THE INSTRUCTOR'S GUIDE TO REAL INDUCTION

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Abstract. We introduce Real Induction, a proof technique analogous to Mathematical Induction but applicable to statements indexed by an interval on the real line. More generally we give an inductive principle applicable in a Dedekind complete ordered set. We apply these principles to give streamlined, conceptual proofs of basic results in elementary real analysis and topology.

1. Real Induction

1.1. "Induction is fundamentally discrete..."

We will introduce, prove and apply the **Principle of Real Induction**, an analogue of Mathematical Induction that applies to a continuous variable. For some it is a truism that such a thing is impossible: how many times have you heard – or said! – that induction is fundamentally discrete? I held this idea myself for many years.

1.2. ...is dead wrong!

Remarkably, this idea has been refuted *repeatedly* in the literature, going back at least 93 years. The earliest instance I know of is a 1919 work of Y. R. Chao.

Theorem 1.1. *(Chao* [Ch19]*) Let $a \in \mathbb{R}$, and let $S \subseteq \mathbb{R}$. Suppose that:*
(CI1) We have that $a \in S$.
(CI2) There is $\Delta > 0$ such that $\forall x \in \mathbb{R}$, if $x \in S$ then $(x - \Delta, x + \Delta) \cap [a, \infty) \subseteq S$.
Then $[a, \infty) \subseteq S$.

Often for a mathematical "principle," implicit use predates explicit formulation, sometimes quite a long time. The first explicit use of Mathematical Induction was in Pascal's 1665 *Traité du triangle arithmétique*, but most agree that Euclid's celebrated Proposition IX.20 – "There are infinitely many primes" – of circa 300 BCE contains the crucial implicit use of an inductive principle.[1] Later we will encounter an important implicit use of Real Induction that predates Chao's work.

1.3. Real Induction.

Consider "conventional" Mathematical Induction. To use it, one thinks in terms of *predicates* – i.e., statements $P(n)$ indexed by the natural numbers – but the cleanest statement is in terms of *subsets* of $\mathbb{N}$. The same goes for Real Induction.

Let $a < b$ be real numbers. We define a subset $S \subseteq [a, b]$ to be **inductive** if:

[1] Strictly speaking, Euclid assumes there are three primes and produces a fourth. Evidently some more general principle is intended.

(RI1) We have $a \in S$.
(RI2) If $a \leq x < b$, then $x \in S \implies [x, y] \subseteq S$ for some $y > x$.
(RI3) If $a < x \leq b$ and $[a, x) \subseteq S$, then $x \in S$.

Theorem 1.2. *(Principle of Real Induction)*
For $S \subseteq [a, b]$, the following are equivalent:
(i) S is inductive.
(ii) $S = [a, b]$.

Proof. (i) $\implies$ (ii): let $S \subseteq [a, b]$ be inductive. Seeking a contradiction, suppose $S' = [a, b] \setminus S$ is nonempty, so $\inf S'$ exists and is finite.
Case 1: $\inf S' = a$. Then by (RI1), $a \in S$, so by (RI2), there exists $y > a$ such that $[a, y] \subseteq S$, and thus y is a greater lower bound for S' then $a = \inf S'$: contradiction.
Case 2: $a < \inf S' \in S$. If $\inf S' = b$, then $S = [a, b]$. Otherwise, by (RI2) there exists $y > \inf S'$ such that $[\inf S', y'] \subseteq S$, contradicting the definition of $\inf S'$.
Case 3: $a < \inf S' \in S'$. Then $[a, \inf S') \subseteq S$, so by (RI3) $\inf S' \in S$: contradiction!
(ii) $\implies$ (i) is immediate. □

Theorem 1.2 was first published by D. Hathaway [Ha11]. I came up with the concept independently, early on September 10, 2010, as a variation on the *Induction over the Continuum* of Kalantari [Ka07, §3]. Later that morning I saw a question on `math.stackexchange.com` asking whether induction on real numbers was possible, and I answered it [Cl10]. I was scheduled to give a seminar in my department that afternoon on a different topic, but instead I spoke on Real Induction.

I want to stress that the enunciation of an inductive principle for subintervals of $\mathbb{R}$ and its use as a unifying principle for proving results in basic real analysis is far from new. In addition to the works [Ch19], [Ka07], [Ha11] mentioned above, each of the following papers introduces some form of "continuous induction," often without reference to past precedent: [Kh23], [Pe26], [Kh49], [Du57], [Fo57], [MR68], [Sh72], [Be82], [Le82], [Sa84], [Do03], [Ka07], [Ha11].

We continue to pursue this topic for several reasons: first, to increase the awareness of these prior results, which remain little known; second, to explore applications thoroughly enough to allow the interested instructor to make use of them in real analysis and topology; and third, to present and apply our (new) *Principle of Ordered Induction*, a simultaneous generalization of Real Induction and Transfinite Induction (hence also Mathematical Induction). This perspective has been gaining proponents since 2010, so now may be the time to publish an updated version.

1.4. **Applications in analysis.**

Let us see Real Induction in action.

Theorem 1.3. *(Intermediate Value Theorem)*
Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let L be any number in between $f(a)$ and $f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = L$.

Proof. It is easy to reduce the theorem to the following special case: if $f : [a, b] \to \mathbb{R} \setminus \{0\}$ is continuous and $f(a) > 0$, then $f(b) > 0$. Put

$$S \coloneqq \{x \in [a, b] \mid f(x) > 0\},$$

so $f(b) > 0$ iff $b \in S$. We will use Real Induction to show that $S = [a, b]$. Thus $f(b) > 0$, completing the proof.

