On Hodge Theory and DeRham Cohomology of Variétiés

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Chapter 1

Introduction: Etale Cohomology Pro and Con

Today's lecture has a somewhat schizophrenic goal: we come to praise étale cohomology *and* to bury it.

To fix ideas, all varieties will be nonsingular, projective and connected over an algebraically closed field k.

1.1 ℓ-adic cohomology is the bomb...

Let X/k be a variety and ℓ a prime number. We can define the **ℓ-adic cohomology groups of** X as follows:

$$H^i(X, \mathbb{Z}_l) := \varprojlim_n H^i_{\text{ét}}(X, \underline{\mathbb{Z}/l^n\mathbb{Z}}).$$

$$H^i(X, \mathbb{Q}_l) := H^i(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Here $H^i_{\text{ét}}(X, \underline{\mathbb{Z}/l^n\mathbb{Z}})$ denotes the ith etale cohomology group – i.e., the ith right derived functor of the global section functor – for the constant sheaf $\mathbb{Z}/l^n\mathbb{Z}$ on the étale site of X, as defined in Andrew's lectures. (Indeed the abelian sheaves on $X_{\text{ét}}$ form an abelian category with sufficiently many injectives.) It is easy to see that $H^i(X, \mathbb{Z}_l)$ is a $\mathbb{Z}_l$-module, so $H^i(X, \mathbb{Q}_l)$ is a $\mathbb{Q}_l$-vector space, so its only invariant is its dimension (possibly infinite), which we denote by $b_{i,l}(X)$. Now we have some wonderful news:

Theorem 1 *(Comparison Theorem) Suppose $k = \mathbb{C}$ and let $X(\mathbb{C})$ denote the corresponding compact complex manifold with its classical ("analytic") topology.*
a) There is a canonical isomorphism $H^i(X, \mathbb{Z}_l) \cong H^i(X(\mathbb{C}), \mathbb{Z}_l)$, where the right hand side is singular cohomology with coefficients in the abelian group $\mathbb{Z}_l$. It follows that:
b) Each $H^i(X, \mathbb{Z}_l)$ is a finitely generated $\mathbb{Z}_l$-module;

c) For all primes ℓ, $b_{i,\ell}(X) = b_i(X(\mathbb{C})) = H^i(X, \mathbb{Q})$, the ith Betti number; and
d) Knowing all the ℓ-adic cohomology groups, we can recover the singular cohomology $H^i(X, \mathbb{Z})$.

Indeed part a) implies that all the pleasant properties of the singular cohomology groups will be satisfied by the ℓ-adic cohomology groups: Poincaré duality, Kunneth formula, Lefschetz trace formula...

An analogous – but easier in the proper case – comparison theorem holds for the étale fundamental group $\pi_1^{\text{ét}}(X)$. We do not want to go into the details of the definition, but: just as the fundamental group of a topological space is the group of deck transformations of the universal cover, the étale fundamental group would be the automorphism group of the universal étale covering of a variety X, except that since étale covers have finite fibers by definition, such a universal covering, if of infinite degree, will not exist as an étale (or even algebraic) map. So instead we consider only the automorphism groups of finite Galois covers and define $\pi_1^{\text{ét}}(X)$ as the inverse limit of these. (In particular – considering for just a moment a not-necessarily algebraically closed field k, $\pi_1^{\text{ét}}(\operatorname{Spec} k)$ is nothing else but the absolute Galois group of k.) To be sure, $\pi_1^{\text{ét}}(X/k)$ makes sense for varieties over fields of any characteristic. When $k = \mathbb{C}$ we can ask for the relationship between $\pi_1^{\text{ét}}(X)$ and $\pi_1(X(\mathbb{C}))$. The relevant fact here is that if $\mathcal{Y} \to X(\mathbb{C})$ is a finite unramified covering of the compact manifold $X(\mathbb{C})$ in the topological sense, then $\mathcal{Y} = Y(\mathbb{C})$ for a unique projective variety Y such that $Y \to X$ is a finite étale morphism (a theorem of Grauert-Remmert). Then:

Proposition 2 *The étale fundamental group of $X/\mathbb{C}$ is the profinite completion of the usual fundamental group $\pi_1(X(\mathbb{C}))$.*

