A NOTE ON RINGS OF FINITE RANK

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Abstract. The rank $\mathrm{rk}(R)$ of a ring R is the supremum of minimal cardinalities of generating sets of I as I ranges over ideals of R. Matsuda and Matson showed that every $n \in \mathbb{Z}^+$ occurs as the rank of some ring R. Motivated by the result of Cohen and Gilmer that a ring of finite rank has Krull dimension 0 or 1, we give four different constructions of rings of rank n (for all $n \in \mathbb{Z}^+$). Two constructions use one-dimensional domains. Our third construction uses Artinian rings (dimension zero), and our last construction uses polynomial rings over local Artinian rings (dimension one, irreducible, not a domain).

1. Introduction

For a module M over a ring[1] R, let $\mu(M)$ be the minimal cardinality of a set of generators of M as an R-module, and let $\mu_*(M)$ be the supremum of $\mu(N)$ as N ranges over all R-submodules of M. We say M has **finite rank** if $\mu_*(M) < \aleph_0$. This implies that M is a Noetherian R-module. We define the **rank** $\mathrm{rk}(R)$ as $\mu_*(R)$. Thus for $n \in \mathbb{N}$, $\mathrm{rk}(R) = n$ means that every ideal of R can be generated by n elements and some ideal of R cannot be generated by fewer than n elements.

1.1. Motivation and Main Results.

This note is directly motivated by the following result.

Theorem 1.1.
a) (Matsuda [Ma84]*) For all $N \in \mathbb{Z}^+$, there is a domain R_N with* $\mathrm{rank}(R_N) = N$.
b) (Matson [Ma09]*) One may take for R_N a subring of the ring of integers of* $\mathbb{Q}(2^{\frac{1}{N}})$.

Here we will explore the class of rings of finite rank with an eye to constructing further families with all possible finite ranks. We begin in §2 with the case of domains. We review the pioneering work of Cohen and use it to deduce a local-global principle for domains of finite rank: Theorem 2.6. In §3 we show that given any PID A with fraction field $F \supsetneq A$ and any field extension K/F of degree $N \in \mathbb{Z}^{\geq 2}$, there is a nonmaximal A-order in K of rank N, generalizing Theorem 1.1. We also construct, for any $2 \leq n \leq N$, a $\mathbb{Z}$-order in a degree N number field with rank n. In §4 we consider the case of rings which are not domains. We give a general discussion of Artinian rings (which always have finite rank) and show that there are Artinian rings of rank n for any $n \in \mathbb{Z}^+$. Finally we determine when a polynomial ring has finite rank, show that for any local Artinian ring $\mathfrak{r}$, the rank of $\mathfrak{r}[t]$ is bounded above by the length of $\mathfrak{r}$, and show that we have equality when $\mathfrak{r}$ is moreover principal, so that e.g. for all $n \in \mathbb{Z}^+$, $\mathbb{Z}/2^n\mathbb{Z}[t]$ has rank n.

1.2. Preliminaries on Change of Rings.

Remark 1.2. *Let $\iota : R \to S$ be a ring map. For an R-module M, let $\iota_*(M)$ be the S-module $M \otimes S$. Then $\mu(\iota_*(M)) \leq \mu(M)$. If every ideal of S is $\iota_*(I)$ for some I, we get* $\mathrm{rk}(S) \leq \mathrm{rk}(R)$. *This holds when ι is a quotient or a localization map.*

[1] Here all rings are commutative and with multiplicative identity.

For an R-module M and a prime ideal $\mathfrak{p}$ of R, we put $M_{\mathfrak{p}} = M \otimes R_{\mathfrak{p}}$ and $\mu_{\mathfrak{p}}(M) = \mu(M_{\mathfrak{p}})$. Remark 1.2 gives $\mu_{\mathfrak{p}}(M) \leq \mu(M)$. By way of a converse, we have:

Theorem 1.3. *(Forster-Swan* [Fo64] [Sw67]*)*
Let M be a finitely generated module over the Noetherian ring R. Then

$$\mu(M) \leq \sup_{\mathfrak{p} \in \operatorname{Spec} R} \left(\mu_{\mathfrak{p}}(M) + \dim R/\mathfrak{p}\right).$$

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2. Domains of Finite Rank

I.S. Cohen initiated the study of ranks of domains. We recall some of his results.

Theorem 2.1. *(Cohen* [Co50]*) If R is a domain of finite rank, then* $\dim R \leq 1$.

For R Noetherian and $\mathfrak{p} \in \operatorname{Spec} R$, let $k(\mathfrak{p})$ be the fraction field of $R/\mathfrak{p}$; put

$$z_{\mathfrak{p}}(R) = \dim_{k(\mathfrak{p})} \mathfrak{p}R_{\mathfrak{p}}/\mathfrak{p}^2 R_{\mathfrak{p}}.$$

Then $z_{\mathfrak{p}}(R) \geq \dim R_{\mathfrak{p}}$; $\mathfrak{p}$ is **regular** if equality holds, otherwise **singular**. Put

$$z(R) = \sup_{\mathfrak{p} \in \operatorname{MaxSpec} R} z_{\mathfrak{p}}(R).$$

Suppose R is a one-dimensional Noetherian *domain*. Then by Krull-Akizuki [CA, Thm. 18.7] the normalization $\overline{R}$ is Dedekind and $\operatorname{Spec} \overline{R} \to \operatorname{Spec} R$ is surjective and finite-to-one. The following well known result dispenses with the normal case.

