SOLUTION TO THE INVERSE MORDELL-WEIL PROBLEM FOR ELLIPTIC CURVES

PETE L. CLARK

1. Introduction

1.1. **From Inverse Picard to Inverse Mordell-Weil.**

In [Ro76], M. Rosen showed that for any countable commutative group G, there is a field K, an elliptic curve $E_{/K}$ and a surjective group homomorphism $E(K) \to G$. From this he deduced that any countable commutative group whatsoever is the ideal class group of an **elliptic Dedekind domain** – the ring of all functions on an elliptic curve which are regular away from some (fixed, possibly infinite) set S of closed points of E. Rosen left open the question of whether every uncountable commutative group can be achieved as the class group of an elliptic Dedekind domain. This was answered in the affirmative in [Cl09], by showing that for any *free* commutative group G, there is a field K and an elliptic curve $E_{/K}$ such that the Mordell-Weil group[1] $E(K)$ is isomorphic to G.

Both constructions involve passing to quotients of the Mordell-Weil group. Leaving behind the application to class groups, we may consider the following question.

Question 1.1. *Which commutative groups G are Mordell-Weil groups of elliptic curves? More precisely, for which commutative groups G is there a field K and an elliptic curve $E_{/K}$ such that $E(K) \cong G$?*

There are many natural variants.

Question 1.2. *Let $g \in \mathbb{Z}^+$, and let p be either a prime number or zero.*
a) For which commutative groups G is there a field K of characteristic p and a g-dimensional abelian variety $A_{/K}$ with $A(K) \cong G$?
b) Suppose $p > 0$, and let A_0 be a g-dimensional abelian variety defined over an algebraically closed field of characteristic p. For which commutative groups G is there a field K of characteristic p and a g-dimensional abelian variety A with the same Newton polygon as A_0 with $A(K) \cong G$?
c) Let A_0 be a g-dimensional abelian variety defined over an algebraically clossed field. For which commutative groups G is there a field K and a g-dimensional abelian variety $A_{/K}$ with $\mathrm{End}_K A \cong \mathrm{End}\, A_0$ and $G \cong A(K)$?

Thus for instance when $g = 1$, Question 1.2b) is asking for a description of the possible Mordell-Weil groups of **ordinary** elliptic curves in characteristic p and then a separate description for **supersingular** elliptic curves in characteristic p.

[1] Here we employ the convention that a "Mordell-Weil group" is the group $A(K)$ of K-rational points of an abelian variety A defined over a field K. No conditions on K are imposed.

The motivation to consider the positive characteristic case separately as well as restriction to a given Newton polygon stratum comes from the following simple result, which gives the only "obvious" nececessary conditions on Mordell-Weil groups of abelian varieties.

Let $f, g, p \in \mathbb{Z}$ with $g \geq 1$, $0 \leq f \leq g$ and p prime. We define commutative groups $T(g)$, $T(g,p)$ and $T(g,p,f)$ as follows:

$$T(g) = (\mathbb{Q}/\mathbb{Z})^{2g} = \bigoplus_{\ell} (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g},$$

$$T(g,p) = \bigoplus_{\ell \neq p} (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{g},$$

$$T(g,p,f) = \bigoplus_{\ell \neq p} (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{f}.$$

Let T be a torsion commutative group. We say a commutative group G is **T-constrained** if there is an injection $G[\text{tors}] \hookrightarrow T$.

Proposition 1.3. *With f, g, p as above, let G be a commutative group.*
a) If G is the Mordell-Weil group of a g-dimensional abelian variety, then G is $T(g)$-constrained.
b) If G is the Mordell-Weil group of a g-dimensional abelian variety defined over a field of characteristic p, then G is $T(g,p)$-constrained.
c) If G is the Mordell-Weil group of an abelian variety defined over a field of characteristic p and with p-rank f, then G is $T(g,p,f)$-constrained.

Proposition 1.3 is almost immediate from the well known structure of the torsion subgroup of an abelian variety over an algebraically closed field. We will give precise references in § 2.3, where in fact we will determine the structure of Mordell-Weil groups of abelian varieties over algebraically closed fields.

A commutative group G is **torsion-split** if $G[\text{tors}]$ is a direct summand of G.

The main result of this paper is the following partial converse of Proposition 1.3.

Theorem 1.4. *Let $g \in \mathbb{Z}^+$, let G be a commutative group, and let K be a field.*
a) If G is torsionfree, there is a field extension L/K and a g-dimensional abelian variety $A_{/L}$ with $A(L) \cong G$.
b) If G is $T(g)$-constrained and torsion split and $\operatorname{char} K = 0$*, there is a field extension L/K and a g-dimensional abelian variety $A_{/L}$ with $A(L) \cong G$.*
c) If G is $T(1)$-constrained and $\operatorname{char} K = 0$*, there is a field extension L/K and an elliptic curve $E_{/L}$ with $E(L) \cong G$.*

1.2. **Acknowledgments.**

I am crucially indebted to B. Poonen who, in an email correspondence in the summer of 2008, provided most of the essential ideas and many of the key techniques leading up to the main result of this paper. Also D. Ulmer played a key role by assuring me that Shioda's landmark paper [Sh72] contained some needed technical results. The proof of Lemma 3.4 was kindly provided by P. Müller via the MathOverflow site. Also Lemma 2.2 was supplied by Y. Cornulier as the answer to a

question I asked on MathOverflow. Thanks also to R. Kloosterman, A. Silverberg and J.F. Voloch for some helpful remarks.

2. Some preliminary results

2.1. Terminology and Notation.

For any commutative group G, we define its **rank** as

$$r(G) = \dim_{\mathbb{Q}} G \otimes_{\mathbb{Z}} \mathbb{Q}.$$

If G is finitely generated, then $G \cong \mathbb{Z}^{r(g)} \oplus G[\mathrm{tors}]$.

2.2. More on T-Constrained Groups.

Lemma 2.1. *Let $g \in \mathbb{Z}^+$, and let T be a $T(g)$-constrained torsion commutative group. Then there are $T(1)$-constrained torsion commutative groups $T_1, \ldots, T_g$ such that $T \cong \bigoplus_{i=1}^n T_i$.*

Proof. For a commutative group A, let $A^* = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$; thus A is torsion iff A^* is profinite. Primary decomposition reduces us to the case in which T embeds in $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g}$ for some prime number ℓ. Taking duals we get an epimorphism $\mathbb{Z}_\ell^{2g} \to T^*$. Since $\mathbb{Z}_\ell$ is a PID, it follows that $T*$ is a direct sum of $2g$ monogenic $\mathbb{Z}_\ell$-modules, and thus $T^* \cong \bigoplus_{i=1}^g C_i$ with each C_i a quotient of $\mathbb{Z}_\ell^2$. Taking duals again gives $T \cong \bigoplus_{i=1}^g C_i^*$ with each C_i^* a sugroup of $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^2$ and thus $T(1)$-constrained. □

Lemma 2.2. *(Y. Cornulier) Let $g \geq 2$. There is a $T(g)$-constrained commutative group G which is* not *isomorphic to a direct product $\prod_{i=1}^g G_i$ with each G_i a $T(1)$-constrained commutative group.*

Proof. See [Co13]. □

Lemma 2.3. *Let G be a commutative group. Suppose there exists an injection $\iota : G[\mathrm{tors}] \to T(g,p,f)$. Then there exists a $\mathbb{Q}$-vector space V and an embedding $\varphi : G \hookrightarrow T(g,p,f) \oplus V$.*

Proof. Since the $\mathbb{Z}$-module $T(g,p)$ is divisible, it is injective, and thus ι extends to a homomorphism $\varphi_1 : G \to T(g,p)$. Let $\varphi_2 : G \to G \otimes \mathbb{Q}$. Finally put $\varphi = \varphi_1 \oplus \varphi_2 : G \to T(g,p) \oplus (G \otimes \mathbb{Q})$. □

2.3. Torsion-Split Groups.

A torsion commutative group T is **bounded** if $T = T[n]$ for some $n \in \mathbb{Z}^+$. Thus every finite group is bounded. The converse does not hold generally, but it does hold for subgroups of $(\mathbb{Q}/\mathbb{Z})^{2g}$, which are the torsion groups of interest to us here.

For any commutative group G, we have the torsion sequence

$$0 \to G[\mathrm{tors}] \to G \to G' \to 0, \tag{1}$$

where $G[\mathrm{tors}]$ is the torsion subgroup of G and $G \to G'$ is the quotient map. The quotient G' is torsionfree (or equivalently, flat). As we will see, dealing with G – in particular, attempting to realize it as a Mordell-Weil group – is easier if (1) splits: when this occurs we will say that G is **torsion-split**.