(Despite our benevolent intentions, we should point out that the canonical map $\pi_1(X(\mathbb{C})) \to \pi_1\widehat{(X(\mathbb{C}))} = \pi_1^{\text{ét}}(X)$ need not be an injection – among combinatorial group theorists the property of a group injecting into its profinite completion is called *residual finiteness*, and not every finitely presented group has this property. Serre has constructed an example of a variety $X/\mathbb{C}$ and a discontinuous field automorphism σ of the complex numbers such that $\pi_1(X(\mathbb{C})) \neq \pi_1(\sigma(X)(\mathbb{C})$, although nevertheless the profinite completions are isomorphic.)

For example, if $X/\mathbb{C}$ is either a genus g curve or a g-dimensional abelian variety, then $\pi_1^{\text{ét}}(X) = \hat{\mathbb{Z}}^{2g}$.

One can also show that if $k_0 \leq k$ is an extension of algebraically closed fields of characteristic zero, then base change from k_0 to k induces isomorphisms both on the ℓ-adic cohomology groups and on the finite étale sites.[1]

[1] This becomes false in positive characteristic for nonprojective varieties: the size of the fundamental group of the affine line over an algebraically closed field of characteristic p depends upon the size of the field!

Well, after all we already have topological cohomology in characteristic zero; though it is certainly interesting to see to what extent this topological information can be recaptured algebraically, it is more interesting to ask about ℓ-adic cohomology for varieties over fields k of characteristic $p > 0$.

So that we can continue to say nice things about ℓ-adic cohomology, we take $\ell \neq p$ for the remainder of the section.

Theorem 3 *(Grothendieck,Deligne) Let X/k be a smooth projective variety of dimension d over an algebraically closed field of characteristic p, and fix any $\ell \neq p$. The ℓ-adic cohomology groups $H^\bullet(X, \mathbb{Q}_l)$ have the following properties:*
a) Each $H^i(X, \mathbb{Q}_l)$ is finite-dimensional, and $H^i(X, \mathbb{Q}_l) = 0$ for $i > 2d$.
b) There is a cup product operation making $H^\bullet(X, \mathbb{Q}_l)$ into a graded $\mathbb{Q}_l$-algebra.
c) (Poincaré duality) There is a natural isomorphism $H^{2d}(X, \mathbb{Q}_l) \cong \mathbb{Q}_l$; composing with this automorphism gives a pairing of $\mathbb{Q}_l$-vector spaces $H^i(X, \mathbb{Q}_l) \times H^{2d-i}(X, \mathbb{Q}_l) \to \mathbb{Q}_l$ which is nondegenerate.
d) (Kunneth formula) If Y/k is another variety, there is a natural isomorphism $H^\bullet(X, \mathbb{Q}_l) \otimes H^\bullet(Y, \mathbb{Q}_l) \xrightarrow{\sim} H^\bullet(X \times Y)$.
e) (Lefschetz trace formula) For a morphism $f : X \to X$ with finitely many fixed points of multiplicity one, the number of fixed points is given by the usual alternating sum $\sum_{i=1}^{2d} (-1)^i Tr(H^i(f))$ of the traces of f acting on the $H^i(X, \mathbb{Q}_l)$.
f) (Lefschetz embedding theorem) If $Y \subset X$ is a nonsingular subvariety of codimension one, the canonical maps $H^i(X, \mathbb{Q}_l) \to H^i(Y, \mathbb{Q}_l)$ are bijective for $i \leq d-2$ and injective for $i = d-1$.
g) (Hard Lefschetz theorem) Let $h \in H^2(X, \mathbb{Q}_l)$ correspond by Poincaré duality to a hyperplane section of a projective embedding of X. Let $L : H^i(X, \mathbb{Q}_l) \to H^{i+2}(X, \mathbb{Q}_l)$ be the map obtained by cupping with h. Then for $i \leq d$, the iterated map $L^{d-i} : H^i(X, \mathbb{Q}_l) \to H^{2d-i}(X, \mathbb{Q}_l)$ is an isomorphism.
h) (Smooth base change) If X/k admits a lift to characteristic zero – i.e., if X can be viewed as the special fiber of a smooth scheme $\mathcal{X}/W(k)$, then $H^i(X, \mathbb{Q}_l)$ coincides with the ith Betti number of (the analytic space associated to a complex embedding of) the generic fiber of $\mathcal{X}$.