Proposition 2.2. *We have* $\operatorname{rk}(\overline{R}) \leq 2$*, with equality iff $\overline{R}$ is not principal.*

Proof. By Forster-Swan, a non-principal Dedekind domain has rank 2. Or: let I be an ideal of $\overline{R}$, let $0 \neq x \in I$, and factor (x) as $\mathfrak{p}_1^{a_1} \cdots \mathfrak{p}_r^{a_r}$. Then $\overline{R}/(x) \cong \prod_{i=1}^r \overline{R}/\mathfrak{p}_i^{a_i} \cong \prod_{i=1}^r \overline{R}_{\mathfrak{p}_i}/(\mathfrak{p}_i \overline{R}_{\mathfrak{p}_i})^{a_i}$ is principal, so $I = \langle x, y \rangle$ for some $y \in I$. $\square$

Henceforth we suppose R is *not* normal. Let

$$\mathfrak{c} = (R : \overline{R}) = \{x \in K \mid x\overline{R} \subset R\}$$

be the conductor of R: it is the largest ideal of $\overline{R}$ which is also an ideal of R. If $\mathfrak{p}$ is regular then $\mathfrak{p}\overline{R}$ is also prime and $R_{\mathfrak{p}} \xrightarrow{\sim} \overline{R}_{\mathfrak{p}\overline{R}}$. We say R is **nearly Dedekind** if $\mathfrak{c} \neq 0$; equivalently, if $\overline{R}$ is a finitely generated R-module. In a nearly Dedekind domain, the singular primes are characterized as: the primes $\mathfrak{p}$ such that $\mathfrak{p} + \mathfrak{c} \subsetneq R$; the radicals of the ideals in a primary decomposition of $\mathfrak{c}$; or the primes of R lying under primes of $\overline{R}$ which divide $\mathfrak{c}$. They are finite in number.

Theorem 2.3. *(Cohen* [Co50]*) A nearly Dedekind domain has finite rank.*

Let $(R, \mathfrak{m})$ be a one-dimensional local Noetherian domain. Then the sequence $\{\dim_{R/\mathfrak{m}} \mathfrak{m}^i/\mathfrak{m}^{i+1}\}_{i=1}^{\infty}$ is eventually constant [Sa78, p. 40]; its eventual value is the **multiplicity** $e(R)$ of R. If R is a one-dimensional Noetherian domain and $\mathfrak{q} \in \operatorname{MaxSpec} R$, then we put $e_{\mathfrak{q}}(R) = e(R_{\mathfrak{q}})$ and $e(R) = \sup_{\mathfrak{p} \in \operatorname{MaxSpec} R} e_{\mathfrak{p}}(R)$.

Example 2.4. *(Sally* [Sa78, p. 5]*) For a field k, put $R = k[[t^3, t^4]]$. Then R is a one-dimensional Noetherian local domain with maximal ideal $\mathfrak{p} = \langle t^3, t^4 \rangle$ and $R/\mathfrak{p} = k$. Moreover we have $\mathfrak{p}^i = \langle t^{3i}, t^{3i+1}, t^{3i+2} \rangle = t^{3i}k[[t]]$ for all $i \geq 2$, so*

$$\mu(\mathfrak{p}) = \dim_k \mathfrak{p}/\mathfrak{p}^2 = 2;\ \forall i \geq 2,\ \mu(\mathfrak{p}^i) = \dim_k \mathfrak{p}^i/\mathfrak{p}^{i+1} = 3.$$

Thus $z(R) = 2 < 3 = e(R)$.

Proposition 2.5. *Let R be a one-dimensional* local *Noetherian domain. Then* $\mathrm{rk}(R) = e(R)$.

Proof. Nakayama's Lemma gives $e(R) \leq \mathrm{rk}(R)$. Since R is Cohen-Macaulay, the inequality $\mathrm{rk}(R) \leq e(R)$ follows from [Sa78, Thm. 3.1]. □

The conclusion of Proposition 2.5 need not hold if R is not assumed to be local. Indeed, if R is a nonprincipal Dedekind domain, then $\mathrm{rk}(R) = 2$ by Proposition 2.2, while $e(R) = 1$ since R is locally principal. However, this is the only problem case, as is shown by the following result.

Theorem 2.6. *Let R be a one-dimensional, non-normal Noetherian domain. Then*

$$\mathrm{rk}(R) = e(R) = \sup_{\mathfrak{p} \in \mathrm{MaxSpec}\, R} \mathrm{rk}(R_{\mathfrak{p}}).$$

Proof. By Proposition 2.5, for all $\mathfrak{p} \in \mathrm{MaxSpec}\, R$ we have $e_{\mathfrak{p}}(R) = \mathrm{rk}(R_{\mathfrak{p}})$ and thus

$$e(R) = \sup_{\mathfrak{p} \in \mathrm{MaxSpec}\, R} \mathrm{rk}(R_{\mathfrak{p}}).$$

By Remark 1.2 we have

$$\sup_{\mathfrak{p} \in \mathrm{MaxSpec}\, R} \mathrm{rk}(R_{\mathfrak{p}}) \leq \mathrm{rk}(R).$$

Since R is not normal, there is some $\mathfrak{p} \in \mathrm{MaxSpec}\, R$ such that $R_{\mathfrak{p}}$ is not principal, and thus by Proposition 2.5 we have

$$e(R) \geq e_{\mathfrak{p}}(R) = \mathrm{rk}(R_{\mathfrak{p}}) \geq 2.$$

Let I be an ideal of R. Applying Forster-Swan and Proposition 2.5, we get

$$\mu(I) \leq \max\left(\sup_{\mathfrak{p} \in \mathrm{MaxSpec}\, R} \mu(IR_{\mathfrak{p}}), 2\right) \leq \max\left(\sup_{\mathfrak{p} \in \mathrm{MaxSpec}\, R} e_{\mathfrak{p}}(R), 2\right) \leq e(R),$$

so $\mathrm{rk}(R) \leq e(R)$. □

Remark 2.7. *In Theorem 2.6 one can have* $\mathrm{rk}(R) = \aleph_0$ [Co50, pp. 38-40].