Theorem 2.4. *(Baer's Theorem) Let T be a torsion commutative group. TFAE:*
(i) Every commutative group G with $G[\mathrm{tors}] \cong T$ is torsion-split.
(ii) T is the direct sum of a bounded group and a divisible group.

Proof. See [Ba36, Thm. 8;5]. □

Lemma 2.5. *A commutative group G is torsion-split if* any *of the following hold:*
a) $G[\mathrm{tors}]$ is divisible (equivalently, injective).
b) G' is projective (equivalently, free).
c) There is an exact sequence

$$0 \to G_1 \to G \to G_2$$

with G_1 torsion-split and $G_2[\mathrm{tors}]$ bounded.

Proof. Conditions a) and b) are immediate from the definitions. As for c): passing to torsion subgroups we get an exact sequence

$$0 \to G_1[\mathrm{tors}] \to G[\mathrm{tors}] \to B_1 \to 0$$

with B_1 a bounded order torsion group. By Baer's Theorem, $G_1[\mathrm{tors}] = D \oplus B_2$ with D divisible and B_2 bounded. Since D is divisible, it is injective, so $G[\mathrm{tors}] = D \oplus C$ for some subgroup C. If $B_1 = B_1[a]$ and $B_2 = B_2[b]$, then $C = C[ab]$, so G is torsion-split by Baer's Theoem. □

Example 2.6. *The group $G = \prod_\ell \mathbb{Z}/\ell\mathbb{Z}$ is $T(1)$-constrained but not torsion-split.*

Thus if every $T(1)$-constrained commutative group is the Mordell-Weil group of an elliptic curve, then there are non-torsion-split Mordell-Weil groups. On the other hand, it is challenging to find non-torsion-split Mordell-Weil groups "in nature".

Example 2.7. *Let $A_{/K}$ an abelian variety. Then $A(K)$ is torsion-split if any of the following hold.*
a) K is algebraic over a finite field: then $A(K)$ is a torsion group.
b) K is finitely generated: then $A(K)$ is finitely generated [Né52]*.)*
c) K is a locally compact, nondiscrete topological field: then if $K \cong \mathbb{C}$, then $A(K)[\mathrm{tors}]$ is divisible. If $K \cong \mathbb{R}$ then $A(K)[\mathrm{tors}]$ is the direct product of a divisible group and a finite group. If K is non-Archimedean, then $A(K)[\mathrm{tors}]$ is finite.
d) K is a finitely generated field extension of a field k over which all Mordell-Weil groups of abelian varieties are torsion-split (e.g. any of the fields of parts a) through c)): then by the Lang-Néron Theorem [LN59]*, $A(K)$ is the extension of a finitely generated group by a torsion-split group, so is torsion-split by Lemma 2.5c).*

2.4. **The Continuity Lemma.**

Example 2.7 shows that in order to prove Theorem 1.4c) we will need to consider fields which are *not* finitely generated (or even finitely generated over a locally compact field). However every field is the direct limit of its finitely generated subfields, so it is natural to start with a field k, an abelian variety $A_{/k}$, and a directed system of field extensions $\{k_i\}_{i \in I}$ (by this we mean to include the "filtered" condition: for all $i, j \in I$, there is $k \in I$ with $k \geq i$ and $k \geq j$) and ask how the Mordell-Weil groups $A(k_i)$ compare to the Mordell-Weil group of $A(\varinjlim k_i)$. Fortunately for us, this question has an extremely simple answer.

Lemma 2.8. *Let k be a field, $X_{/k}$ a finite-type k-scheme, and let $\{k_i\}_{i\in I}$ be a directed system of field extensions of k. Then the natural map*

$$\varinjlim X(k_i) \to X(\varinjlim k_i)$$

is an isomorphism.

Proof. Let $K = \varinjlim k_i$. Then K is a field. We may view each $X(k_i)$ as a subset of $X(K)$, and the task is to show that each $P \in X(K)$ lies in $X(k_i)$ for some $i \in I$. Since each $P \in X(K)$ lies in $U(K)$ for some affine open subscheme of X, we may assume that X is itself affine, say $X = \operatorname{Spec} R$, where $R = k[t_1, \ldots, t_n]/I$. Then $P \in X(K)$ corresponds to a k-algebra map $\iota_P : R \to K$. For $1 \leq j \leq n$, choose $i_j \in I$ such that $\iota_P(t_j) \in k_i$. Let $i_\bullet \in I$ be such that $i_j \leq i_\bullet$ for all i; then ι_P lies in the image of the natural map $X(k_{i_\bullet}) \hookrightarrow X(K)$. □

Remark 2.9. *Lemma 2.8 is a generalization of* [Cl09, Lemma 16]*, in which $X_{/k}$ is taken to be an elliptic curve. The proof of that result reads "E.g. by abstract nonsense: this holds for any representable contravariant functor from the category of affine K-schemes to the category of abelian groups"; thus the finite-type hypothesis of Lemma 2.8 is being ignored. This is a mistake: take $k = \mathbb{Q}$, $X = \operatorname{Spec}\mathbb{C}$, and $\{k_i\}_{i\in I}$ to be the directed system of finitely generated subfields of $\mathbb{C}$. In particular each k_i is countable, so $X(k_i) = \operatorname{Hom}_{\mathbb{Q}}(\mathbb{C}, k_i) = \varnothing$, whereas clearly $X(\mathbb{C}) = \operatorname{Hom}_{\mathbb{Q}}(\mathbb{C},\mathbb{C}) \neq \varnothing$ (in fact it has cardinality $2^{2^{\aleph_0}}$).* Mea culpa.

2.5. **The Shioda-Tate Formula.**

The purpose of this section is to recall a key result from [Sh72] on rational elliptic surfaces, to be applied in the following section.

Let k be a field, and put $K = k(t)$. Here, by a **rational elliptic surface** we will mean an elliptic curve $E_{/K}$ admitting a Weierstrass equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 = a_4x + a_6$ with $a_i \in k[t]$, $\deg a_i \leq i$ for all i, and with j-invariant $j(E) \notin k$ ("non-isotrivial"). The degree conditions ensure that

$$Y^2Z + a_1(t)XYZ + a_3(t)YZ^2 = X^3 + a_2(t)X^2Z + a_4(t)XZ^2 + a_6(t)Z^3,$$

viewed as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$, defines a regular surface, which we denote by $\mathcal{E}$. The nonisotriviality implies that $\mathcal{E}$ is nonsplit and thus $E(K)$ is finitely generated [AECII, Thm. III.6.1]. Let Δ be the discriminant of the given Weierstrass equation and Δ_E be the discriminant of a $k[t]$-minimal Weierstrass equation (which exists since $k[t]$ is a PID). Since j is nonconstant, so is Δ_E and thus there is at least one finite singular fiber.

Theorem 2.10. *(Shioda-Tate Formula) Let $E_{/k(t)}$ be a nonisotrivial rational elliptic surface. Let $S(E)$ be the set of places of $k(t)$ at which E has bad reduction. For each $v \in S(E)$, let m_v be the number of geometric components of the special fiber at v of the minimal regular model of E. Then*

$$r(E) = 8 - \sum_{v\in S(E)} (m_v - 1)\,.$$

Proof. See [Sh72, Cor. 1.5] or [SS10, Cor. 6.3, Prop. 8.1]. □

2.6. **Some Elliptic Curves With Torsion Mordell-Weil Group.**

Let T be a $T(1)$-constrained torsion group. To solve the Inverse Mordell-Weil Problem for elliptic curves in characteristic 0, we need to construct elliptic curves with Mordell-Weil group any group G of the form

$$0 \to T \to G \to G' \to 0$$

with G' torsionfree. An important special case is when $G' = 0$, so we want a field K and an elliptic curve E with $E(K) \cong T$. For each fixed T there are many, many ways to do this: for instance by a rather recent theorem of Mazur-Rudin, for every number field K there is an elliptic curve $E_{/K}$ with $E(K) = 0$. It is both of intrinsic interest and useful for our later construction of elliptic curves with $E(K)[\mathrm{tors}] \cong T$ to do this in a *maximally generic way.*

Let T be a $T(1)$-constrained finite group: $T \cong T_{m,n} = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for positive integers m, n with $m \mid n$. Considering elliptic curves endowed with a T-structure gives a functor from the category of $\mathbb{C}$-schemes to sets. There is an associated coarse moduli space $Y_m(n)_{/\mathbb{C}}$, which is nonsingular, integral affine curve; let $X_m(n)_{\mathbb{C}}$ be its projective completion. By [CP80, Prop. 1.2, Thm. A.1], this moduli space is fine except (precisely) for the following cases:

$$(m, n) \in \{(1, 1),\ (1, 2),\ (1, 3),\ (2, 2)\}. \tag{2}$$

We call these four pairs as well as the corresponding subgroups $T_{m,n}$ **small**. Suppose for now that (m, n) is not small. Then there is a universal family $\mathcal{E}_m(n) \to X_m(n)$, which is an elliptic surface, whose singular fibers occur precisely at the cusps $X_m(n)(\mathbb{C}) \setminus Y_m(n)(\mathbb{C})$. The fiber at the generic point is an elliptic curve $E_m(n)_{/\mathbb{C}(X_m(n))}$ equipped with an embedding $T_{m,n} \hookrightarrow E_m(n)(\mathbb{C}(X_m(n)))$.

