to note that this strategy *cannot* be carried out in the case of a function field F of characteristic p, since in this case all of the residue fields have degree a power of p!

Remark: Conversely, it ought to be possible to eliminate the dependence on the full Ramanujan Conjecture for $GL_2(\mathbb{F}_q(t))$ in favor of transferring from the Hecke module of Brandt matrices to the Hecke module of $S_2(X)$, where X is a "Drinfeld-Shimura curve" (a moduli space of $\mathcal{D}$-elliptic sheaves with quaternionic multiplication) in the sense of Taelman [**?**]. It is hard to claim that this effects any kind of simplification of Morgenstern's construction (we are just asserting our preference for one brand of technology over another), but it has the following intriguing consequence:

Every known instance of a Ramanujan q-family comes from a family of algebraic curves.

5. The main result

Let F be a totally real number field of degree g; fix a real place ∞_1 of F. Let B/F be a division quaternion algebra which is split at ∞_1 but ramified at every other infinite place ∞_i $2 \leq i \leq g$ (we abbreviate this by saying that B if of type $(1, g-1)$); we write $\mathcal{D}$ for the discriminant of B: it is an integral F-ideal. Fix $\mathcal{O}$ a maximal order of B. Let $\Gamma(1)$ be the group of totally positive units in $\mathcal{O}$, and let $\Gamma \subset \Gamma(1)$ be a congruence subgroup, of level prime to $\mathcal{D}$.

Let $X(\Gamma)_{/F}$ be the algebraic curve constructed by Shimura: it is nonsingular, projective and irreducible, but is, in general, geometrically reducible. Indeed, the set of geometric components is a principal homogeneous space for the $(\infty_2 \cdot \ldots \cdot \infty_g)$-ray class group of F, so in particular is at most the narrow class number $h(F)$ of F.

We **assume** that there exists a prime ideal $\mathfrak{p}$ of F dividing $\mathcal{D}$; note that this is automatic when g is odd. Recall that Ceregnik has constructed a canonical model of $X(\Gamma)$ over $\mathcal{O}_{\mathfrak{p}}$, the ring of integers of the completion of F at $\mathfrak{p}$. Let $\mathbb{F}_{\mathfrak{p}}$ be the residue field at $\mathfrak{p}$, say of cardinality q; then the fiber of $X(\Gamma)$ over $\mathbb{F}_{\mathfrak{p}}$ is a semistable curve, such that every geometric component has geometric genus zero. As for any semistable curve, we may consider the dual graph $G(\Gamma)$, whose vertices are the geometric components, and in which two vertices C_i and C_j are connected by $a_{ij} := C_i \cdot C_j$ edges. The graph can be explicitly given as a quotient of the $q+1$-regular tree $\mathcal{T}_{q+1}$ by a congruence subgroup $\Gamma' \subset PGL_2(F_{\mathfrak{p}})$. We shall assume that Γ' is *torsionfree* – i.e., contains no nontrivial elements of finite order (later we will recall some mild conditions on F and Γ which suffice to guarantee this), and in this case a theorem of Ihara asserts that Γ' acts freely on $\mathcal{T}_{q+1}$ (so is itself a free group), so that the map $\mathcal{T}_{q+1} \rightarrow \Gamma' \backslash \mathcal{T}_{q+1} = G(\Gamma)$ is the universal covering. In particular, $G(\Gamma)$ is $(q+1)$-regular.

Claim: The adjacency operator on the vertex set of $G(\Gamma)$ is nothing else than the $T_{\mathfrak{p}}$-Hecke operator from the theory of automorphic forms. (This follows from Ribet's setup; discuss it a bit.)