3. Nonmaximal Orders

3.1. **A First Example.**

One knows examples of local nearly Dedekind domains R with multiplicity $e(R)$ any given $n \in \mathbb{Z}^+$: e.g. [Wa73]. The following is perhaps the most familiar.

Example 3.1. *For a field k and $n \in \mathbb{Z}^+$, let*

$$R_n = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]] = k[[t^n]] + t^n k[[t]] = k + t^n k[[t]].$$

Then R_n is local nearly Dedekind with maximal ideal $\mathfrak{m} = \langle t^n, \ldots, t^{2n-1} \rangle = t^n k[[t]]]$ and $R_n/\mathfrak{m} = k$. For $i \in \mathbb{Z}^+$ we have $\mathfrak{m}^i = \langle t^{in}, \ldots, t^{(i+1)n-1} \rangle = t^{in} k[[t]]$, so

$$\mathrm{rk}(R_n) = e(R_n) = \lim_{i \to \infty} \dim_{R_n/\mathfrak{m}} \mathfrak{m}^i/\mathfrak{m}^{i+1} = \lim_{i \to \infty} n = n = z(R_n).$$

3.2. **Nonmaximal Orders I: Maximal Rank.**

Let A be a PID with fraction field F, and let K/F be a field extension of degree $N \in \mathbb{Z}^{\geq 2}$. An **A-order in K** is an A-subalgebra R of K which is finitely generated as an A-module and such that $F \otimes_A R = K$. We say R is an **A-order of degree N**. The structure theory of modules over a PID implies $R \cong_A A^N$.

Let R be an A-order in K. Then its normalization $\overline{R}$ is the integral closure of A in K. By Krull-Akizuki, $\overline{R}$ is a Dedekind domain. If $\overline{R}$ is finitely generated as an A-module then it is an R-order in K, the unique maximal order. It can happen that $\overline{R}$ is not a finitely generated A-module, but $\overline{R}$ is finitely generated if K/F is separable or A is finitely generated over a field.

Remark 3.2. *Let A be a PID, not a field, with fraction field F, and let K/F be a field extension of degree $N \in \mathbb{Z}^{\geq 2}$. If the integral closure S of A in K is not finitely generated as an A-module, then K admits no normal A-order. But it always admits some A-order: start with an F-basis of K, scale to get an F-basis $\alpha_1, \ldots, \alpha_N$ of elements integral over A, and take $S = A[\alpha_1, \ldots, \alpha_N]$.*

Proposition 3.3. *Let $N \in \mathbb{Z}^{\geq 2}$, let A be a PID, and let R be a non-normal A-order of degree N. If there is $\mathfrak{p} \in \operatorname{MaxSpec} R$ such that $z_{\mathfrak{p}}(R) = N$, then $\operatorname{rk}(R) = N$.*

Proof. Because R is free of rank N as a module over the PID A, every ideal of I of R is a free R-module of rank at most N and thus $\mu(I) \leq N$, so $\operatorname{rk}(R) \leq N$. On the other hand $\operatorname{rk}(R) \geq e_{\mathfrak{p}}(R) \geq z_{\mathfrak{p}}(R) = N$. $\square$

We say an A-order in K has **maximal rank** if $\operatorname{rk}(R) = N = [K : F]$.

Theorem 3.4. *Let A be a PID with fraction field $F \supsetneq A$, and let K/F be a field extension of degree $N \in \mathbb{Z}^{\geq 2}$. Then there is an A-order R in K of maximal rank.*

Proof. By Remark 3.2, there is an A-order S in K. Let x be a nonzero, nonunit in A, so $x = \epsilon p_1^{a_1} \cdots p_r^{a_r}$ where $r \in \mathbb{Z}^+$, $\epsilon \in A^\times$, $p_1, \ldots, p_r$ are nonassociate prime elements and $a_1, \ldots, a_r \in \mathbb{Z}^+$. Put

$$R = R(S, x) = A + xS.$$

We CLAIM: (i) R is an A-order; (ii) for $1 \leq i \leq r$ there is a unique $\mathfrak{p}_i \in \operatorname{Spec} R$ with $\mathfrak{p}_i \cap A = (p_i)$; and (iii) $z_{\mathfrak{p}_i}(R) = N$ for all i. Assuming the claim, Proposition 3.3 gives $\operatorname{rk}(R) = N$. We show the claim:

(i) Certainly R is a subring of K. If $\alpha_1 = 1, \alpha_2, \ldots, \alpha_N$ is an A-basis for S,[2] then $1, x\alpha_2, \ldots, x\alpha_N$ is an A-basis for R. So R is an A-order in K.

(ii) Fix $1 \leq i \leq r$ and define $\mathfrak{p}_i = p_i A + xS$. Then $\mathfrak{p}_i$ is an ideal of R and $R/\mathfrak{p}_i \cong A/(p_i)$ is a field, so $\mathfrak{p}_i$ is a prime ideal of R containing p_i. Let $\mathfrak{q}$ be a prime ideal of R such that $\mathfrak{q} \cap A = (p_i)$. Then since $p_i \mid x$ we have

$$\mathfrak{p}_i^2 = (p_i A + xS)^2 = p_i^2 A + p_i x S + x^2 S = p_i^2 A + p_i x S \subset p_i R \subset \mathfrak{q}.$$

Since $\mathfrak{q}$ is prime we get $\mathfrak{q} \supset \mathfrak{p}_i$ and thus (since $\dim R = 1$) $\mathfrak{q} = \mathfrak{p}_i$.