We should put these results in persective as follows: although we didn't write them down, they are all true when $k = \mathbb{C}$ *because* they are true for the singular cohomology (we regard h) as being trivially true for the singular cohomology.) A more abstract perspective on all of this is the notion of a **Weil cohomology theory** with coefficients in a field K of characteristic zero – it is a functor from smooth projective varieties X/k to graded K-algebras $X \mapsto H^\bullet(X, K)$ satisfying the properties of the previous theorem. So singular cohomology is a Weil cohomology for complex varieties with coefficients in $\mathbb{Q}$, and for any $\ell \neq p$ ℓ-adic cohomology (with $\mathbb{Q}_l$-coefficients) gives a Weil cohomology with coefficients in $\mathbb{Q}_l$. For its intended application, any field K of characteristic zero will do:

Theorem 4 *(Grothendieck) The first two-thirds of the Weil conjectures for smooth projective varieties over a finite field $\mathbb{F}_q$ – i.e., the rationality of the* *zeta function and the existence of a functional equation – follow formally from the existence of a Weil cohomology for varieties over $\mathbb{F}_q$ with coefficients in* any *field K of characteristic zero. In particular, any $\mathbb{Q}_l$ with $l \neq p$ will do.*

Of course, the final third of the Weil conjectures – the Riemann hypothesis – was proved by Deligne *using* ℓ-adic cohomology, but the proof does not (to say the least!) follow immediately from Theorem 3. There is also a comparison theorem for the étale fundamental group. More precisely, let $\mathcal{X}/\overline{W(k)}$ be a scheme which is proper and smooth; write X_η for the generic fiber and $X_0 = X/k$ for the special fiber. By Proposition 2 we can compute $\pi_1^{\text{ét}}(X_\eta)$ as the profinite completion of the classical fundamental group of any associated complex manifold. Grothendieck [SGAI] shows there is a **specialization map** $sp : \pi_1^{\text{ét}} X_\eta) \to \pi_1^{\text{ét}}(X_0)$ which is an "isomorphism away from p":

Theorem 5 *(Grothendieck) For any group G, define the p-tame quotient $G^{\not p}$ to be the inverse limit of all finite quotients of G of order prime to p. Passage to the tame quotient is clearly functorial, and we have:*

$$sp^{\not p} : \pi_1^{\text{ét}}(X_\eta)^{\not p} \xrightarrow{\sim} \pi_1^{\text{ét}}(X_0)^{\not p}.$$

Remark: The relationship between $\pi_1^{\text{ét}}(X)$ and $H^1(X, \mathbb{Z}_l)$ is the same as in the topological case: namely for all n, $\text{Hom}(\pi_1^{\text{ét}}(X), \mathbb{Z}/l^n\mathbb{Z}) \cong H^1(X, \underline{\mathbb{Z}/l^n Z})$, so passing to the inverse limit we get $\text{Hom}_c(\pi_1^{\text{ét}}, \mathbb{Z}_l) = H^1(X, \mathbb{Z}_l)$. Indeed, when $\pi_1^{\text{ét}}(X)$ is abelian, this reduces to the previous comparison theorem, since one needs only to see that for all primes ℓ different from p, there is a canonical isomorphism between the **pro-ℓ** completions of $\pi_1^{\text{ét}}(X_\eta)$ and $\pi_1^{\text{ét}}(X_0)$. Especially, when X is a genus g curve or a g-dimensional abelian variety, this result is part of the basic theory of abelian varieties: for $\ell \neq p$, $\pi_1^{\text{ét}}(X)^\ell \cong \mathbb{Z}_l^{2g}$, obtained by passing to the limit under the Galois groups of $[\ell^n] : A = J(X) \to A = J(X)$, the multiplication by ell^n maps.