Theorem 2.11. *(Shioda* [Sh72, Thm. 5.5]*, Cox-Parry* [CP80, Cor. 4.2]*) Let $m \mid n$ be positive integers such that $T_{m,n}$ is* not *small. Then*

$$E_m(n)(\mathbb{C}(X_m(n)) = T_{m,n}.$$

Now let T be an infinite, $T(1)$-constrained subgroup. Then T is the direct limit of its non-small subgroups $T_{m,n}$. Let (m_1, n_1) and (m_2, n_2) be non-small pairs with $T_{m_1,n_1} \subset T_{m_2,n_2} \subset T$. Then the natural forgetful functor from T_{m_2,n_2}-structured elliptic curves to T_{m_1,n_1}-stuctured elliptic curves induces a morphism of compactified fine moduli spaces $X_{m_2}(n_2) \to X_{m_1}(n_1)$, and it follows easily that the universal elliptic curve $\mathcal{E}_{m_2}(n_2)$ is the fiber product $X_{m_2}(n_2) \times_{X_{m_1}(n_1)} \mathcal{E}_{m_1}(n_1)$, and thus

$$E_{m_1}(n_1)_{/\mathbb{C}(X_{m_2})} \cong E_{m_2}(n_2). \tag{3}$$

Let X_T be the direct limit of the function fields $\mathbb{C}(X_m(n))$, for all non-small $T_m(n) \subset T$. Fix one non-small $T_{m,n} \subset T$. Applying Lemma 2.8, we get:

Theorem 2.12.

$$E_m(n)(X_T) = T.$$

Remark 2.13. *The moduli space $X_m(n)$ can be defined over a more general base than $\mathbb{C}$. For instance it can be defined over $\mathbb{Q}(\mu_n)$ (and, by changing the subgroup scheme from $\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$ to $\mu_m \times \mathbb{Z}/n\mathbb{Z}$, it can be defined over $\mathbb{Q}$, but the interpretation in terms of rational torsion points is lost) as well as any algebraically closed* *field k of characteristic indivisible by n. However, the finiteness of the Mordell-Weil group of the universal curve is not guaranteed. In fact, Shioda has shown that when $k = \overline{\mathbb{F}_3}$, then $E_4(4)(k(X_4(4)) \cong \mathbb{Z}^2 \times (\mathbb{Z}/4\mathbb{Z})^2$. Moreover, conditionally on the conjecture of Birch, Swinnerton-Dyer and Tate, this holds with $k = \overline{\mathbb{F}_p}$ for all* $p \equiv 3 \pmod 4$. *This phenomenon prevents us from solving the Inverse Mordell-Weil problem for elliptic curves in positive characteristic.*

Lemma 2.14. *Let K be a field of characteristic $p > 0$, and let $E_{/K}$ be an elliptic curve. If $E(K)[p^\infty] \neq 0$, then $j(E) \in K^p$.*

Proof. The hypothesis $E(K)[p^\infty] \neq 0$ is equivalent to $E(K)[p] \neq 0$. If E is supersingular then $E(\overline{F})[p] = 0$, so we may assume E is ordinary: $E(\overline{F})$ contains a unique subgroup C_p of order p. If $P \in E(F)$ has order p, then $C_p = \langle P \rangle$ is F-rational, and thus $\varphi : E \to E/C_p$ is a separable isogeny of degree p. Let $\varphi^\vee : E/C_p \to E$ be the dual isogeny, so $\varphi^\vee \circ \varphi = [p]_E$. Since $[p]_E$ is inseparable, $\varphi^\vee$ must be the Frobenius map on E/C_p, and thus $E \cong (E/C_P)^{(p)}$ and $j(E) = j(E/C_P)^p \in F^p$. □

In the next six results we will construct elliptic curves with Mordell-Weil group equal to each of the four *small* torsion groups $T_{m,n}$. The general strategy will always be the same, so it will be more efficient to indicate it now. In each case we will start with a field k. In one case k will have characteristic 2; in another, characteristic 3; in the other four cases $\operatorname{char} k$ will not be 2 or 3; but otherwise k will be arbitrary. We will let $l = k(s)$ be a rational function field in one variable. In each case we will define an elliptic curve over $K = l(t)$ via a Weierstrass equation with $a_i \in l[t]$ with $\deg a_i \leq i$ and $j(E) \notin \bar{l}$. Thus we will be in the setting of § 2.5: E will define a nonisotrivial rational elliptic surface over ℓ. By Lang-Néron, $E(K)$ is finitely generated. Moreover the rank can be computed by Tate's algorithm and the Shioda-Tate formula, and it will turn out to be equal to 0 in all cases. As for the torsion, let $p = \operatorname{char} l$. Then in all cases, when $p > 0$ we will have $j(E) \notin K^p$ – indeed $j(E)$ will be given by a rational function of t which is not a rational function of t^p, which implies the stronger statement $j(E) \notin \overline{\ell}((t^p)) = \overline{\ell}((t))^p$. Thus by Lemma 2.14 we will have no p-torsion in characteristic $p > 0$. Now let v be a place of K which is trivial on l, let K_v be the corresponding completion, a complete DVR with valuation ring R_v, maximal ideal $\mathfrak{m}_v$ and residue field k_v. We denote by $E_{/R_v}$ the Néron model. Then the Néron mapping property gives $E(R_v) = E(K_v)$. Reduction modulo v gives a short exact sequence

$$0 \to K \to E(R_v) \to E(k_v) \to 0.$$

The kernel of reduction K is isomorphic to the $\mathfrak{m}_v$-points of the associated formal group. Then K is torsionfree when $\operatorname{char} l = 0$ and has only p-primary torsion when $\operatorname{char} l = p > 0$ [AECI, Prop. IV.3.2]. In our case we already know that $E(K)[p^\infty] = \{0\}$, so we get an injection $E(K)[\text{tors}] \hookrightarrow E(k_v)[\text{tors}]$. In each of our examples there will be a place v of additive reduction, so (again using the fact that we have no p-primary torsion in characteristic p) we get an injection of $E(K)[\text{tors}]$ into the component group Φ_v, which will be one of our four small groups $T_{m,n}$. By exhibiting a subgroup of $E(K)$ isomorphic to $T_{m,n}$, we conclude $E(K) \cong T_{m,n}$.

Theorem 2.15. *Let k be a field with* $\operatorname{char} k \notin \{2, 3\}$. *Let $K = k(s,t)$ and*

$$E_{/K} : y^2 = x^3 + sx + t.$$

Then $\operatorname{End} E = \mathbb{Z}$ *and* $E(K) = 0$.

Proof. Let $l = k(s)$. We follow the general strategy laid out above. In particular, $j = \frac{256s^3}{t^2+\frac{4}{27}s^3} \notin \overline{\ell}$, so $\operatorname{End} E = \mathbb{Z}$. If $\operatorname{char} k = p > 0$, then $j \notin K^p$ so $E(K)[p^\infty] = 0$ by Lemma 2.14. We may view E as a nonisotrivial rational elliptic surface over l. Tate's algorithm shows that E has two singular fibers, with corresponding uniformizers $\frac{1}{t}$ and $t^2 + \frac{4}{27}s^3$, and with Kodaira symbols II* and I1 (respectively). By the Shioda-Tate Formula,

$$r(E(K)) = 8 - ((1-1) + (9-1)) = 0.$$

The fiber at ∞ is $\mathbb{G}_a$, and the component group is trivial, so $E(K) = 0$. $\square$

Theorem 2.16. *Let k be a field of characteristic 2, let $K = k(s,t)$, and let*

$$E_{/K} : y^2 + xy = x^3 + sx^2 + t.$$

Then $\operatorname{End} E = \mathbb{Z}$ and $E(K) = 0$.