(iii) Since $p_i, x\alpha_2, \ldots, x\alpha_N$ is an A-basis for $\mathfrak{p}_i$ and $p_i^2, p_i x\alpha_2, \ldots, p_i x\alpha_N$ is an A-basis for $\mathfrak{p}_i^2$, we have $\mathfrak{p}_i/\mathfrak{p}_i^2 \cong_A (A/(p_i))^N$. Thus for all $1 \leq i \leq r$ we have

$$z_{\mathfrak{p}_i}(R) = \dim_{R/\mathfrak{p}_i} \mathfrak{p}_i/\mathfrak{p}_i^2 = \dim_{A/(p_i)} (A/(p_i))^N = N. \qquad \square$$

Remark 3.5. *a) If R is a PID with fraction field $F \supsetneq R$, then F admits a degree N field extension for all $N \in \mathbb{Z}^{\geq 2}$: for $(p) \in \operatorname{MaxSpec} R$, we can take $K = F(p^{\frac{1}{N}})$. b) The $R = A + xS$ construction is modelled on* [Co, Thm. 3.15].

[2]We are permitted to take $\alpha_1 = 1$ by "Hermite's Lemma" [CA, Prop. 6.14].

Example 3.6. *Let k be a field. For $n \in \mathbb{Z}^{\geq 2}$ put $A = k[[t^n]]$, so $F = k((t^n))$, and let $K = k((t))$. Let $S = k[[t]]$, the maximal A-order in K. Then $R(S, t^n) = k[[t^n]] + t^n k[[t]]$ is the ring R_n of Example 3.1.*

Example 3.7. *For $n \in \mathbb{Z}^{\geq 2}$ put $A = \mathbb{Z}$, $K = \mathbb{Q}(2^{\frac{1}{n}})$, $S = \mathbb{Z}[2^{\frac{1}{n}}]$ and $x = 2$. Then $R = R(S, x) = \mathbb{Z}[2^{\frac{n+1}{n}}, 2^{\frac{n+2}{n}}, \ldots, 2^{\frac{2n-1}{n}}]$ has rank n, and it is the order in K that Matson used to prove Theorem 1.1b).*

3.3. **Nonmaximal Orders II: Pullbacks and Locally Maximal Orders.**

Let A be a DVR with fraction field F and residue field k. Let l/k be a separable field extension of degree $d \in \mathbb{Z}^{\geq 2}$, let K/F be the corresponding degree d unramified (hence separable) field extension, and let S be the integral closure of A in K, so S is a DVR with maximal ideal $\overline{\mathfrak{m}}$ (say) and $S/\overline{\mathfrak{m}} = l$. Let $q : S \to l$ be the quotient map, and put $R = q^{-1}(k)$. By [AM92, pp. 35-36] and the Eakin-Nagata Theorem [CA, Cor. 8.31], R is local nearly Dedekind with normalization S and an A-order in K. In *loc. cit.*, Anderson and Mott show $\mathfrak{m} = q^{-1}(0) = \overline{\mathfrak{m}}$ and $R/\mathfrak{m} = k$. Finally, $\mathfrak{m}/\mathfrak{m}^2 = \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$ is a one-dimensional l-vector space hence a d-dimensional $k = R/\mathfrak{m}$-vector space, so Proposition 3.3 applies: we have $\mathrm{rk}(R) = d$.

Theorem 3.8. *For $r \in \mathbb{Z}^+$, prime numbers $p_1, \ldots, p_r$ and $n_1, \ldots, n_r \in \mathbb{Z}^{\geq 2}$ with $\max_i n_i = n$, there is a degree N number field K and a $\mathbb{Z}$-order R in K with exactly r singular primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, such that $e_{\mathfrak{p}_i}(R) = n_i$ for all i. Thus $\mathrm{rk}(R) = n$.*

Proof. Step 1: The construction of K and R is essentially given in [CGP15, §3.2-3.3]; the only difference is that in that construction one inverts all the regular primes to get a semilocal domain. So we will content ourselves with a sketch. For $1 \leq i \leq r$, let $\mathbb{Q}_{p_i^{n_i}}$ be the unramified extension of $\mathbb{Q}_{p_i}$ of degree n_i. By weak approximation / Krasner's Lemma there is a number field K of degree N and primes $\mathcal{P}_1, \ldots, \mathcal{P}_r$ such that $K_{\mathcal{P}_i} \cong \mathbb{Q}_{p_i^{n_i}}$ for all i. The local degree $[K_{\mathcal{P}_i} : \mathbb{Q}_{p_i}] = n_i$ is assumed (only) to be at most N; when $n_i < N$ this is handled by having other primes of $\mathbb{Z}_K$ lying over p_i. We have

$$\mathbb{Z}_K/\mathcal{P}_1 \cdots \mathcal{P}_r \cong \prod_{i=1}^r \mathbb{Z}_K/\mathcal{P}_i \cong \prod_{i=1}^r \mathbb{F}_{p_i^{n_i}}.$$