(RI1) Since $f(a) > 0$, we have $a \in S$.
(RI2) Let $x \in S$, $x < b$, so $f(x) > 0$. Since f is continuous at x, there exists $\delta > 0$ such that f is positive on $[x, x + \delta]$, and thus $[x, x + \delta] \subseteq S$.
(RI3) Let $x \in (a, b]$ be such that $[a, x) \subseteq S$, i.e., f is positive on $[a, x)$. We claim that $f(x) > 0$. Indeed, since $f(x) \neq 0$, the only other possibility is $f(x) < 0$, but if so, then by continuity there would exist $\delta > 0$ such that f is negative on $[x - \delta, x]$, i.e., f is both positive and negative at each point of $[x - \delta, x]$: contradiction! □

Theorem 1.4. *A continuous function $f : [a, b] \to \mathbb{R}$ is bounded.*

Proof. Put $S := \{x \in [a, b] \mid f : [a, x] \to \mathbb{R} \text{ is bounded}\}$. We will use Real Induction to show that $S = [a, b]$.
(RI1): Evidently $a \in S$.
(RI2): Suppose $x \in S$, so that f is bounded on $[a, x]$. But then f is continuous at x, so is bounded near x: for instance, there exists $\delta > 0$ such that for all $y \in [x - \delta, x + \delta]$, $|f(y)| \leq |f(x)| + 1$. So f is bounded on $[a, x]$ and also on $[x, x + \delta]$ and thus on $[a, x + \delta]$.
(RI3): Suppose $x \in (a, b]$ and $[a, x) \subseteq S$. Since f is continuous at x, there exists $0 < \delta < x - a$ such that f is bounded on $[x - \delta, x]$. Since $a < x - \delta < x$, f is bounded on $[a, x - \delta]$, so f is bounded on $[a, x]$. □

Remark 1.5. *When using Real Induction, one must beware of the following pitfall. It is often the case that the predicate $P(I)$ is naturally indexed by subintervals I of $[a, b]$. In the proof Theorem 1.4, $P(I)$ is: "f is bounded on I." In the proof of Theorem 1.3, we could take $P(I)$ to be: "f is positive at all points of I."*

It can be tempting construe (RI3) as: for all $a < x \leq b$, assume $P([a, x))$ holds and prove $P([a, x])$. But this is not correct: we must assume $P([a, y])$ holds for all $a \leq y < x$ and prove $P([a, x])$. Sometimes the distinction is immaterial: a function positive on $[a, y]$ for all $a \leq y < x$ is positive on $[a, x)$. But often it matters: a function bounded on $[a, y]$ for all $a \leq y < x$ need not be bounded on $[a, x)$.

Theorem 1.6. *(Integrability Theorem)*
Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is Darboux integrable: for all $\epsilon > 0$, there is a partition $\mathcal{P}$ of $[a, b]$ such that the difference between the associated upper sum $U(f, \mathcal{P})$ and the associated lower sum $L(f, \mathcal{P})$ is less than ϵ.

Proof. Fix $\epsilon > 0$, and put

$$S(\epsilon) := \{x \in [a, b] \mid \text{there is a partition } \mathcal{P}_x \text{ of } [a, x] \mid U(f, \mathcal{P}_x) - L(f, \mathcal{P}_x) < (x - a)\epsilon\}.$$

We will use Real Induction to $S(\epsilon) = [a, b]$. Then $b \in S(\frac{\epsilon}{b-a})$, completing the proof.
(RI1) As usual, this is clear.
(RI2) Suppose that for $x \in [a, b)$ we have $[a, x] \subseteq S(\epsilon)$, so that there is a partition $\mathcal{P}_x$ of $[a, x]$ such that $U(f, \mathcal{P}_x) - L(f, \mathcal{P}_x) < (x - a)\epsilon$. Since f is continuous at x, there is $\delta > 0$ such that $\sup(f, [x, x + \delta]) - \inf(f, [x, x + \delta]) < \epsilon$. Now let $y \in [x, x + \delta]$ and take the partition $\mathcal{P}_y := \mathcal{P}_x \cup \{y\}$ of $[a, y]$. Then

$$U(f, \mathcal{P}_y) - L(f, \mathcal{P}_y) =$$

$$(U(f, \mathcal{P}_x) + (y - x)\sup(f, [a, y])) - (L(f, \mathcal{P}_x) + (y - x)\inf(f, [a, y]))$$

$$< (x - a)(\epsilon) + (y - x)(\epsilon) = (y - a)(\epsilon).$$

(RI3) Suppose that for $x \in (a, b]$ we have $[a, x) \subseteq S(\epsilon)$. Since f is continuous at x, there is $\delta > 0$ such that $\sup(f, [x - \delta, x]) - \inf(f, [x - \delta, x]) < \epsilon$. Since

$x-\delta < x$, $x-\delta \in S(\epsilon)$, there is a partition $\mathcal{P}_{x-\delta}$ of $[a, x-\delta]$ such that $U(f, \mathcal{P}_{x-\delta}) = L(f, \mathcal{P}_{x-\delta}) = (x-\delta-a)\epsilon$. Let $\mathcal{P}_x = \mathcal{P}_{x-\delta} \cup \{x\}$. Then as above we get

$$U(f, \mathcal{P}_x) - L(f, \mathcal{P}_x) < (x-\delta-a)\epsilon + \delta\epsilon = (x-a)\epsilon. \qquad \square$$