Proof. Let $l = k(s)$. We have

$$j(E) = \frac{1}{t},$$

so as above $\operatorname{End} E = \mathbb{Z}$, E has no 2-primary torsion, and E defines a nonisotrivial rational elliptic surface over l. Tate's algorithm shows that E has two singular fibers, with uniformizers $\frac{1}{t}$ and t, and with Kodaira symbols II* and I1 (respectively). The remainder of the argument is exactly the same as in Theorem 2.15. $\square$

Theorem 2.17. *Let k be a field of characteristic 3, let $K = k(s,t)$, and let*

$$E_{/K} : y^2 = x^3 + sx^2 + t.$$

Then $\operatorname{End} E = \mathbb{Z}$ and $E(K) = 0$.

Proof. Let $l = k(s)$. We have

$$j(E) = \frac{2s^3}{t},$$

so as above $\operatorname{End} E = \mathbb{Z}$, E has no 3-primary torsion, and E defines a nonisotrivial rational elliptic surface over l. Tate's algorithm shows that E has two singular fibers, with uniformizers $\frac{1}{t}$ and t, and with Kodaira symbols II* and I1 (respectively). The remainder of the argument is exactly the same as in Theorem 2.15. $\square$

Theorem 2.18. *Let k be a field of characteristic neither 2 nor 3, let $K = k(s,t)$ be a rational function field; put*

$$E_{/K} : y^2 = x(x^2 + sx + t) = x^3 + sx^2 + tx.$$

Then $\operatorname{End} E = \mathbb{Z}$ and $E(K) = \mathbb{Z}/2\mathbb{Z}$.

Proof. Let $l = k(s)$. We have

$$j(E) = \frac{256s^6 - 2304s^4t + 6912s^2t^2 - 6912t^3}{s^2t^2 - 4t^3},$$

so as above $\operatorname{End} E = \mathbb{Z}$, E has no p-primary torsion, and E defines a nonisotrivial rational elliptic surface over l. Tate's algorithm shows there are three singular fibers, of Kodaira types I1, I2, III*. By the Shioda-Tate Formula,

$$r(E(K)) = 8 - ((1-1) + (2-1) + (8-1)) = 0.$$

The fiber of type III* is additive with component group $\mathbb{Z}/2\mathbb{Z}$. Since $\operatorname{char} k = 0$, this shows $\#E(K)[\text{tors}] \leq 2$. The point $(0,0)$ has order 2, so $E(K) = \mathbb{Z}/2\mathbb{Z}$. $\square$

Theorem 2.19. *Let k be a field, not of characteristic 3, let $K = k(s,t)$ be a rational function field, and let*

$$E_{/K} : y^2 + sxy + ty = x^3.$$

Then $\operatorname{End} E = \mathbb{Z}$ *and* $E(K) = \mathbb{Z}/3\mathbb{Z}$.

Proof. Let $l = k(s)$. We have

$$j(E) = \frac{512s^3t^3 - 64s^6t^2 + \frac{8}{3}s^9t - \frac{1}{27}s^{12}}{t^4 - \frac{1}{27}s^3t^3},$$

so as above $\operatorname{End} E = \mathbb{Z}$, E has no p-primary torsion, and E defines a nonisotrivial rational elliptic surface over l. Tate's algorithm shows there are three singular fibers, of Kodaira types I1, I3, IV*. By the Shioda-Tate Formula,

$$r(E(K)) = 8 - ((1-1) + (3-1) + (7-1)) = 0.$$

The fiber of type IV* is additive with component group $\mathbb{Z}/3\mathbb{Z}$. Since $\operatorname{char} k = 0$, this shows $\#E(K)[\text{tors}] \mid 3$. The point $(0,0)$ has order 3 on E, so $E(K) = \mathbb{Z}/3\mathbb{Z}$. □

Theorem 2.20. *Let k be a field of characteristic different from 2. Let $K = k(t)$ be a rational function field, and let*

$$E_{/K} : y^2 = x^3 + s(-1-t)x^2 + s^2tx.$$

Then $\operatorname{End} E = \mathbb{Z}$ *and* $E(K) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Proof. Let $l = k(s)$. We have

$$j(E) = \frac{256t^6 - 768t^5 + 1536t^4 - 1792t^3 + 1536t^2 - 768t + 256}{t^4 - 2t^3 + t^2},$$

so as above $\operatorname{End} E = \mathbb{Z}$, E has no p-primary torsion, and E defines a nonisotrivial rational elliptic surface over l. Tate's algorithm shows there are three singular fibers, of Kodaira types I2, I2, I2*. By the Shioda-Tate Formula,

$$r(E(K)) = 8 - ((2-1) + (2-1) + (7-1)) = 0.$$

The fiber of type I2* is additive with component group $(\mathbb{Z}/2\mathbb{Z})^2$. Since $\operatorname{char} k = 0$, this shows $E(K)[\text{tors}] \subset (\mathbb{Z}/2\mathbb{Z})^2$. The points $(0,0)$, $(1,0)$, $(t,0)$ have order 2, so $E(K) = (\mathbb{Z}/2\mathbb{Z})^2$. □

2.7. **Mordell-Weil Groups Over Function Fields.**

Let k be a field. For k-schemes A and B, we write $\operatorname{Map}_k(A,B)$ for the set of morphisms of k-schemes from A to B. If A and B are k-group schemes, we write $\operatorname{Hom}_k(A,B)$ for the set of homomorphisms of k-group schemes from A to B.

Proposition 2.21. *Let A, B be abelian varieties over k. There is an exact sequence*

$$0 \to \operatorname{Hom}_k(B,A) \to A(k(B)) \xrightarrow{q} A(k) \to 0.$$

This sequence is split by the injection $\iota : A(k) \to A(k(B))$, so

$$A(k(B)) = \operatorname{Hom}_k(B,A) \times A(k).$$

Proof. We have $A(k(B)) = \mathrm{Map}_k(\mathrm{Spec}\, k(B), A)$, which is the set of all rational k-maps from B to A, under pointwise addition. Since every rational map from a nonsingular variety to an abelian variety is everywhere defined [M-AV, Thm. 3.2],

$$A(k(B)) = \mathrm{Map}_k(B, A).$$

Evaluation at $0 \in B$ gives a homomorphism $q : \mathrm{Map}_k(B, A) \to A(k)$, with kernel $\mathrm{Hom}_k(B, A)$. The homomorphism $\sigma : A(k) \to \mathrm{Map}_k(B, A)$ which sends $x \in A(K)$ to the constant map with image x satisfies $q \circ \sigma = 1_{A(k)}$. The result follows. $\square$

3. Mordell-Weil Groups Over Algebraically Closed Fields

In general our goal in this paper is to explore which groups arise as the Mordell-Weil group of some g-dimensional abelian variety over some field, or some restricted class of fields. We freely acknowledge the more interesting (and certainly, more difficult) question which arises if we *fix* a field K and ask for all Mordell-Weil groups of abelian varieties over that field. For instance, the possible Mordell-Weil group of elliptic curves over $\mathbb{Q}$ is one of the main open questions in elliptic curve theory: the structure of the torsion subgroup was determined by B. Mazur, but even whether the rank is uniformly bounded – let alone whether all natural numbers can arise as the rank – remains open. To the best of my knowledge it remains open whether for every natural number n, there is a number field $K = K(n)$ and an elliptic curve E/K with $\mathrm{rank}\, E(K) = n$.

However, there are some fields K for which the possible structures of Mordell-Weil groups of g-dimensional abelian varieties can be determined. For instance, the Mordell-Weil group of an elliptic curve over $\mathbb{R}$ is $S^1 \times \mathbb{Z}/2\mathbb{Z}$ if the discriminant (of any Weierstrass equation) is positive and S^1 if the discriminant is negative. And especially, for any g-dimensional abelian variety $A_{/C}$, $A(\mathbb{C})$ is a g-dimensional complex torus so has Mordell-Weil group $(S^1)^{2g}$. As an abstract group, the latter is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{2g} \oplus V$, where V is a $\mathbb{Q}$-vector space (equivalently, a uniquely divisible commutative group) of rank $2^{\aleph_0} = \#\mathbb{C}$.