Let $q : \mathbb{Z}_K \to \prod_{i=1}^r \mathbb{F}_{p_i^{n_i}}$ be the corresponding quotient map. Then we take

$$R = q^{-1}(\prod_{i=1}^r \mathbb{F}_{p_i}).$$

For $1 \leq i \leq r$, let $\mathfrak{p}_i = \mathcal{P}_i \cap R$ and let $q_i : (\mathbb{Z}_K)_{\mathcal{P}_i} \to (\mathbb{Z}_K)_{\mathcal{P}_i}/\mathcal{P}_i(\mathbb{Z}_K)_{\mathcal{P}_i} \cong \mathbb{F}_{p_i^{n_i}}$. Then (as is shown in detail in *loc. cit.*; see also [CGP15, Thm. 2.6]) for all i, the ring $R_{\mathfrak{p}_i}$ is the pullback $q_i^{-1}(\mathbb{F}_{p_i})$, and thus is local nearly Dedekind with $z_{\mathfrak{p}_i}(R) = z(R_{\mathfrak{p}_i}) = [\mathbb{F}_{p_i^{n_i}} : \mathbb{F}_{p_i}] = n_i$.
Step 2: We have – e.g. using CRT as above – that $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the only singular primes of R, so

$$z(R) = \max_{1 \leq i \leq r} n_i = n.$$

For $1 \leq i \leq r$, let $\widehat{R_{\mathfrak{p}_i}}$ denote the $\mathfrak{p}_i$-adic completion of $R_{\mathfrak{p}_i}$. Because there is a unique prime of $\overline{R}$ lying over $\mathfrak{p}_i$ ("analytically irreducible"), $\widehat{R_{\mathfrak{p}_i}}$ is itself a one-dimensional local Noetherian domain, with fraction field $K_{\mathfrak{p}_i}$, and moreover we have $e(R_{\mathfrak{p}_i}) = e(\widehat{R_{\mathfrak{p}_i}})$ and $z(\widehat{R_{\mathfrak{p}_i}}) = z(R_{\mathfrak{p}_i}) = n_i$. But $\widehat{R_{\mathfrak{p}_i}}$ is a $\mathbb{Z}_{p_i}$-order in $K_{\mathfrak{p}_i}$ of degree n_i, so Proposition 3.3 applies to give

$$\mathrm{rk}(R_{\mathfrak{p}_i}) = e(R_{\mathfrak{p}_i}) = e(\widehat{R_{\mathfrak{p}_i}}) = \mathrm{rk}(\widehat{R_{\mathfrak{p}_i}}) = z(\widehat{R_{\mathfrak{p}_i}}) = z(R_{\mathfrak{p}_i}) = z_{\mathfrak{p}_i}(R)$$

and thus

$$\operatorname{rk}(R) = \max_{1 \le i \le r} \operatorname{rk}(R_{\mathfrak{p}_i}) = \max_{1 \le i \le r} z_{\mathfrak{p}_i}(R) = \max_{1 \le i \le r} n_i = n. \qquad \square$$

Remark 3.9. *a) There is an analogue of Theorem 3.8 with $\mathbb{Z}$ replaced by $\mathbb{F}_q[t]$.*
b) One would like to take $n = 1$ in Theorem 3.8! Unfortunately at present we cannot prove that there are infinitely many number fields with class number one: this is perhaps the most (in)famous open problem in algebraic number theory.

4. More Rings of Finite Rank

4.1. A Trichotomy.

Theorem 4.1. *a) Let $R = R_1 \times \cdots \times R_n$ be a finite direct product of rings. Then*

$$\operatorname{rk}(R) = \max_i \operatorname{rk}(R_i).$$

b) (Gilmer) For a Noetherian ring R, the following are equivalent:
(i) R has finite rank.
(ii) For all minimal primes $\mathfrak{p} \in \operatorname{Spec} R$, the ring $R/\mathfrak{p}$ has finite rank.
c) (Cohen-Gilmer) A ring of finite rank has dimension zero or one.

Proof. a) Every ideal in $R_1 \times \cdots \times R_n$ is of the form $I_1 \times \cdots \times I_n$ for ideals I_i of R_i. b) This is [Gi72, Thm. 2]. c) Apply Theorem 2.1 and part b). $\square$

So if R is a ring with $\operatorname{rk}(R) \in \mathbb{Z}^+$, exactly one of the following holds:
- R is Noetherian of dimension zero, i.e., Artinian;
- R is a Noetherian domain of dimension one;
- R is Noetherian of dimension one and not a domain.

We treated domains in §3, and we will discuss Artinian rings in §4.2. This leaves us with rings which are one-dimensional Noetherian and not domains. One can show that there are such rings of all ranks quite cheaply: if R is a domain of rank $n \in \mathbb{Z}^+$ then Theorem 4.1a) gives $\operatorname{rk}(R \times R) = n$. The more interesting case is when the localization of R at some minimal prime is not a domain, so in particular when R has a unique minimal prime, which is nonzero. A good example is a polynomial ring over a local Artinian ring. In §4.3 we study the ranks of polynomial rings.

4.2. Artinian Rings.

Let $\mathfrak{r}$ be an Artinian ring. We denote by $\ell(\mathfrak{r})$ the length of $\mathfrak{r}$ as an $\mathfrak{r}$-module, which is finite. We denote by $n(\mathfrak{r})$ the **nilpotency index** of $\mathfrak{r}$: the least $n \in \mathbb{Z}^+$ such that if $x \in \mathfrak{r}$ is nilpotent then $x^n = 0$.

Proposition 4.2. *Let $\mathfrak{r}$ be an Artinian ring.*
a) We have $\operatorname{rk}(\mathfrak{r}) \le \ell(\mathfrak{r})$.
b) If $\ell(\mathfrak{r}) \ge 2$ (i.e., if $\mathfrak{r}$ is nonzero and not a field), then $\operatorname{rk}(\mathfrak{r}) \le \ell(\mathfrak{r}) - 1$.