1.2 . . . unless ℓ is the characteristic of k

If we look instead at the p-adic cohomology groups $H^i(X, \mathbb{Z}_p)$ of a (smooth, proper, connected) variety over an algebraically closed field of characteristic p, we find that they are miserable: none of the properties of Theorem 3 hold! First, the most dramatic:

Proposition 6 $H^i(X, \mathbb{Z}_p) = 0$ *for $i > d$.*

Idea of proof: The first step is to show that $H^i(X, \mathbb{Z}/p\mathbb{Z}) = 0$ when $i > d$, and we will give a reasonably complete discussion of this. Afterwards we discuss the modification necessary to show $H^i(X, \mathbb{Z}/p^n\mathbb{Z}) = 0$ for all n.

Let $\mathbb{G}_{a\mathbb{F}_p}$ be the additive group scheme, i.e., $\mathbb{A}^1/\mathbb{F}_p$ with its natural group law. The map $x \mapsto x^p - x$ gives an endomorphism $\varphi_p : \mathbb{G}_a \to \mathbb{G}_a$, called the **Artin-Schreier isogeny** – indeed its kernel is just the fixed points of Frobenius on

$\mathbb{A}^1$, or $\mathbb{Z}/p\mathbb{Z}$. We have an exact sequence of group schemes

$$0 \to \underline{\mathbb{Z}/p\mathbb{Z}} \to \mathbb{G}_a \xrightarrow{\rho_p} \mathbb{G}_a.$$

If X is any scheme of characteristic p – i.e., a scheme over $\mathbb{F}_p$ – we can (flatly) basechange this sequence to X, getting

$$0 \to \underline{\mathbb{Z}/p\mathbb{Z}}_X \to \mathbb{G}_{aX} \xrightarrow{\rho_p} \mathbb{G}_{aX}$$

On the other hand, we can consider any groupscheme G/X as a "representable sheaf" on the étale site: if $Y \to X$ is an étale map, then $G(Y)$, the set of Y-valued points $Maps_X(Y, G)$ of the X-scheme G, is canonically endowed with the structure of an abelian group. So we may view the Artin-Schreier sequence as an exact sequence of étale sheaves.

In particular it is an exact sequence of Zariski sheaves, but at the level of Zariski sheaves φ_p is not surjective: as a morphism of Zariski sheaves over $\mathbb{F}_p$, $\varphi_p = 0$. However it is surjective as a morphism of étale sheaves: to solve the equation $x^p - x = y$ in $\mathbb{G}_a(U) = \mathcal{O}_U$, we need only pass to the étale cover $\mathcal{O}_U[t]/(t^p - t - y)$ of U (by construction we have a separable polynomial!).

On the other hand, for any $Y \to X$, $\mathbb{G}_a(Y) = \mathcal{O}_Y(Y)$ is just the global functions on Y, in other words, as étale sheaves we have $\mathbb{G}_{aX} = \mathcal{O}_X$. At last we have a short exact sequence of sheaves on $X_{'et}$

$$0 \to \underline{\mathbb{Z}/p\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{\varphi_p} \mathcal{O}_X \to 0.$$

Taking cohomology, we get

$$\ldots H^d(X, \mathcal{O}_X) \overset{\varphi_p}{ra} H^d(X, \mathcal{O}_X) \to H^{d+1}(X, \mathbb{Z}/p\mathbb{Z}) \to H^{d+1}(X, \mathcal{O}_X) = 0 \to \ldots$$

and certainly $H^i(X, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geq d+2$. Moreover we get $H^{d+1}(X, \mathbb{Z}/p\mathbb{Z}) = H^d(X, \mathcal{O}_X)/(F-1)H^d(X, \mathcal{O}_X)$, and as a matter of (semi)linear algebra one can show that if F is any p-linear map on a finite-dimensional k-vector space, then $F - 1$ is surjective. This completes the proof for $\mathbb{Z}/p\mathbb{Z}$.