In fact we can give a similar description of the Mordell-Weil group of an abelian variety over any algebraically closed field K.

Theorem 3.1. *Let K be an algebraically closed field, and let $A_{/K}$ be a g-dimensional abelian variety. Put*

$$T(g) = (\mathbb{Q}/\mathbb{Z})^{2g}.$$

For a prime number p and $0 \leq a \leq g$, put

$$T(g, p, a) = \left(\prod_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell \right)^{2g} \times (\mathbb{Q}_p/\mathbb{Z}_p)^a.$$

Let V be a $\mathbb{Q}$-vector space of dimension $\#K$.
a) If $\mathrm{char}\, K = 0$, then

$$A(K) \cong (\mathbb{Q}/\mathbb{Z})^{2g} \times V.$$

b) If $K \cong \overline{\mathbb{F}_p}$, then there is $0 \leq a \leq g$ such that

$$A(K) \cong T(g, p, a).$$

c) If K has characteristic $p > 0$ and is not algebraic over $\mathbb{F}_p$ then there is some $0 \leq a \leq g$ such that

$$A(K) \cong T(g, p, a) \times V.$$

In particular, if K is algebraically closed of characteristic 0, then if A, B are abelian varieties over K then $A(K) \cong B(K) \iff \dim A = \dim B$.

Theorem 3.1 is not an essentially new result: special cases carrying the essential content have appeared several times in the literature. In particular, the part of Theorem 3.1 which asserts that if K is not algebraic over a finite field then $r(A(K)) = \#K$ – from which the full statement of Theorem 3.1 is an easy consequence – is an old result of Frey-Jarden [FJ74, Thm. 10.1]. Their proof uses valuations and Greenberg's generalization of Hensel's Lemma.

The following result is a consequence of Theorem 3.1; conversely, Theorem 3.1 follows easily from it (as we will see).

Theorem 3.2. *Let K be an infinite, finitely generated field, and let $A_{/K}$ be a nontrivial abelian variety. Then $r(A(\overline{K})) = \aleph_0$.*

Theorem 3.2 appears as Theorem A in the appendix of [Ro73]. The proof given there is short and elegant, making use of the fact that K is either a number field or a univariate function field over some constant field; in some ways it is reminiscent of the proof of the weak Mordell-Weil Theorem. Theorem 3.2 also appears are [FPS07, Thm. 7]. (The statements of [Ro73, Thm. A], [FPS07, Thm. 7] and Theorem 3.2 are all slightly different but plainly equivalent.) This third proof uses a specialization theorem of A. Néron.

Before we learned of the work of Rosen, Frey-Jarden and Falcone-Plaumann-Strambach we gave our own proof of Theorem 3.2, which relies on a group-theoretic lemma whose proof was supplied to us by P. Müller. In fact Müller had previously used this lemma to show that any nontrivial abelian variety over an algebraically closed field which is not algebraic over a finite field has non-torsion Mordell-Weil group.

Our approach proceeds by first establishing following intermediate result.

Theorem 3.3. *Let K be a Hilbertian field with separable closure K^{sep}, let $A_{/K}$ be a nonzero abelian variety, and for $n \in \mathbb{Z}^+$, let $K_n = A([n]^{-1}A(K))$ be the field obtained by adjoining to K all residue fields of closed points $Q \in A(\overline{K})$ such that $[n]Q \in A(K)$, and let $K_\infty = \bigcup_{n=1}^{\infty} K_n$. Then there is $P \in A(K^{\text{sep}}) \setminus A(K_\infty)$. In particular $K^{\text{sep}} \not\subseteq K_\infty$.*

We feel that Theorem 3.3 is of some independent interest, so asking the reader's indulgence, we will give a fourth proof of Theorem 3.2.

3.1. **A Group-Theoretic Lemma.**

Lemma 3.4. *For all $m \in \mathbb{Z}^+$, there are only finitely many integers $k \geq 5$ such that A_k is a subquotient of $\mathrm{GL}_m(\mathbb{Z}/n\mathbb{Z})$ for some $n \in \mathbb{Z}^+$.*

Proof. Step 1: Let G be a finite group. We claim that every simple subquotient of G is a subquotient of some Jordan-Hölder factor of G. Indeed, by induction it suffices to show that if N is a normal subgroup of G, then every simple subquotient

of G is a subquotient of either N or G/N. Let G' be a subgroup of G, let N' be a normal subgroup of G', and let $\pi : G \to G/N$ be the quotient map. Then G' is an extension of $\pi(G')$ by $G' \cap N$, so the Jordan-Hölder factor G'/N' of G' must appear up to isomorphism as a Jordan-Hölder factor of either $G' \cap N$ or of $\pi(G')$, and it follows that G'/N' is a subquotient of either N or G/N.
Step 2: Let $m \in \mathbb{Z}^+$. Let $n \in \mathbb{Z}^+$ have prime power factorization $n = p_1^{a_1} \cdots p_r^{a_r}$. Then $\mathrm{GL}_m(\mathbb{Z}/n\mathbb{Z}) \cong \prod_{i=1}^r \mathrm{GL}_m(\mathbb{Z}/p_i^{a_i}\mathbb{Z})$. Further, for $1 \leq i \leq r$, the kernel of the natural map $\mathrm{GL}_m(\mathbb{Z}/p_i^{a_i}\mathbb{Z}) \to \mathrm{GL}_m(\mathbb{Z}/p_i\mathbb{Z})$ is a p-group. It follows that every noncyclic simple subquotient of $\mathrm{GL}_m(\mathbb{Z}/n\mathbb{Z})$ is a simple subquotient of $\mathrm{GL}_m(\mathbb{Z}/p_i\mathbb{Z})$ for some i. Thus we assume henceforth that $n = p$ is prime.
Step 3: We apply the following theorem of Larsen-Pink [LP11, Thm. 0.2]: there is a positive integer $J'(m)$ such that every subgroup $\Gamma \subset \mathrm{GL}_m(\mathbb{Z}/p\mathbb{Z})$ admits a subnormal series $\Gamma_3 \subset \Gamma_2 \subset \Gamma_1 \subset \Gamma$ such that:
• $[\Gamma : \Gamma_1] \leq J'(m)$;
• Γ_1/Γ_2 is a direct product of finite simple groups of Lie type;
• Γ_2/Γ_3 is commutative; and
• Γ_3 is a p-group.
In view of Step 1, the result follows immediately from this and the fact that for all $k \geq 9$, A_k is *not* isomorphic to a finite simple group of Lie type. □

3.2. **Proof of Theorem 3.3.**

Step 1: Let $g = \dim A$. The extension K_n/K is normal algebraic; let K_n° be its maximal separable subextension. Then K_n°/K is Galois, and if $G_n = \mathrm{Aut}(K_n^\circ/K)$, then we have

$$1 \to C_n \to G_n \to Q_n \to 1,$$

where $Q_n = \mathrm{Aut}(K(A[n])/K)$ embeds in $\mathrm{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$ and C_n is an inverse limit of finite commutative groups of exponent dividing n. By Lemma 3.4, there is $k \in \mathbb{Z}^+$ such that for no $n \in \mathbb{Z}^+$ does G_n admit A_k as a composition factor. Since K is Hilbertian, there is a Galois extension L/K with $\mathrm{Aut}(L/K) \cong S_k$, and the above analysis shows that L is not contained in K_∞. By Noether normalization, if A' is an affine open subset of A, then (since A' is smooth) there is a finite morphism $\varphi : A' \to \mathbb{A}^g$. Since L/K is separable, it is of the form $K(\alpha)$, and then $\overline{P} = (\alpha, \ldots, \alpha) \in \mathbb{A}^g$ has residue field L and thus any $P \in \varphi^{-1}(\overline{P})$ has residue field containing L. It follows that $P \in A(\overline{K}) \setminus A(K_\infty)$.
Step 2: We refine the above argument to show that P can be chosen to lie in $A(K^{\mathrm{sep}})$. Since $A_{/K}$ is smooth, by Separable Noether Normalization the map φ can be chosen to be generically separable. Since the set of points of $\mathbb{A}^n(\overline{K})$ with residue field L is Zariski dense, by the openness of the étale locus it follows that we may choose $\overline{P}$ so that $P \in A(K^{\mathrm{sep}}) \setminus A(K_\infty)$.