Proof. The result is clear when $\mathfrak{r}$ is the zero ring or is a field, so assume $\ell(\mathfrak{r}) \ge 2$. If $\mathfrak{r}$ is principal then $\operatorname{rk}(\mathfrak{r}) = 1 \le \ell(\mathfrak{r}) - 1$. If $\mathfrak{r}$ is not principal, then there is an ideal I with $2 \le \mu(I) \le \ell(I)$, and such an ideal is necessarily proper, so $\ell(\mathfrak{r}) \ge \ell(I) + 1$. $\square$

The following result shows that the bound of Proposition 4.2b) is sharp.

Corollary 4.3. *Let $n \in \mathbb{Z}^{\ge 2}$. Then:*
a) For any field k, there is an Artinian k-algebra of rank n and length $n + 1$.
b) There is a finite ring R of rank n and length $n + 1$.

Proof. Let R be a non-normal domain of finite rank n with a maximal ideal $\mathfrak{p}$ such that $\dim_{R/\mathfrak{p}} \mathfrak{p}/\mathfrak{p}^2 = n$. Then $R/\mathfrak{p}^2$ is Artinian, of rank n and length $n+1$. Taking R as in Example 3.6 gives part a), and taking R as in Example 3.7 gives part b). □

4.3. **Polynomial Rings of Finite Rank.**

Next we explore polynomial rings of finite rank. For any nonzero ring $\mathfrak{r}$, a polynomial ring over $\mathfrak{r}$ in at least two indeterminates has Krull dimension at least 2 and thus has infinite rank. So we are reduced to the case of $\mathfrak{r}[t]$.

Theorem 4.4. *For a ring $\mathfrak{r}$, the following are equivalent:*
(i) The polynomial ring $\mathfrak{r}[t]$ has finite rank.
(ii) The ring $\mathfrak{r}$ is Artinian.

Proof. (i) $\implies$ (ii): Suppose $\mathfrak{r}[t]$ has finite rank. Then $\mathfrak{r}[t]$ is Noetherian, hence so is $\mathfrak{r}$. Thus $\dim \mathfrak{r}[t] = 1 + \dim \mathfrak{r}$. By Theorem 4.1c) we have $\dim \mathfrak{r} = 0$, so $\mathfrak{r}$ is Artinian.
(ii) $\implies$ (i): If $\mathfrak{r}$ is Artinian, there are local Artinian rings $\mathfrak{r}_1, \ldots, \mathfrak{r}_r$ such that $\mathfrak{r} \cong \mathfrak{r}_1 \times \cdots \times \mathfrak{r}_r$, and then $\mathfrak{r}[t] \cong \mathfrak{r}_1[t] \times \cdots \times \mathfrak{r}_r[t]$. So Theorem 4.1a) reduces us to showing: a polynomial ring $\mathfrak{r}[t]$ over a local Artinian ring $(\mathfrak{r}, \mathfrak{m})$ has finite rank. The ring $\mathfrak{r}[t]$ is Noetherian, with unique minimal prime $\mathfrak{m}\mathfrak{r}[t]$. Since $\mathfrak{r}[t]/\mathfrak{m}\mathfrak{r}[t] = (R/\mathfrak{m})[t]$ is a PID, the ring $\mathfrak{r}[t]$ has finite rank by Theorem 4.1b). □

Theorem 4.5. *Let $\mathfrak{r}$ be an Artinian local ring of length ℓ, and let $R = \mathfrak{r}[t]$. Then*

$$\operatorname{rk} R \leq \ell.$$

Proof. Let $\mathfrak{p}$ be the unique prime ideal of $\mathfrak{r}$, let $k = \mathfrak{r}/\mathfrak{p}$. Then $\mathcal{P} = \mathfrak{p}[t]$ is the unique minimal prime of R. By Theorem 4.4, R has finite rank. The main input was Theorem 4.1b), and we will get the upper bound $\operatorname{rk} R \leq \ell$ by following Gilmer's proof. For $1 \leq i \leq \ell$, let

$$R_i = R/\mathcal{P}^i = \mathfrak{r}[t]/\mathfrak{p}^i\mathfrak{r}[t] = \mathfrak{r}/\mathfrak{p}^i[t].$$

We will show inductively that $\operatorname{rk} R_i$ is at most the length $\ell(\mathfrak{r}/\mathfrak{p}^i)$ of $\mathfrak{r}/\mathfrak{p}^i$.
Base Case ($i = 1$): The ring $R_1 = \mathfrak{r}/\mathfrak{p}[t] = k[t]$ is a PID, thus $\operatorname{rk} R_1 = 1 = \ell(k)$.
Inductive Step: Suppose $1 \leq i < \ell$ and $\operatorname{rk} R_i \leq \ell(\mathfrak{r}/\mathfrak{p}^i)$. Consider the exact sequence of R-modules

$$0 \to \mathcal{P}^i/\mathcal{P}^{i+1} \to R/\mathcal{P}^{i+1} \to R/\mathcal{P}^i \to 0.$$