To handle the $\mathbb{Z}/p^n\mathbb{Z}$ in the same way, one needs to find a commutative group scheme G_n over $\mathbb{F}_p$ such that the kernel of kernel of the Artin-Schreier isogeny $\varphi_p : G_n \to G_n$, $x \mapsto x^p - x$ is isomorphic to the constant groupscheme $\mathbb{Z}/p^n\mathbb{Z}$. Such a group scheme exists – it is the scheme representing the functor which sends an $\mathbb{F}_p$-scheme X to the additive group of **Witt vectors of length** n $W_n(X)$. The corresponding sheaf, which we denote $\mathcal{W}_n$, is *not* an $\mathcal{O}_X$-module when $n > 1$ (it is a sheaf of rings of characteristic p^n, not characteristic p), but since $\mathcal{W}_n$ is a repeated extension of $\mathcal{W}_1 = \mathcal{O}_X$, it equally well has vanishing cohomology in degree greater than the dimension of X, and the proof can be completed as above.

Because of this there is certainly no Poincaré duality, none of the Lefschetz theorems, and no comparison of $H^i(X, \mathbb{Q}_p)$ with the Betti numbers of a lifting to characteristic zero. Moreover, it need not even be the case that $H^i(X, \mathbb{Q}_p)$ is finite-dimensional for a proper smooth variety.

So really the p-adic étale cohomology groups have nothing to recommend them.

1.3 So what?

We can restate the title of this section a bit more carefully as follows: given that we can get a wonderful cohomology theory by taking any $\ell \neq p$, what does the pathological nature of p-adic étale cohomology mean we are missing out on?

Well, for one thing, the p-torsion. We begin by performing the following thought experiment: suppose that for a variety in characteristic zero, we were for some reason forbidden to use the 17-adic cohomology, but we could use the ℓ-adic cohomology for all $\ell \neq 17$. We would still know the Betti numbers, but we would *not* know the finitely generated groups $H^i(X(\mathbb{C}), \mathbb{Z})$ e.g. if $H^1(X(\mathbb{C}), \mathbb{Z}_l) = 0$ for all allowable ℓ, is $H^1(X(\mathbb{C}), \mathbb{Z}) = 0$, or is it equal to $\mathbb{Z}/17\mathbb{Z}$ (or to $(\mathbb{Z}/289\mathbb{Z})^3 \oplus (\mathbb{Z}/17)^5$ or $\ldots$). From a purely topological perspective this is unacceptable.

But this is exactly what happens when we work over a field of characteristic p. [2] Of course, this is a sort of philosophical reasoning by analogy that would make Plato proud ("Yes, of course Socrates, that's how it is for horses, so it must be exactly the same for men"). But there are reasons both obvious and deep to believe that there *is* "topological p-torsion in characteristic p", and many of the "pathologies" of algebraic geometry of characteristic p are related to p-torsion existing in a way which is not seen by étale cohomology.

This is most clear for (algebraic curves and) abelian varieties: let A/k be a g-dimensional principally polarized abelian variety over a field of characteristic p which can be lifted to characteristic zero.[3] As we discussed above, for every $\ell \neq p$, we have $\mathrm{Hom}(pi_1(X), \mathbb{Z}_l)) = H_1(X, \mathbb{Z}_l) = \mathbb{Z}_l^{2g}$, essentially due to the fact that the finite flat group schemes $A[\ell^n]$ are étale. But the finite flat group scheme $A[p]$ is *never* étale; indeed the (log to the base p of the) maximum rank of an étale subgroup scheme, a, is called the **a-number** of A, and satisfies $0 \leq a \leq g$, the case $a = g$ being called **ordinary**. It does *not* follow that we should ignore the nonétale part of $A[p]$ (and indeed of the p-divisible group) – e.g. when one is trying to calculate the image of a p-adic Galois representation of an $A/\mathbb{Q}$, the most important single case to examine is the local representation

[2] Okay, it's *exactly* what happens when we work over a field of characteristic 17, but allow a little poetic license.

[3] Every abelian variety can be lifted, by Hensel's Lemma and the smoothness of the Siegel moduli space over $\mathbb{Z}_p$.

at $\mathbb{Q}_p$, and we do not give up if the abelian variety has a-number zero!