3.3. **Proof of Theorem 3.2.**

We will show $r(A(K)) = \aleph_0$ inductively: for $n \in \mathbb{N}$, let $P_1, \ldots, P_n \in A(K)$ be $\mathbb{Z}$-linearly independent. There is an infinite, finitely generated subfield k of K such that A and $P_1, \ldots, P_n$ are all defined over k. Since k is finitely generated over either $\mathbb{Q}$ or $\mathbb{F}_p(t)$ it is Hilbertian, and applying Proposition 3.3 to $A_{/k}$ we get $P \in A(\overline{k}) \setminus A(k_\infty) \in A(K) \setminus A(k_\infty)$, and it follows that $P_1, \ldots, P_n, P$ are $\mathbb{Z}$-linearly independent in $A(K)$.

3.4. **Proof of Theorem 3.1.**

Step 1: Let $T = A(K)[\text{tors}]$ and $G' = A(K)/T$. Since K is algebraically closed, for all $n \in \mathbb{Z}^+$, the multiplication by n map $[n] : A(K) \to A(K)$ is surjective, i.e., $A(K)$ is a divisible commutative group. Thus G' is torsionfree and divisible, hence a $\mathbb{Q}$-vector space. Also T is divisible, so $A(K)[\text{tors}]$ is torsionsplit:

$$A(K) \cong T \times G'.$$

Step 2: The structure of T is part of the standard theory of abelian varieties: e.g. [M-AV, § I.7]. Thus it suffices to show that the $\mathbb{Q}$-vector space G' has dimension 0 when K is algebraic over $\mathbb{F}_p$ and dimension $\#K$ otherwise.

Step 3: If K is algebraic over $\mathbb{F}_p$ for some prime p, then A is defined over some finite field $\mathbb{F}_q$ and $A(K) = \varinjlim A(\mathbb{F}_{q^n})$ is a direct limit of finite commutative groups, hence $A(K) = T$ and $G' = 0$. Next observe that for any algebraic variety V of positive dimension over an algebraically closed field K, $\#V(K) = \#K$. To see this: it is no loss of generality to assume that V is irreducible. To get the upper bound, observe that V can be covered by finitely many affine varieties and for all $n \geq 1$, $\#\mathbb{A}^n(K) = \#K$. To get the lower bound: let U be an affine open subvariety. By Noether normalization, U is a branched covering of $\mathbb{A}^n$ for some $n \geq 1$ and thus has at least $\#K$ points. If K is uncountable, then since T is countable, $\#G' = \#A(K) = \#K$, and it follows that $\dim_{\mathbb{Q}} G' = \#K$.

Step 4: Suppose finally that K is countable and not algebraic over a finite field. Since K is countably infinite, so is $A(K)$, and thus $r(A(K)) \leq \aleph_0$. Moreover K contains an infinite, finitely generated subfield k, so by Theorem 3.2,

$$\aleph_0 = r(A(k)) \leq r(A(K)) \leq \aleph_0.$$

Thus $r(A(K)) = \aleph_0$.

3.5. **Subgroups of Mordell-Weil Groups.**

The following is an immediate consequence of Theorem 3.1.

Corollary 3.5. *For $g \in \mathbb{Z}^+$, $p \geq 0$ and a commutative group G, TFAE:*
(i) There is a field K of characteristic p and a g-dimensional abelian variety $A_{/K}$ and an injective group homomorphism $G \hookrightarrow A(K)$.
(ii) G is $T(g,p)$-constrained.

In particular Corollary 3.5 establishes Proposition 1.3. The implication (ii) $\Longrightarrow$ (i) may be regarded as dual to the main ressult of [Cl09], which determines which commutative groups are quotients of Mordell-Weil groups (all of them).

4. Abelian Varieties With Prescribed Mordell-Weil Group

4.1. **Claborn's Theorem Revisited.**

In [Cl09] we constructed, for any *free* commutative group G, an elliptic curve $E_{/K}$ with $E(K) \cong G$ by starting with a field k and $E_{/k}$ with $\operatorname{End} E = \mathbb{Z}$ and $E(k) = 0$, observing that by Proposition 2.21 we have $E(k(E)) \cong \mathbb{Z}$, $E(k(E)(E_{/k(E)}) \cong \mathbb{Z}^2$, and using transfinite induction to iterate this process. This uses the "fact" that every infinite set can be well-ordered and thus the Axiom of Choice (AC).

We would like to present a variant of this argument suggested by B. Poonen.

This is logically superfluous, since we will soon present a refinement of this variant argument (again due to Poonen) which will be used to prove Theorem 1.4a). Our reasons for doing this will be explained below.

Let k be a field and $A_{/k}$ an abelian variety with $\mathrm{End}_k(A) = \mathrm{Hom}_k(A, A) = \mathbb{Z}$ and $A(k) = 0$. Let S be a finite set, put $A^S = A \times \ldots \times A$ ($\#S$ factors), and put $k_S = k(A^S)$. Applying Proposition 2.21 with $(A, B) = (A, A^S)$, we get

$$A(k^S) = \mathrm{Hom}(A^S, A) \times A(k) \cong \mathbb{Z}^S.$$

Let κ be any set, and let $F(\kappa)$ be the collection of finite subsets of κ, which is directed under inclusion. If $S \subset S'$ are finite subsets, let $\pi_{S',S} : A^{S'} \to A^S$ be the projection map. Then $\{A^S\}_{S \in F(\kappa)}$ is an inverse system of abelian varieties and $\{k_S\}_{S \in F(\kappa)}$ is a directed system of fields. By Lemma 2.8,

$$A(\varinjlim_S k_S) = \varinjlim_S A(k_S) = \varinjlim_S \mathbb{Z}^S = \mathbb{Z}[\kappa],$$

the free abelian group on the set κ. Taking $A_{/k}$ to be (e.g.) the elliptic curve $E_{/\mathbb{Q}(s,t)}$ of Theorem 2.15 above, we find once again that every free commutative group is the Mordell-Weil group of an elliptic curve.

As mentioned in the introduction, following M. Rosen the construction of free commutative groups as Mordell-Weil groups of elliptic curves shows that every commutative group occurs up to isomorphism as the ideal class group of a Dedekind domain, a celebrated result which was first proved by L.E. Claborn [Cl66] and then again by C.R. Leedham-Green [LG72]. Upon seeing a preprint of [Cl09] in 2008, B. Poonen immediately suggested a variant which does not use transfinite induction and thus gives a proof of Claborn's Theorem without using (AC). Leedham-Green's argument also uses the well-ordering theorem, and it seems to me that Claborn's proof also uses (AC). Thus it seems that Poonen's argument shows – apparently for the first time – that Claborn's Theorem holds without (AC).

On the other hand, even the basic result that divisible commutative groups are injective uses Baer's Criterion and thus (AC). So (AC) is used in Lemma 2.3 and thus in Theorem 1.4c). It is also used in the proof of Baer's Theorem (Theorem 2.4) and thus in many of the examples and auxiliary results of this paper.

4.2. **Acquiring a Torsion-Free Summand in the Mordell-Weil Group.**

Theorem 4.1. *Let $A_{/k}$ be an abelian variety with $\mathrm{End}_k A = \mathbb{Z}$. Let G be a torsionfree commutative group. Then there is a field extension l/k with*

$$A(l) \cong G \times A(k).$$

Proof. Let $\{H_i\}_{i \in I}$ be the set of all finitely generated subgroups of G, directed under inclusion. Then each H_i is free abelian of finite rank, hence (non-canonically) isomorphic to $\mathbb{Z}^{S_i}$ for some finite set S_i. The functor $\mathrm{Hom}(H_i, A)$ which takes a k-algebra R to $\mathrm{Hom}(H_i, A_{/R}(R))$ is representable by an abelian variety A_i:[2] indeed, choosing an isomorphism $H_i \cong \mathbb{Z}^n$ yields an isomorphism of functors to

$$\mathrm{Hom}(\mathbb{Z}^n, A_{/R}(R)) = A_{/R}(R)^n = A^n_{/R}(R),$$

[2]This construction goes back (at least) to Serre [Se67], and it is further developed in the 2000 Strasbourg thesis of Christoph Cornut. It sometimes goes by the name "$\mathfrak{a}$-transform".

so $\mathrm{Hom}(H_i, A)$ is non-canonically isomorphic to A^n. The Yoneda Lemma gives a canonical isomorphism $\mathrm{Hom}(\mathrm{Hom}(H_i, A), A) = H_i$. If $H_i \subset H_{i'}$ there are canonical maps $A_{i'} \to A_i$. Therefore

$$A(\lim_i k(A_i)) = \lim_i A(k(A_i)) = \lim_i \mathrm{Hom}(\mathrm{Hom}(H_i, A), A)) \times A(k)$$
$$= \lim_i H_i \times A(k) = G \times A(k).$$