Theorem 1.7. *(Bolzano-Weierstrass) An infinite $\mathcal{A} \subseteq [a, b]$ has a limit point.*

Proof. Let S be the set of x in $[a, b]$ such that *if* $\mathcal{A} \cap [a, x]$ is infinite, it has a limit point. It suffices to show $S = [a, b]$, which we will do by Real Induction.
(RI1) is clear.
(RI2) Suppose $x \in [a, b) \cap S$. If $\mathcal{A} \cap [a, x]$ is infinite, then it has a limit point and hence so does $\mathcal{A} \cap [a, b]$: thus $S = [a, b]$. If for some $\delta > 0$, $\mathcal{A} \cap [a, x+\delta]$ is finite, then $[x, x+\delta] \subseteq S$. Otherwise $\mathcal{A} \cap [a, x]$ is finite but $\mathcal{A} \cap [a, x+\delta]$ is infinite for all $\delta > 0$, and then x is a limit point for $\mathcal{A}$ and $S = [a, b]$ as above.
(RI3) If $[a, x) \subseteq S$, then: either $\mathcal{A} \cap [a, y]$ is infinite for some $y < x$, so $x \in S$; or $\mathcal{A} \cap [a, x]$ is finite, so $x \in S$; or $\mathcal{A} \cap [a, y]$ is finite for all $y < x$ and $\mathcal{A} \cap [a, x]$ is infinite, so x is a limit point of $\mathcal{A} \cap [a, x]$ and $x \in S$. $\square$

1.5. **Applications in topology.**

Theorem 1.8. *The interval $[a, b]$ is connected.*

Proof. Suppose $[a, b] = A \cup B$, with A and B open and closed and with $A \cap B = \emptyset$. We assume $a \in A$ and prove by Real Induction that $A = [a, b]$: (RI1) is immediate, (RI2) holds because A is open, and (RI3) holds because A is closed. We're done! $\square$

Theorem 1.9. *(Heine-Borel) The interval $[a, b]$ is compact.*

Proof. For an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $[a, b]$, let

$$S := \{x \in [a, b] \mid \mathcal{U} \cap [a, x] \text{ has a finite subcovering}\}.$$

We prove $S = [a, b]$ by Real Induction. (RI1) is clear. (RI2): If $U_1, \ldots, U_n$ covers $[a, x]$, then some U_i contains $[x, x+\delta]$ for some $\delta > 0$. (RI3): If $[a, x) \subseteq S$, then $x \in U_i$ for some $i \in I$; let $y < x$ be such that $(y, x] \in U_i$. There is a finite $J \subseteq I$ with $\bigcup_{i \in J} U_i \supset [a, y]$, so $\{U_i\}_{i \in J} \cup U_i$ covers $[a, x]$. We're done! $\square$

1.6. **Some Real Induction proofs for the reader.**

Here are more results amenable to Real Induction. The proofs are left to you.

Theorem 1.10. *(Mean Value Inequality) Let $f : [a, b] \to \mathbb{R}$ be differentiable. Suppose that there exists $M \in \mathbb{R}$ such that for all $x \in [a, b]$ we have $f'(x) \geq M$. Then for all $x < y \in \mathbb{R}$, we have $f(y) - f(x) \geq M(y - x)$.*

Theorem 1.11. *(Uniform Continuity Theorem) Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous on $[a, b]$.*

Theorem 1.12. *(Cantor Intersection Theorem) Let $\{F_n\}_{n=1}^{\infty}$ be a decreasing sequence of closed subsets of $[a, b]$. Put $F = \bigcap_n F_n$. Then* either *$F \neq \varnothing$ or there exists $n \in \mathbb{Z}^+$ such that $F_n = \varnothing$.*

Theorem 1.13. *Each open covering $\{U_i\}_{i \in I}$ of $[a, b]$ has a Lebesgue number: there is $\delta > 0$ such that if $A \subseteq [a, b]$ has diameter at most δ, then $A \subseteq U_i$ for some $i \in I$.*

Theorem 1.14. *(Dini's Lemma) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous real-valued functions on the interval $[a, b]$ that is* pointwise decreasing*: for all $x \in [a, b]$ and all $n \in \mathbb{Z}^+$, $f_{n+1}(x) \leq f_n(x)$. If $f : [a, b] \to \mathbb{R}$ is continuous and $f_n \to f$ pointwise, then $f_n \to f$ uniformly.*

Theorem 1.15. *(Arzelà-Ascoli)*
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $[a, b]$ such that:
(i) There is $M \in \mathbb{R}$ such that for all $n \in \mathbb{Z}^+$ and all $x \in [a, b]$, $|f_n(x)| \leq M$, and
(ii) For all $x \in [a, b]$ and all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$, then for all $n \in \mathbb{Z}^+$, $|f_n(x) - f_n(y)| < \epsilon$.
Then there is a subsequence $\{f_{n_k}\}$ that is uniformly convergent on $[a, b]$.

Theorem 1.10 is a consequence of the Mean Value Theorem. It is one of several results that have been advocated (by some; we will not weigh in on this issue) as being pedagogically preferable to the Mean Value Theorem. The other variants can also be proven by Real Induction. But what about the Mean Value Theorem itself?