For any surjective homomorphism of rings $R \to S$, the rank $\mu_*(S)$ as an R-module is equal to its rank $\operatorname{rk} S$ as a ring. Thus our inductive hypothesis gives $\mu_*(R/\mathcal{P}^i) \leq \ell(\mathfrak{r}/\mathfrak{p}^i)$. Moreover the rank of $\mathcal{P}^i/\mathcal{P}^{i+1}$ as an R-module is its rank as an $R/\mathcal{P} = k[t]$-module. By [Gi72, Prop. 2] we have

$$\mu_*(\mathcal{P}^i/\mathcal{P}^{i+1}) \leq \operatorname{rk}(k[t])\mu_R(\mathcal{P}^i/\mathcal{P}^{i+1}) = \mu_{\mathfrak{r}}(\mathfrak{p}^i/\mathfrak{p}^{i+1}) = \ell(\mathfrak{p}^i/\mathfrak{p}^{i+1}).$$

By [Gi72, Prop. 1] we have

$$\operatorname{rk} R_{i+1} = \mu_*(R/\mathcal{P}^{i+1}) \leq \mu_*(\mathcal{P}^i/\mathcal{P}^{i+1}) + \mu_*(R/\mathcal{P}^i) \leq \ell(\mathfrak{p}^i/\mathfrak{p}^{i+1}) + \ell(\mathfrak{r}/\mathfrak{p}^i) = \ell(\mathfrak{r}/\mathfrak{p}^{i+1}),$$

completing the induction. Since the nilpotency index of $\mathfrak{r}$ is at most ℓ, we have

$$\operatorname{rk} R = \operatorname{rk} \mathfrak{r}[t] = \operatorname{rk} \mathfrak{r}/\mathfrak{p}^\ell[t] = \operatorname{rk} R_\ell \leq \ell(\mathfrak{r}/\mathfrak{p}^\ell) = \ell(\mathfrak{r}) = \ell.$$ □

Since an Artinian ring is a finite product of local Artinian rings, Theorem 4.5 implies: for any Artinian ring $\mathfrak{r}$, the rank of $\mathfrak{r}[t]$ is bounded above by the maximum length of the local Artinian factors of $\mathfrak{r}$.

Corollary 4.6. *Let $\mathfrak{r}$ be a nonzero principal Artinian ring, which we may decompose as $\mathfrak{r} = \prod_{i=1}^r \mathfrak{r}_i$, with each $\mathfrak{r}_i$ a local Artinian principal ring. For $1 \leq i \leq r$, let n_i be the length of $\mathfrak{r}_i$ (which coincides with its nilpotency index), and let $n = \max_{1 \leq i \leq r} n_i$. Then $\operatorname{rk} \mathfrak{r}[t] = n$.*

Proof. We easily reduce to the case in which $\mathfrak{r}$ is local with maximal ideal $\mathfrak{p} = (\pi)$. Theorem 4.5 gives $\operatorname{rk} \mathfrak{r}[t] \leq n$. Let $\mathfrak{m} = \langle \pi, t \rangle$, so $\mathfrak{m}$ is a maximal ideal of $\mathfrak{r}[t]$ and $R/\mathfrak{m} = \mathfrak{r}/\pi\mathfrak{r} = k$, say. We claim that the ideal $\mathfrak{m}^{n-1} = \langle \pi^{n-1}, \pi^{n-2}t, \ldots, \pi t^{n-2}, t^{n-1} \rangle$ requires n generators. Indeed, for $a_1, \ldots, a_n \in k$, we have that $a_1\pi^{n-1} + a_2\pi^{n-2}t + \ldots + a_n t^{n-1}$ cannot be expressed as an $\mathfrak{r}$-linear combination of terms $\pi^i t^j$ with $i + j \geq n$ unless $a_1 = \ldots = a_n = 0$, so $\dim_k \mathfrak{m}^{n-1}/\mathfrak{m}^n = n$ and $\mu(\mathfrak{m}^{n-1}) = n$. $\square$

Remark 4.7. *a) Corollary 4.6 implies: if A is a PID, $\pi \in R$ a prime element and $n \in \mathbb{Z}^+$, then* $\operatorname{rk} A/\pi^n A[t] = n$. *In fact this is equivalent: every local principal Artinian ring is a quotient of a PID* [Hu68, Cor. 11]. *P. Pollack has shown me a thoroughly elementary proof that* $\operatorname{rk} A/\pi^n A[t] \leq n$.
b) In particular, for a prime number p and $n \in \mathbb{Z}^+$ we have

$$\operatorname{rk} \mathbb{Z}/p^n\mathbb{Z}[t] = n.$$

The case $p = n = 2$ appears in Matson's thesis [Ma08, p. 44, Example 1.3.15]. *Her proof uses that $\mathbb{Z}/4\mathbb{Z}$ has a unique nonzero zero-divisor. The only other ring with this property is $\mathbb{Z}/2\mathbb{Z}[t]/(t^2)$, so this argument is rather specialized. Matson's result was our motivation for finding Theorem 4.5.*

5. Work of Matsuda

After this paper was first written we learned of relevant of R. Matsuda [Ma77] [Ma79], [Ma84]. We end with a brief discussion of its pertinence to the current work.

5.1. Matsuda's work on polynomial rings.

A ring R has the **n-generator property** if for all ideals I of R, if $\mu(I) < \aleph_0$ then $\mu(I) \leq n$. Thus a ring with the n-generator property has rank at most n iff it is Noetherian. In [Ma77] and [Ma84], Matsuda studied these properties in polynomial rings. Our Theorem 4.4 follows from [Ma84, Thm. 15]. The other results of §4 are complementary to Matsuda's work rather than overlapping. For instance:

Theorem 5.1. *(Matsuda* [Ma84, Prop. 18]*) Suppose $\mathfrak{r}$ is a ring that is not a field.*[3] *If $\mathfrak{r}[t]$ has the n-generator property, then $\mathfrak{r}$ has the $(n-1)$-generator property.*

It follows that if $\mathfrak{r}$ is Artinian and not a field, then $\operatorname{rk}(\mathfrak{r}) \leq \operatorname{rk}(\mathfrak{r}[t]) - 1$. Corollary 4.6 shows that the gap between the rank of $\mathfrak{r}$ and $\mathfrak{r}[t]$ can be arbitrarily large.