□

4.3. **Torsionfree Groups as Mordell-Weil Groups.**

Lemma 4.2. *Let k be a field, let l/k be a purely transcendental field extension, and let $A_{/k}$ be an abelian variety. Then $A(l) = A(k)$.*

Proof. Step 1: Suppose $l = k(t_1, \ldots, t_n)$. Since every rational map from a smooth variety to an abelian variety is a morphism [BG, Cor. 8.2.22] and every morphism from a rational variety to an abelian variety is constant [BG, 8.2.18], we have

$$A(l) = A(k(\mathbb{P}^n)) = \mathrm{Map}_p(\mathbb{P}^n, A) = A(k).$$

Step 2: Every purely transcendental field extension is a direct limit of finitely generated purely transcendental subextensions, so the general case follows from Step 1 and the Continuity Lemma. □

Theorem 4.3. *Let $g \in \mathbb{Z}^+$, let k be a field, and let G be a torsionfree commutative group. Then there is a field extension K/k and a g-dimensional abelian variety $A_{/K}$ with $A(K) \cong G$.*

Proof. Step 1: Suppose first that $g = 1$, and put $l = k(s, t)$. By Theorem 2.15, there is an elliptic curve $E_{/l}$ with $\mathrm{End}\, E = \mathbb{Z}$ and $E(l) = 0$. By Theorem 4.1, there is an extension K/l such that $E(K) = G \times E(l) = G$.
Step 2: Suppose now that $g \geq 1$. By Step 1, there is a field extension K/k and an elliptic curve $E_{/K}$ with $E(K) \cong G$. Let $L = K(s, t)$. By Theorem 2.15, there is an elliptic curve $E'_{/L}$ with $E'(L) = 0$. Put $A = E \times (E')^{g-1}$. By Lemma 4.2,

$$A(L) = E(L) \times (E')^{g-1}(L) \cong G \times \{0\}^{g-1} \cong G.$$

□

4.4. **Torsion-Split Groups as Mordell-Weil Groups.**

Theorem 4.4. *Let $g \in \mathbb{Z}^+$, let T be a $T(g)$-constrained torsion group, and let k be a field of characteristic zero. Then there is a field extension K/k and a g-dimensional abelian variety $A_{/k}$ with $A(K) \cong T$.*

Proof. It is no loss of generality to assume that $\mathbb{C} \subset k$. By Lemma 2.1, there are $T(1)$-constrained torsion groups $T_1, \ldots, T_g$ such that $T \cong \bigoplus_{i=1}^g T_i$.
Step 1: Suppose $g = 1$. By the results of § 2.4, there is a field extension K_1/k and an elliptic curve $E_{/K_1}$ with $E(K_1) \cong T_1$. More precisely, if T is small – i.e., $T \cong T_{1,1}$, $T_{1,2}$, $T_{1,3}$, or $T_{2,2}$ – we apply one of Theorems 2.15, 2.18, 2.19, or 2.20; otherwise we apply Theorem 2.12. For later use we note that in all cases $(E_1)_{/K_1}$ has transcendental j-invariant.
Step 2: Suppose inductively that we have constructed $k \subset K_1 \subset \ldots \subset K_{g-1}$ and elliptic curves E_i/K_i for all $1 \leq i \leq g-1$ each with transcendental j-invariant and such that if $A_{g-1} = E_1 \times \ldots \times E_{g-1}$, then $A_{g-1}(K_{g-1}) \cong \bigoplus_{i=1}^{g-1} T_i$. Let

$K_g = (K_{g-1})_{/T_g}$, and let $(E_g)_{/K_g}$ be the elliptic curve constructed in Step 1, so that $E_g(K_g) \cong T_g$. Put $A_g = (A_{g-1})_{/K_g} \times E_g$, so $A_g(K_g) \cong A_{g-1}(K_g) \times T_g$, and it suffices to show that $A_{g-1}(K_g) = A_{g-1}(K_{g-1})$. The Continuity Lemma reduces us to the case in which T is finite, in which case K_g/K_{g-1} is the function field of a variety $V_{/K_{g-1}}$, where V is either $\mathbb{P}^2$ (if T is small) or a modular curve $X_m(n)$ (otherwise). Thus

$$A_{g-1}(K_g) = A_{g-1}(K_{g-1}(V)) = \mathrm{Map}_{K_{g-1}}(V, A_{g-1}) = \mathrm{Map}_{K_{g-1}}(\mathrm{Alb}\, V, A_{g-1}),$$

where $\mathrm{Alb}\, V$ is the Albanese abelian variety of V.[3] When $V \cong \mathbb{P}^2$, $\mathrm{Alb}\, V = \{0\}$, so every morphism from $\mathrm{Alb}\, V$ to A_{g-1} is constant, giving the result in this case. When $V \cong X_m(n)$, every isogeny factor of $X_m(n)$ can be defined over $\overline{\mathbb{Q}}$, but no isogeny factor of A_{g-1} can be defined over $\overline{\mathbb{Q}}$, so again every morphism from $\mathrm{Alb}\, V$ to A_{g-1} is constant, completing the proof. □

Remark 4.5. *In the proof of Theorem 4.4, the hypothesis* $\mathrm{char}\, k = 0$ *was used in two places: first in order to apply Theorem 2.12 on elliptic curves with Mordell-Weil group any* $T(1)$*-constrained torsion group. But it was also used in the distinction between isogeny factors with algebraic and transcendental moduli: if* $A_{/k}$ *is an abelian variety,* K/k *is an extension field, then every* separable *isogeny of* A/K *can already be defined over* $\overline{k}$*. But this does not hold for inseparable isogenies: it is possible to gain them in passing from one algebraically closed field to another.*

Theorem 4.6. *Let* $g \in \mathbb{Z}^+$*, let* G *be a* $T(g)$*-constrained torsion-split commutative group, and let* k *be a field of characteristic* 0*. There is a field extension* L/k *and an abelian variety* $A_{/L}$ *with* $A(L) \cong G$*.*

Proof. Let $T = G[\mathrm{tors}]$, so that $G = T \times G'$ for a torsionfree commutative group G'. By Theorem 4.4 there is a field extension K/k and a g-dimensional abelian variety $A_{/K}$ with $A(K) \cong T$. By Theorem 4.2, there is a field extension L/K such that $A(L) \cong G' \times A(K) \cong G$. □

4.5. **The Mordell-Weil Groups of Elliptic Curves.**

In this section we will establish Theorem 1.4c), completing the proof of the main result of this note.

Let G be a $T(1)$-constrained commutative group. If $G[\mathrm{tors}]$ is finite, then G is torsion-split and we know that G is the Mordell-Weil group of an elliptic curve over a field of characteristic 0, so we may assume that $G[\mathrm{tors}]$ is infinite: in particular we may assume $T_{m,n} \subset G[\mathrm{tors}]$ for some non-small pair (m, n). By Lemma 2.3 there is a $\mathbb{Q}$-vector space V and an injection

$$\varphi : G \to (\mathbb{Q}/\mathbb{Z})^2 \oplus (G \otimes \mathbb{Q}).$$

Let B be a $\mathbb{Q}$-basis of $G \otimes \mathbb{Q}$; scaling each basis element by a suitable positive integer we get a $\mathbb{Q}$-basis B' of $G \otimes \mathbb{Q}$ which is a subset of G. Let $F = \langle B' \rangle$. Thus F is a free $\mathbb{Z}$-module and

$$F \subset G \hookrightarrow (\mathbb{Q}/\mathbb{Z})^2 \oplus (G \otimes \mathbb{Q}).$$

Further, for every element $P \in (\mathbb{Q}/\mathbb{Z})^2 \oplus (G \otimes \mathbb{Q})$, there is $n \in \mathbb{Z}^+$ such that $nP \in F$.

[3]We note that both $\mathbb{P}^2$ and $X_m(n)$ have rational points over K_{g-1}.