Problem 1.16. *Either prove the Mean Value Theorem directly by Real Induction or explain why it is not possible to do so.*

Problem 1.17. *Find other theorems that can be proved via Real Induction.*

1.7. **Comments and complements.**

Our proof of Theorem 1.3 is not so different from the usual proof using suprema. That proof is probably even cleaner: it suffices to assume that $f(a) > 0$ and $f(b) < 0$ and show that there is $c \in (a, b)$ with $f(c) = 0$. For this, put

$$c := \sup \{x \in [a, b] \mid f(x) \leq 0\}.$$

Then – as follows easily from the definition of continuity – we must have $f(c) = 0$.

But this proof has within it a germ of the idea for Real Induction. In fact, one can motivate Real Induction in a classroom setting by asking for a proof of Theorem 1.4 along the lines of the above proof of Theorem 1.3: i.e., we start by putting

$$c := \sup \{x \in [a, b] \mid f \text{ is bounded on } [a, x]\}.$$

Then it emerges naturally that we want to show that $c = b$ and that we can establish this by showing (RI2) and (RI3). (Here, as in every application I know of, (RI1) is obvious.) It is then an interesting exercise to see how to modify the standard proof of Theorem 1.3 to get the proof by Real Induction.

Real Induction has a natural role in basic real analysis because of the large number of key results that are "interval theorems" – assertions that every continuous function $f : [a, b] \to \mathbb{R}$ has a certain property: e.g. Theorems 1.3, 1.4, 1.6, 1.11. The main interval theorem we are missing is the Extreme Value Theorem, a strengthening of Theorem 1.4 asserting that every continuous $f : [a, b] \to \mathbb{R}$ assumes its maximum and minimum values. This is however deduced from Theorem 1.4 by an easy argument using suprema: by Theorem 1.4, $M := \sup(f, [a, b])$ is finite. If M were not attained on the interval $[a, b]$, then the function $g : x \mapsto \frac{1}{M - f(x)}$ would be continuous and unbounded on $[a, b]$, contradicting Theorem 1.4. This seems like a good time to mention that we do not advocate using Real Induction *in place of* the least upper bound axiom but rather (when helpful) as a proof technique.

Theorem 1.6 is usually proved using the fact that any continuous function f :

$[a, b] \to \mathbb{R}$ is uniformly continuous. In [Sp, pp. 292-293], Spivak gives a different proof by establishing equality of the upper and lower integrals by differentiation. This method goes back at least to M.J. Norris [No52]. The proof given above seems quite different from both of these.

Standard proofs of Theorem 1.7 use monotone subsequences, dissection / nested intervals, or the compactness of $[a, b]$. Our proof appears to be new.

Perhaps the strongest justification for using Real Induction in the classroom is the proofs it affords for Theorems 1.8 and 1.9. Not only are these proofs short and simple, but an initiate of the Principle of Real Induction will find them easily.

On the one hand this suggests that the concepts of connectedness and compactness may be inherently inductive in some sense. There seems to be something to this: see e.g. "induction on connectedness" and "induction on compactness" in [VINK]. We will give a different kind of generalization in the next section when we explore connectedness and compactness in order topologies.

On the other hand, it raises the question of why this proof technique – which, recall, has appeared in many variations in more than a dozen prior works – is not more popular. The situation becomes even more curious once one learns (as I did only several years after this article was first written) that our proof of Theorem 1.9 is essentially the same as one given in 1904 by Henri Lebesgue! In [Le04], Lebesgue proves the result as follows: he says that $x \in [a, b]$ is "reached" if there is a finite subcovering of the interval $[a, x]$, and proceeds by considering the supremum of the set of all points x that are reached. This is the last of the early proofs of Heine-Borel surveyed in [AEP13]: they also discuss proofs by Borel, Cousin, Schoenflies and Young. The authors are quite enthusiastic about Lebesgue's proof, writing "This is the one! The proof is thoroughly modern and simple to follow. In comparison, all previous arguments are cumbersome and overly complicated."

Of course Theorem 1.3 and the Extreme Value Theorem are quick consequences of Theorems 1.8 and 1.9, via the following truly basic result, whose proof we omit.

Proposition 1.18. *Let $f : X \to Y$ be a continuous surjection of topological spaces.*
a) If X is connected, then so is Y.
b) If X is compact, then so is Y.

2. Ordered Induction

In this section we broaden our perspective to induction in ordered sets. One motivation is "dialectic": some people who speak against continuous induction do so because they view the essence of induction as lying in the well-ordering of the natural numbers $\mathbb{N}$, from which perspective the ultimate generalization of "ordinary" Mathematical Induction is Transfinite Induction.

In fact, we will give a result (Theorem 2.2) that simultaneously generalizes both Real Induction and Transfinite Induction.

2.1. Ordered sets.

An **ordered set**[2] is a set X endowed with a binary relation $\leq$ that satisfies:
• reflexivity: for all $x \in X$, $x \leq x$;

[2] Other common terminologies include: **linearly ordered set** and **totally ordered set**.

- anti-symmetry: for all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$; and
- transitivity: for all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.
- totality: for all $x, y \in X$, at least one of $x \leq y$ and $y \leq x$ holds.

Our distinguished example is the interval $[a, b] \subseteq \mathbb{R}$.