5.2. Matsuda's work on semigroup rings.

A **numerical semigroup** is a nonempty subset of the positive integers $\mathbb{Z}^+$ that is closed under addition. A numerical semigroup S is **primitive** if there are elements $n_1, \ldots, n_k \in S$ such that $\gcd(n_1, \ldots, n_k) = 1$. For any numerical semigroup S, let d be the greatest common divisor of the elements of S. Then $\overline{S} := \{\frac{n}{d} \mid n \in S\}$ is a primitive numerical semigroup and $\overline{S} \stackrel{\cdot d}{\to} S$ is a semigroup isomorphism. A famous result of Frobenius (see e.g. [RA]) asserts that for a primitive numerical semigroup S, the set $\mathbb{Z}^+ \setminus S$ is finite, and thus there is a unique element $\mathfrak{c} \in S$, the **conductor of S**, such that $\mathfrak{c} \in S$, $\mathfrak{c} - 1 \notin S$ and for all $n \in \mathbb{Z}^+$, we have $\mathfrak{c} + n \in S$.

Let S be a primitive numerical semigroup with least element n_1, and put $S_1 = n\mathbb{Z}^+$. If $n_1 = 1$ then $S = S_1$. Otherwise S_1 is not primitive and thus cannot equal S, and we take n_2 to be the least element of $S \setminus S_1$. We continue in this manner: if the semigroup S_k generated by $n_1, \ldots, n_k$ is not all of S, let n_{k+1} be the least element of $S \setminus S_k$. Every element of S lies in some S_k, and there is some $k_0 \in \mathbb{Z}^+$ such that $\gcd(n_1, \ldots, n_{k_0}) = 1$, so Frobenius's theorem implies that $S \setminus S_{k_0}$

[3] The hypothesis that $\mathfrak{r}$ not be a field does not explicitly appear in Matsuda's work, but it must be intended.

is finite. Thus $S = S_k$ for some k, and then $\{n_1, \ldots, n_k\}$ is the unique minimal generating set.

For a field k and a numerical semigroup S, the **semigroup ring** $k[S]$ is the subring of $k[t]$ consisting of polynomials $\sum_{s \in S \cup \{0\}} a_s t^s$ – i.e., the monomial exponents must lie in $S \cup \{0\}$.

Theorem 5.2. *(Matsuda* [Ma79, Prop. 5.6]*) Let k be a field, and let S be a primitive numerical semigroup with least element n_1. Then* $\mathrm{rk}(k[S]) = n_1$.

For $n_1 \in \mathbb{Z}^+$, there are infinitely many primitive numerical semigroups with least element n_1: e.g. n_2 can be prescribed to be any integer greater than n_1 and not divisible by it. Thus we attribute Theorem 1.1a) to Matsuda.

Perhaps the simplest example is $S_{n_1} := \{n \in \mathbb{Z} \mid n \geq n_1\}$. The minimal generating set of S_{n_1} is $\{n_1, n_1+1, \ldots, 2n_1 - 1\}$, so $k[S_{n_1}] = k[t^{n_1}, \ldots, t^{2n_1-1}]$. (The completion of $k[S_{n_1}]$ at the maximal ideal $\langle t^{n_1}, \ldots, t^{2n_1-1} \rangle$ is the ring R_{n_1} of Examples 3.1 and 3.6, and one can easily deduce that $\mathrm{rk}\, R_{n_1} = n_1$ from $\mathrm{rk}\, k[S_{n_1}] = n_1$.) This suggests a connection between Theorem 5.2 and the nonmaximal orders of §3.2. Indeed we can get a quick proof of Theorem 5.2 using these ideas:

The ring $k[t]$ is a free $k[t^{n_1}]$-module of dimension n_1. The ring $k[t^{n_1}] \cong k[t]$ is a PID, so it follows by the usual PID structure theory that $k[S]$ is a free $k[t^{n_1}]$-module of rank $r \leq n_1$. Thus any ideal I of $k[S]$ has rank at most $r \leq n_1$ as a $k[t^{n_1}]$-module. It follows that $\mathrm{rk}(k[S]) \leq n_1$.

Let $\mathfrak{c}$ be the conductor of S, and let J be the ideal $\langle t^{\mathfrak{c}}, t^{\mathfrak{c}+1}, \ldots, t^{\mathfrak{c}+n_1-1} \rangle$ of $k[S]$. We claim that $\mathrm{rk}(J) \geq n_1$, which will complete the proof of Theorem 5.2. Let $\mathfrak{m}$ be the ideal $\langle t^n \mid n \in S \rangle$ of $k[S]$. It is enough to show that $\dim_k J/\mathfrak{m}J \geq n_1$, which amounts to showing: for $\alpha_1, \ldots, \alpha_{n_1} \in k$,

$$\alpha_1 t^{\mathfrak{c}} + \alpha_2 t^{\mathfrak{c}+1} + \ldots + \alpha_{n_1} t^{\mathfrak{c}+n_1-1} \in \mathfrak{m}J \implies \alpha_1 = \ldots = \alpha_{n_1} = 0.$$

But every element of $\mathfrak{m}J \setminus \{0\}$ has degree at least $\mathfrak{c} + n_1$, so this is clear.

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