Let $k = \mathbb{C}(X_m(n))$ and let $E = E_m(n)$ be the universal elliptic curve, so $E(k) = T_{m,n} \subset G$. For each finitely generated subgroup $H \subset F$, let $(\operatorname{Hom}(H,E))_{/k}$ be the abelian variety described in the proof of Theorem 4.1 and let k_H be its function field. Let $l = \varinjlim k_H$. Then, as shown in the proof of Theorem 4.1, we have

$$E(l) \cong T_{m,n} \oplus F.$$

Since $E(\bar{l})$ is a divisible group with torsion subgroup $(\mathbb{Q}/\mathbb{Z})^2$, the subgroup of $E(\bar{k})$ obtained by adjoining all points P such that $nP \in E(l)$ for some $n \in \mathbb{Z}^+$ is isomorphic to $(\mathbb{Q}/\mathbb{Z})^2 \oplus (G \otimes \mathbb{Q})$. Thus we have

$$T_{m,n} \oplus F = E(l) \subset G \subset (\mathbb{Q}/\mathbb{Z})^2 \oplus (G \otimes \mathbb{Q}) \subset E(\bar{l}).$$

Let $l(G)$ be the subextension of $\bar{l}/l$ obtained by adjoining the fields $l(P)$ for all $P \in G$. Tautologically we have

$$G \subset E(l(G)).$$

We CLAIM that also $E(l(G)) \subset G$, which will complete the proof of the theorem.
PROOF OF CLAIM By the Continuity Lemma, if $P \in E(l(G))$ there are $P_1, \ldots, P_r \in G$, a finitely generated subgroup H of F and $n \in \mathbb{Z}^+$ such that

$$P \in E(k_H(\langle P_1, \ldots, P_r \rangle))$$

and

$$nP_1, \ldots, nP_r \in E(k_H).$$

To establish the claim it is enough to show

$$E(k_H(\langle P_1, \ldots, P_r \rangle)) = \langle E(k_H), P_1, \ldots, P_r \rangle.$$

Without changing the subgroup $\langle P_1, \ldots, P_r \rangle$, we may assume that $P_1, \ldots, P_a$ have finite order and $P_{a+1}, \ldots, P_r \in \frac{1}{n}H$. (We may further assume that $a \leq 2$, but there is no particular advantage in doing so.)
Torsion points: Let P be a torsion point. Then it follows from (3) and Theorem 2.12 that if $T_{m',n'} = \langle T_{m,n}, P_1, \ldots, P_a \rangle$ then $E(k(P)) = T_{m',n'}$. Applying Theorem 4.1 over $\mathbb{C}(X_{m'}(n'))$, we get the claim in this case.
Having treated the torsion case, to ease notation we may as well assume $a = 0$ and thus $P_1, \ldots, P_r \in \frac{1}{n}H$. In this case, let $\tilde{H}$ be the subgroup of $\frac{1}{n}H$ generated by H and $P_1, \ldots, P_r$. There is $\Psi \in M_s(\mathbb{Z}) \cong \operatorname{End} \mathbb{Z}^s \cong \operatorname{End} A_H$ such that $\Psi \tilde{H} = H$. There is a corresponding extension of function fields $\Psi^* : k_H \hookrightarrow k_H$; writing $\frac{1}{\Psi} k_H$ for k_H viewed as an extension of itself via Ψ^*, we have

$$E(\frac{1}{\Psi} k_H) = \langle E(k_H), P_1, \ldots, P_r \rangle.$$

This completes the proof of the claim and of Theorem 1.4c).

4.6. **The Mordell-Weil Groups of Products of Elliptic Curves.**

Let $g \in \mathbb{Z}^+$. A $T(g)$-constrained commutative group G is **of product type** if there are $T(1)$-constrained commutative groups $G_1, \ldots, G_g$ such that $G \cong \prod_{i=1}^g G_i$.

Let k be a field. We say a g-dimensional abelian variety $A_{/k}$ is **of product type** if there are elliptic curves $E_1, \ldots, E_g$ deined over k such that $A \cong \prod_{i=1}^g$. Thus when $g = 1$ every abelian variety is of product type; when $g \geq 2$, in any of several reasonable senses, "most" g-dimensional abelian varieties $A_{/k}$ are not of product type (e.g.

this holds for the universal abelian varieties over most PEL-type Shimura varieties).

The following result, a modest generalization of Theorem 1.4c), can be proven by combining the arguments used to prove that result together with those used to prove Theorem 4.3. We omit the details.

Theorem 4.7. *Let $g \in \mathbb{Z}^+$. For a commutative group G, TFAE:*
(i) There is a field K of characteristic zero and a g-dimensional abelian variety $A_{/K}$ of product type with $A(K) \cong G$.
(ii) G is $T(g)$-constrained of product type.

On the other hand, Lemma 2.2 asserts that for each $g \geq 2$ there is a $T(g)$-constrained commutative group G which is *not* of product type. It seems at least plausible that G is the Mordell-Weil group of an abelian variety in dimension g (e.g. this would follow from Siegel-modular analogues of the results of Shioda and Cox-Parry). If so, one finds that one cannot build all Mordell-Weil groups of abelian varieties using products of elliptic curves.

References

[AECI] J. Silverman, *The Arithmetic of Elliptic Curves.* Second edition. Graduate Texts in Mathematics, 106. Springer, Dordrecht, 2009.

[AECII] J. Silverman, *Advanced topics in the arithmetic of elliptic curves.* Graduate Texts in Mathematics, 151. Springer-Verlag, New York, 1994.

[Ba36] R. Baer, *The Subgroup of the Elements of Finite Order of an Abelian Group.* Annals of Mathematics, Second Series, Vol. 37, No. 4 (Oct., 1936), pp. 766–781.

[BG] E. Bombieri and W. Gubler, *Heights in Diophantine geometry.* New Mathematical Monographs, 4. Cambridge University Press, Cambridge, 2006.

[Cl66] L. Claborn, *Every abelian group is a class group.* Pacific J. Math. 18 (1966), 219–222.

[Cl09] P.L. Clark, *Elliptic Dedekind domains revisited.* Enseign. Math. (2) 55 (2009), 213-225.

[Co13] Y. Cornulier, `http://mathoverflow.net/questions/135181`.

[CP80] D.A. Cox and W.R. Parry, *Torsion in elliptic curves over $k(t)$.* Compositio Math. 41 (1980), 337-354.

[FPS07] G. Falcone, P. Plaumann and K. Strambach, *Monothetic algebraic groups.* J. Aust. Math. Soc. 82 (2007), 315-324.

[FJ74] G. Frey and M. Jarden, *Approximation theory and the rank of abelian varieties over large algebraic fields.* Proc. London Math. Soc. (3) 28 (1974), 112-128.

[Le68] M. Levin, *On the group of rational points on elliptic curves over function fields.* Amer. J. Math. 90 (1968), 456-462.

[LG72] C.R. Leedham-Green, *The class group of Dedekind domains.* Trans. Amer. Math. Soc. 163 (1972), 493-500.

[LN59] S. Lang and A. Néron, *Rational points of abelian varieties over function fields.* Amer. J. Math. 81 (1959), 95-118.

[LP11] M.J. Larsen and R. Pink, *Finite subgroups of algebraic groups.* J. Amer. Math. Soc. 24 (2011), 1105-1158.

[M-AV] J. Milne, *Abelian varieties*, course notes available at `http://wwww.jmilne.org` .

[Né52] A. Néron, *Problèmes arithmétiques et géométriques rattachés à la notion de rang d'une courbe algébrique dans un corps.* Bull. Soc. Math. France 80 (1952), 101-166.

[Ro73] M. Rosen, *S-units and S-class group in algebraic function fields.* J. Algebra 26 (1973), 98-108.

[Ro76] M. Rosen, *Elliptic curves and Dedekind domains.* Proc. Amer. Math. Soc. 57 (1976), 197–201.

[SS10] M. Schütt and T. Shioda, *Elliptic surfaces.* Algebraic geometry in East AsiaSeoul 2008, 51160, Adv. Stud. Pure Math., 60, Math. Soc. Japan, Tokyo, 2010.

[Se67] J.-P. Serre, *Complex Multiplication. In: Algebraic Number Theory.* (J.W.S. Cassels,A. Fröhlich, eds), Academic Press (1967), 292-296.

[Sh72] T. Shioda, *On elliptic modular surfaces*. J. Math. Soc. Japan 24 (1972), 20-59.
[Sh73] T. Shioda, *On rational points of the generic elliptic curve with level N structure over the field of modular functions of level N*. J. Math. Soc. Japan 25 (1973), 144-157.
.

Department of Mathematics, Boyd Graduate Studies Research Center, University of Georgia, Athens, GA 30602-7403, USA
E-mail address: pete@math.uga.edu