A **top element** (resp. a **bottom element**) of an ordered set $(X, \leq)$ is an element $\mathbb{T}$ (resp. $\mathbb{B}$) such that $x \leq \mathbb{T}$ (resp. $\mathbb{B} \leq x$) for all $x \in X$. Clearly X can have at most one top (resp. bottom) element. If X lacks a top element, then we can simply adjoin such an element, denoted $\mathbb{T}$ – i.e., $\mathbb{T}$ is not an element of X and decreed to satisfy $x \leq \mathbb{T}$ for all $x \in X$. Similarly, if X lacks a bottom element we can adjoin one, denoted $\mathbb{B}$. We denote by $\tilde{X}$ the set X extended by a top element if it lacks one and extended by a bottom element if it lacks one. Applying this construction to the real numbers, we get the extended real numbers $[-\infty, \infty]$, in which *every* subset has a supremum and an infimum.

Our next order of business is to define intervals in an ordered set. The empty set is decreed to be an open interval in X. A **closed, bounded interval** in X is either the empty set or a subset of the form $[a, b] := \{x \in X \mid a \leq x \leq b\}$ for elements $a \leq b$ in X. A nonempty subset $I \subseteq X$ is an interval if $\inf I$ and $\sup I$ both exist in $\tilde{X}$ and $I \cup \{\inf I, \sup I\}$ is a closed, bounded interval in $\tilde{X}$. A nonempty interval I is **open** if it does not contain either $\inf S$ or $\sup S$.

The open intervals form a base for a topology on X, called the **order topology**. The bounded open intervals $(a, b) := \{x \in X \mid a < x < b\}$ are another base for the same topology. If $X = \mathbb{R}$ this is the usual Euclidean topology.

In some ways, order topologies are closer relatives to $\mathbb{R}$ than an arbitrary metric space, while in other ways they are more exotic: e.g. they need not be metrizable or even first countable. Order topologies are always Hausdorff (indeed normal), so a compact subset must be closed. Moreover a compact subset C must be **bounded** – i.e., contained in a closed, bounded interval: for each $x \in C$, let I_x be a bounded open interval containing x. Then there is a finite subset $Y \subseteq X$ such that $C \subseteq \bigcup_{x \in Y} I_x$ and $C \subseteq [\min_{x \in Y} \inf I_x, \max_{x \in Y} \sup I_x]$.

An ordered set is **Dedekind complete** if every nonempty subset that is bounded above has a supremum. This holds iff every nonempty subset that is bounded below has a greatest lower bound. An ordered set is **complete** if *every* subset has a supremum. One sees easily that X is Dedekind complete iff $\tilde{X}$ is complete and thus X is complete iff it is Dedekind complete and has top and bottom elements.

Example 2.1. *a) The most important example of a Dedekind complete ordered set is the real numbers $\mathbb{R}$. Indeed $\mathbb{R}$ is the unique (up to unique isomorphism) Dedekind complete ordered field (see e.g.* [Cl-FT, §515.8]*).*
b) A closed bounded interval $[a, b] \subseteq \mathbb{R}$ is complete. Every other kind of interval in $\mathbb{R}$ is Dedekind complete but not complete. (We will shortly give a more general result along these lines: Proposition 2.3.)
c) A ***well-ordered set*** *is an ordered set in which each nonempty subset has a bottom element. Well-orderedness implies Dedekind completeness but is much stronger.*

All of the above is standard, but the following definition is new. A subset S of an ordered set $(X, \leq)$ is **inductive** if it satisfies all of the following:

(IS1) There exists $a \in X$ such that $(-\infty, a] := \{z \in X \mid z \leq a\} \subseteq S$.

(IS2) For all $x \in S$, either $x = \mathbb{T}$ or there is $y > x$ such that $[x, y] \subseteq S$.
(IS3) For all $x \in X$, if $(-\infty, x) \coloneqq \{z \in X \mid z < x\} \subseteq S$, then $x \in S$.

Theorem 2.2. *(Principle of Ordered Induction)*
For a nonempty ordered set X, the following are equivalent:
(i) X is Dedekind complete.
(ii) The only inductive subset of X is X itself.

Proof. (i) $\implies$ (ii): Let $S \subseteq X$ be inductive. Seeking a contradiction, we suppose $S' = X \setminus S$ is nonempty. Fix $a \in X$ satisfying (IS1). Then a is a lower bound for S', so by hypothesis S' has an infimum, say y. Any element less than y is strictly less than every element of S', so $(-\infty, y) \subset S$. By (IS3), $y \in S$. If $y = \mathbb{T}$, then $S' = \{\mathbb{T}\}$ or $S' = \varnothing$: both are contradictions. So $y < \mathbb{T}$, and then by (IS2) there exists $z > y$ such that $[y, z] \subseteq S$ and thus $(-\infty, z] \subseteq S$. Thus z is a lower bound for S' which is strictly larger than y, contradiction.
(ii) $\implies$ (i): Let $T \subseteq X$ be nonempty and bounded below by a. Let S be the set of lower bounds for T. Then $(-\infty, a] \subseteq S$, so S satisfies (IS1).
Case 1: Suppose S does not satisfy (IS2): there is $x \in S$ with no $y \in X$ such that $[x, y] \subseteq S$. Since S is downward closed, x is the top element of S and $x = \inf(T)$.
Case 2: Suppose S does not satisfy (IS3): there is $x \in X$ such that $(-\infty, x) \in S$ but $x \notin S$, i.e., there exists $t \in T$ such that $t < x$. Then also $t \in S$, so t is the least element of T: in particular $t = \inf T$.
Case 3: If S satisfies (IS2) and (IS3), then $S = X$, so $T = \{\mathbb{T}\}$ and $\inf T = \mathbb{T}$. $\square$

Taking $X = [a, b] \subseteq \mathbb{R}$, we recover Real Induction. Taking X to be a well-ordered set, we recover the Principle of Transfinite Induction and, in particular, the Principle of Mathematical Induction.