Note that we could have started with a level $\mathcal{N}$ congruence subgroup Γ' of the totally definite quaternion algebra D' – with $(\mathcal{N}, D') = 1$ – and considered, for any prime $\mathfrak{p}$ prime to $\mathcal{N}D'$, the Hecke operator $T_{\mathfrak{p}}$ on $\Gamma' \backslash \mathcal{T}_{q+1}$, so our earlier assumption on the assumed divisibility of $\mathcal{D}$ by $\mathfrak{p}$ was no loss of generality – we can perform the construction for all but finitely many prime $\mathfrak{p}$. However, we will now assume more: that there exists another prime ideal $\mathfrak{q} \neq \mathfrak{p}$ dividing $\mathcal{D}$ – or equivalently, that $\mathcal{D}' = \mathcal{D}/\mathfrak{p}$ is divisible by some finite prime. Then:

Theorem 23. *Let F be a totally real number field, $\mathcal{D} = \mathfrak{p}\mathfrak{q}\mathcal{D}''$ be a squarefree product of at least two primes of F, let B be the unique quaternion algebra of F of discriminant $\mathcal{D}$ and type $(1, g-1)$, let Γ be a congruence subgroup of B of* *level $\mathcal{N}$ prime to $\mathcal{D}$. Then the dual graph of the special fiber of the Cerednik-Drinfeld canonical model of the Shimura curve $X(\Gamma)$ is Ramanujan. Moreover, the sequence of all isomorphism classes of connected components of such graphs forms a Ramanujan q-family.*

Proof: Let B'' be the unique quaternion algebra over F of type $(1, g-1)$ and discriminant $\mathcal{D}'' = \mathcal{D}/(\mathfrak{p}\mathfrak{q})$, and let Γ'' be the congruence subgroup of B'' which is, locally at every prime $\mathfrak{r}$ different from $\mathfrak{q}$, equal to the congruence subgroup Γ', and which is locally at $\mathfrak{q}$ equal to $\Gamma_0(q)$. Then the Jacquet-Langlands correspondence gives an isomorphism of $T_{\mathfrak{p}}$-modules from [introduce the notation for the two Hecke algebras]. Moreover, by the Eichler-Shimura congruence relation, the characteristic polynomial of $T_{\mathfrak{p}}$ on $S_2(X(\Gamma''))$ is equal to the numerator of the Hasse-Weil zeta function of the (good!) reduction of $X(\Gamma'')$ modulo $\mathfrak{p}$.

Recall that $X(\Gamma'')_{/\mathbb{F}_q}$ need not be geometrically irreducible: nevertheless, with just a little care we may apply the Weil bounds to conclude that the cuspidal spectrum of the adjacency matrix of $G(\Gamma)$ satisfies the Riemann hypothesis. Namely, whether or not the i-dimensional part of a zeta function of a scheme over $\mathbb{F}_q$ satisfies the Riemann hypothesis is faithfully preserved by base extension to $\mathbb{F}_{q^r}$, and after a suitable finite base extension we get that $X(\Gamma'')_{/\mathbb{F}_{q^r}}$ is a disjoint union of smooth, projective, geometrically irreducible curves C_i, so that the numerator of $Z(X(\Gamma''))/\mathbb{F}_{q^r}$ is just the product of the numerators of the $Z(C_i)$'s, each of which does, by Weil, satisfy the Riemann hypothesis. Thus $X(\Gamma'')_{/\mathbb{F}_{q^r}}$ and hence also $X(\Gamma)_{/}/\mathbb{F}_q$ satisfies the Riemann hypothesis.

To prove the second part of the theorem ....

The various components of $G(\Gamma)$ all have the same number of vertices (morally, this occurs because the connected components of the Shimura curve $X(\Gamma)$ are Galois conjugates).

Proof of Theorem 20: For any $q \geq 2$, consider the factorization of $q+1$ into prime powers:

$$q + 1 = \prod_{i=1}^{r} p_i^{a_i}.$$

For each i, choose a field extension $K_i/\mathbb{Q}_{p_i}$ of degree $a = \prod_i a_i$ whose maximum unramified subextension has degree a_i over $\mathbb{Q}_p$. For all i, let $P_i \in \mathbb{Q}[T]$ be such that $\mathbb{Q}_p[T]/(P_i) = K_i$. By a standard weak approximation / Krasner's Lemma argument, one can find a degree a polynomial $P \in \mathbb{Q}[T]$ which is for all i p_i-adically close enough to P_i and which is ∞-adically close enough to $(T-1)^a$ such that $K := \mathbb{Q}[T]/(P)$ is totally real and has residue degree $p_i^{a_i}$ at each p_i, for all $1 \leq i \leq r$.

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