2.2. **Completeness of subsets.**

Let X be a Dedekind complete ordered set, and let $\varnothing \neq Y \subseteq X$. Then Y is an ordered set in its own right – when is it Dedekind complete? The analogy with metric spaces (and "Cauchy completeness": i.e., every Cauchy sequence converges) suggests that this holds iff Y is closed, but a little thought shows that this cannot be quite right: e.g. for any nonempty open interval I in $\mathbb{R}$ there is an order-preserving bijection to $\mathbb{R}$, so I is also Dedekind complete, even though it is not closed. The precise answer is as follows: as usual, let $\tilde{X}$ be X augmented with a bottom element and/or a top element iff X lacks them. Then $\inf Y$ and $\sup Y$ exist in $\tilde{X}$, so we may view $\tilde{Y} = Y \cup \{\inf Y, \sup Y\}$ as a subset of the complete ordered set $\tilde{X}$.

Proposition 2.3. *Let Y be a subset of a Dedekind complete ordered set $(X, \leq)$ Then Y is Dedekind complete iff $\tilde{Y}$ is closed in $\tilde{X}$. It follows that every interval in a Dedekind complete ordered set is Dedekind complete.*

Proof. Suppose that Y is Dedekind complete, and let $x \in \tilde{X}$ be a point such that every neighborhood U of x in $\tilde{X}$ meets Y. Seeking a contradiction, we suppose that $x \notin \tilde{Y}$. Then at least one of the following holds: (i) for all elements $x' < x$ of X, we have $(x', x) \cap Y \neq \varnothing$ – in which case $x = \sup\{y \in Y \mid y \leq x\} \in \tilde{Y}$; or (ii) for all elements $x < x''$ of X we have $(x, x'') \cap Y \neq \varnothing$ – in which case $x = \inf\{y \in Y \mid x \leq y\} \in \tilde{Y}$. This contradiction shows that $\tilde{Y}$ is closed in $\tilde{X}$.

Conversely, suppose $\tilde{Y}$ is closed in $\tilde{X}$, and let $\varnothing \neq A \subseteq Y$ be bounded above by

$M \in Y$. Then $\sup A$ exists in X. Every open interval centered at $\sup A$ contains points of Y, so $\sup A \in \tilde{Y} \cap (-\infty, M] \subseteq Y$ and Y is Dedekind complete.

Finally, the empty interval is Dedekind complete, and for every nonempty interval $I \subseteq X$, the set $\tilde{I}$ is a closed interval in $\tilde{X}$, so I is Dedekind complete. □

Ordered Induction is a characteristic property of $\mathbb{R}$ among ordered fields:

Corollary 2.4. *In an ordered field $(F, +, \cdot, \leq)$, the following are equivalent:*
(i) The ordered set F is Dedekind complete.
(ii) For all $a < b$ in F, the interval $[a, b]$ is complete.
(ii′) For all $a < b$ in F, the only inductive subset of $[a, b]$ is $[a, b]$.
(iii) There exists $a < b$ in F such that the interval $[a, b]$ is complete.
(iii′) There exists $a < b$ in F such that the only inductive subset of $[a, b]$ is $[a, b]$.

Proof. (i) $\implies$ (ii) by Proposition 2.3.
(ii) $\iff$ (ii′) and (iii) $\iff$ (iii′) by Theorem 2.2.
(ii) $\implies$ (iii) is clear.
(iii) $\implies$ (i): Suppose $[a, b]$ is complete. Let $S \subset F$ be any subset that is nonempty and bounded above, say by B. Let $A \in S$. Then S has a supremum in F iff

$$T \coloneqq \{x \in S \mid A \leq x\} \subseteq [A, B]$$

does. The following is an order-preserving bijection:

$$\ell : [a, b] \to [A, B], \ x \mapsto \frac{B - A}{b - a}(x - a) + A.$$

Thus $[A, B]$ is also Dedekind complete, so T has a supremum in F. □

Corollary 2.4 fits into Propp's program of **real analysis in reverse** [Pr13]: given a result of real analysis that can be enunciated in any ordered field, one seeks to determine whether it implies Dedekind completeness – or more generally, determine the class of ordered fields in which it holds. In the remainder of this article we explore such inverse problems with "ordered field" replaced by "ordered set."

Problem 2.5. *Characterize the inductive subsets of $[a, b] \cap \mathbb{Q}$.*

2.3. **Completeness and connectedness.**

A subset Y of an ordered set $(X, \leq)$ is **convex** if for all $x, z, y \in X$ with $x < z < y$, if $x, y \in Y$ then also $z \in Y$. In any ordered set $(X, \leq)$, both intervals and connected sets are convex. The former is clear; as for the latter, if $Y \subseteq X$ is not convex, there are $x < z < y \in X$ with $x, y \in Y$ and $z \notin Y$, and then $Y_1 \coloneqq (-\infty, z) \cap Y$, $Y_2 \coloneqq (z, \infty) \cap Y$ is a separation of Y. The converse implications depend on completeness. Indeed:

Proposition 2.6. *In an ordered set $(X, \leq)$, the following are equivalent:*
(i) X is Dedekind complete.
(ii) Every convex subset $Y \subseteq X$ is an interval.

Proof. (i) $\implies$ (ii): We may assume that Y is nonempty. Consider $\tilde{Y} \subseteq \tilde{X}$. We have $\tilde{Y} \subseteq [\inf Y, \sup Y]$. Conversely, if $\inf Y < z < \sup Y$ then there are $x, y \in Y$ with $x < z < y$, so $z \in Y$. Thus $\tilde{Y} = [\inf Y, \sup Y]$, so Y is an interval.

(ii) $\implies$ (i): We go by contrapositive: suppose X is not Dedekind complete, and let $Y \subseteq X$ be nonempty, bounded above and without a supremum in X. Put

$$D(Y) := \{x \in X \mid x \leq y \text{ for some } y \in Y\}.$$

Then $D(Y)$ is convex, bounded above and has no supremum, so not an interval. $\square$

The next question is when intervals are connected. For this even completeness is not sufficient: e.g. a finite ordered set with more than one element is complete but not connected: the order topology is discrete. The extra condition we need is as follows: an ordered set $(X, \leq)$ is **densely ordered** if for all $x < y$ in X there is $z \in (x, y)$. A convex subset of a densely ordered set is again densely ordered.

Theorem 2.7. *For an ordered set X, the following are equivalent:*
(i) X is densely ordered and Dedekind complete.
(ii) X is connected in the order topology.

Proof. (i) $\implies$ (ii): Step 1: We suppose $\mathbb{B} \in X$. Since X is densely ordered, a subset $S \subseteq X$ which contains $\mathbb{B}$ and is both open and closed in the order topology is inductive. Since X is Dedekind complete, by Theorem 2.2, $S = X$. This shows X is connected!
Step 2: We may assume $X \neq \varnothing$ and choose $a \in X$. By Lemma 2.3, Step 1 applies to show $[a, \infty)$ connected. A similar downward induction argument shows $(-\infty, a]$ is connected. Since $X = (-\infty, a] \cup [a, \infty)$ and $(-\infty, a] \cap [a, \infty) \neq \varnothing$, X is connected.
(ii) $\implies$ (i): If X is not densely ordered, there are $a < b$ in X with $[a, b] = \{a, b\}$, so $A = (-\infty, a]$, $B = [b, \infty)$ is a separation of X. Suppose we have $S \subseteq X$, nonempty, bounded below by a and with no infimum. Let L be the set of lower bounds for S, and put $U = \bigcup_{b \in L} (-\infty, b)$, so U is open and $U < S$. We have $a \neq \inf(S)$, so $a \in U$, and thus $U \neq \varnothing$. If $x \notin U$, then $x \geq L$ and, indeed, since L has no top element, $x > L$, so there exists $s \in S$ such that $s < x$. Since X is densely ordered, there is y with $s < y < x$, and then the entire open set (y, ∞) lies in the complement of U. Thus U is also closed. Since X is connected, $U = X$, contradicting $U < S$. $\square$

Corollary 2.8. *Let $(X, \leq)$ be densely ordered and Dedekind complete. For a subset $Y \subseteq X$, the following are equivalent:*
(i) Y is connected in the order topology.
(ii) Y is convex.
(iiii) Y is an interval.

Proof. (i) $\implies$ (ii) was shown above for any order topology.
(ii) $\implies$ (iii) by Proposition 2.6.
(iii) $\implies$ (i): Being an interval, Y is a convex subset of a densely ordered set, so Y is densely ordered. By Proposition 2.3, Y is Dedekind complete, so by Theorem 2.7, Y is connected in the order topology. $\square$

Above we saw that an ordered field F is Dedekind complete iff there are $a < b$ in F such that the interval $[a, b]$ is complete. This has the following consequence.

Corollary 2.9. *Let $(F, +, \cdot, <)$ be an ordered field. The following are equivalent:*
(i) F is Dedekind complete.
(ii) Every closed interval $[a, b]$ of F is connected in the order topology.
(iiii) For some $a < b$ in F, the interval $[a, b]$ is connected in the order topology.

Remark 2.10. *It follows that if $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is continuous, then $f(I)$ is again an interval. If I is closed and bounded, then it is compact, so $f(I)$ is again closed and bounded. Conversely, it is a nice exercise to show that if $I, J \subseteq \mathbb{R}$ are intervals, each consisting of more than one point, and J is closed and bounded if I is, then there is a continuous function $f : I \to \mathbb{R}$ with $f(I) = J$.*

Remark 2.11. *A subtlety arises when considering the topology on a subset Y of an ordered set $(X, \leq)$. On the one hand, Y is an ordered set in its own right, so has an order topology. On the other hand, we can endow Y with the topology it inherits as a subspace of the order topology on X. In general the order topology is coarser than the subspace topology, and they need not coincide: e.g. consider*

$$Y := \{0\} \cup (1/2, 1] \subseteq \mathbb{R}.$$

Then $\{0\}$ is open in the subspace topology on Y but not in the order topology – with the order topology, Y is homeomorphic to $[\frac{1}{2}, 1]$. (Moreover there is no ordering on Y that induces the subspace topology.) Thus in the above results we were careful to specify "in the order topology."

For a convex subset $Y \subseteq X$ the two topologies coincide. We give an example in which the distinction matters: an ordered field F that is not *Dedekind complete is totally disconnected in the order topology: that is, if $Y \subseteq F$ consists of more than a single point, then Y is not connected* in the subspace topology*. However, if $F \supset \mathbb{R}$ – e.g. $F = \mathbb{R}(t)$ – then the subset $\mathbb{R}$ is connected* in the order topology*.*

2.4. **Completeness and compactness.**

Theorem 2.12. *(Frink* [Fr42]*) For an ordered set X, the following are equivalent:*
(i) X is complete.
(ii) X is compact in the order topology.

Proof. (i) $\Longrightarrow$ (ii): Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X. Let S be the set of $x \in X$ such that the covering $\mathcal{U} \cap [\mathbb{B}, x]$ of $[\mathbb{B}, x]$ admits a finite subcovering. We have $\mathbb{B} \in S$, so S satisfies (IS1). Suppose $U_1 \cap [\mathbb{B}, x], \ldots, U_n \cap [\mathbb{B}, x]$ covers $[\mathbb{B}, x]$. If there exists $y \in X$ such that $[x, y] = \{x, y\}$, then adding to the covering any element U_y containing y gives a finite covering of $[\mathbb{B}, y]$. Otherwise some U_i contains x and hence also $[x, y]$ for some $y > x$. So S satisfies (IS2). Now suppose that $x \neq \mathbb{B}$ and $[\mathbb{B}, x) \subseteq S$. Let $i_x \in I$ be such that $x \in U_{i_x}$, and let $y < x$ be such that $(y, x] \subseteq U_{i_x}$. Since $y \in S$, there is a finite $J \subseteq I$ with $\bigcup_{i \in J} U_i \supset [a, y]$, so $\{U_i\}_{i \in J} \cup U_{i_x} \supset [a, x]$. Thus $x \in S$ and S satisfies (IS3). Thus S is inductive; since X is Dedekind complete, we have $S = X$. In particular $\mathbb{T} \in S$, hence the covering has a finite subcovering.
(ii) $\Longrightarrow$ (i): For each $x \in X$ there is a bounded open interval I_x containing x. If X is compact, $\{I_x\}_{x \in X}$ has a finite subcovering, so X is bounded, i.e., has $\mathbb{B}$ and $\mathbb{T}$. Let $S \subseteq X$. Since $\inf \varnothing = \mathbb{T}$, we may assume $S \neq \varnothing$. Let L be the set of lower bounds for S. For each $(b, s) \in L \times S$, consider the closed interval $C_{b,s} := [b, s]$. For any finite subset $\{(b_1, s_1), \ldots, (b_n, s_n)\}$ of $L \times S$, we have $\bigcap_{i=1}^{N} [b_i, s_i] \supset [\max b_i, \min s_i] \neq \varnothing$. Since X is compact there is $y \in \bigcap_{L \times S} [b, s]$ and then $y = \inf S$. $\square$

Corollary 2.13. *(Generalized Heine-Borel)*
a) For an ordered set X, the following are equivalent:
(i) X is Dedekind complete.
(ii) A subset S of X is compact in the order topology iff it is closed and bounded.

b) For an ordered field F, the following are equivalent:
(i) F is Dedekind complete.
(ii) Every closed bounded interval $[a, b] \subseteq F$ is compact.
(iii) For some $a < b$ in F, the interval $[a, b]$ is compact.

Proof. a) (i) $\implies$ (ii): A compact subset of any ordered space is closed and bounded. Conversely, if X is Dedekind complete and $S \subseteq X$ is closed and bounded, then by Lemma 2.3, S is complete and then by Theorem 2.12, S is compact.
(ii) $\implies$ (i): If $S \subseteq X$ is nonempty and bounded above, let $a \in S$. Then $S' = S \cap [a, \infty)$ is bounded, so $\overline{S'}$ is compact and thus $\overline{S'}$ is complete by Theorem 2.12. The least upper bound of $\overline{S'}$ is also the least upper bound of S.
b) This follows immediately from part a) and Corollary 2.4. □

2.5. **Completeness and sequential compactness.**

Recall that a Hausdorff topological space X is **sequentially compact** if every sequence in X admits a convergent subsequence and is **limit point compact** if every infinite subset of X has a limit point. Compactness implies limit point compactness, as does sequential compactness. Compactness does not imply sequential compactness: the standard counterexample is $[0, 1]^{[0,1]}$. Neither does sequential compactness imply compactness: the least uncountable ordinal ω_1 and the long line L [Mu, §24] are counterexamples: both of these are order topologies.

The proof of Theorem 1.7 goes over verbatim to show that if $(X, \leq)$ is a complete ordered set, then it is limit point compact in the order topology. Via Theorem 2.12, this is a special case of compactness implying limit point compactness. In fact, let $(X, \leq)$ be a complete ordered set, and let $\{x_n\}$ be a sequence in X. As usual – see e.g. [NP88] – there is a monotone subsequence $\{x_{n_k}\}$, which we lose no generality by assuming is increasing and thus converges to its supremum. This shows:

Proposition 2.14. *A compact order topology is sequentially compact.*

As seen above, completeness of an ordered set is sufficient for sequential compactness, but it is not necessary, as the least uncountable ordinal ω_1 and the long line L show: both are Dedekind complete but not complete. In fact, consider $L \times \{0, 1\}$ lexicographically ordered: in plainer terms, we take a disjoint union of two copies of the long line, with every element of the first copy $L \times \{0\}$ less than every element of the second copy $L \times \{1\}$. Then $L \times \{0, 1\}$, being a union of two sequentially compact subsets, is sequentially compact. However $L \times \{0, 1\}$ is not Dedekind complete: $L \times \{0\}$ is bounded above but has no supremum.

Problem 2.15. *Characterize sequential compactness in ordered spaces.*

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