Rather in this case, the theory of p-divisible groups provides a solution to this: we study the p-divisible group $A[p^\infty]$ of A via its **Dieudonné** module DA, which is a free $W(k)$-module of dimension $2g$ endowed with certain extra semi-linear structure (the F and V). In other words, this is exactly the sort of object that $H^1(A, \mathbb{Z}_l)$ (aka, up to duality, $T_l(A)$, the ℓ-adic Tate module) provides at every other prime! In fact it is quite standard in the theory of abelian varieties to define the "full Tate module"

$$TA = \prod_{\ell \neq p} T_l A \times \ DA.$$

In summary, in the case of an abelian variety in characteristic p, we know what "the first cohomology group with $W(k)$-coefficients" should be – it should be DA – and we know that this is *always* larger (in rank) than the étale cohomology group $H^1_{ét}(X, \mathbb{Z}_p)$.

But there is more to the story than this – indeed there must be, because I am quite literally here to tell you that one can study abelian varieties in characteristic p without needing a wholly new cohomology theory. Indeed, although we found torsion in the sense that the "first cohomology with coefficients in $\mathbb{Z}/p\mathbb{Z}$" should always have rank $2g$, because of the group structure the "first cohomology with coefficients in $\mathbb{Z}_p$" is torsionfree so in topological terms is captured by its Betti number b_1, which we already know how to compute in terms of $b_{1,\ell}$ for any other ℓ. On the other hand, there are varieties – with liftings to characteristic zero – for which the "true $H^1(X, \mathbb{Z}_p)$" has p-torsion, and this reification is justified by certain behavior of varieties in characteristic p which seems completely "pathological" compared to characteristic zero – for instance, (liftable) varieties which have $b_1 = 0$ but $H^1(X, \mathcal{O}_X) \neq 0$.

This brings up a key point: differentials, although purely algebraically defined, behave strangely in positive characteristic – for instance in characteristic zero one has $h^p(X, \Omega^q) = h^q(X, \Omega^p)$, but this can fail in characteristic p. This phenomenon is both the problem and the beginning of the cure: if X/k has a smooth lifting to $\mathcal{X}/\overline{W(k)}$, then $H^\bullet(\mathcal{X}, \Omega^p_{\mathcal{X}/\overline{W(k)}})$ is a graded algebra of $\overline{W(k)}$-modules, so (loosely speaking; as we shall discuss, we want really to take the hypercohomology) they are candidates for giving both p-torsion and, by tensoring with $\mathbb{Q}$, giving a p-adic Weil cohomology theory. The problems are twofold: i) Are these $\overline{W(k)}$-modules are independent of the lifting? ii) Is there always some (smooth!) lifting? It shall turn out that the answer to ii) is "no," and that this requires construction of a more complicated theory, the **crystalline cohomology**. After the fact, we shall see that the answer to i) is "yes," and that crystalline cohomology can be viewed as an "interpolation" of DeRham cohomology to nonliftable varieties.

In fact the issue of liftability itself is interesting and in turn related to further topological invariants in characteristic p: over the complex numbers one knows that the coherent sheaf cohomology groups $H^q(X, \Omega^p)$ are related to the DeRham cohomology groups by the **Hodge theorem**. This is expressed most baldly as a key "refinement" of the Betti numbers:

$$b_n = \bigoplus_{p+q=n} h^q(X(\mathbb{C}), \Omega^p).$$

The second real merit of crystalline cohomology – beyond its giving topological p-torsion a place to live – is its intimate relationship with Hodge theory and DeRham cohomology in characteristic p.

In order to further motivate the definition of crystalline cohomology, we are going to **review** the classical (i.e., complex-analytic and even differential-geometric) DeRham cohomology and Hodge theory for complex manifolds. We want to, at least on paper, describe enough of this theory so that the all-important notion of a family of Hodge structures varying over a base can at least be defined, as this is a basic and fundamental notion in both complex and arithmetic algebraic geometry (e.g., Shimura varieties arise as moduli of Hodge structures). We want to then show how these deep analytic theorems (due to Hodge, Atiyah, Kodaira,...) can be reformulated so as to make sense as statements completely algebraically – this will involve a discussion of spectral sequences and hypercohomology[4] Finally, we want to examine some examples – curves, abelian varieties, and surfaces – in characteristic p which show that what holds classically over $\mathbb{C}$ can fail in characteristic p, but that this failure is not random or "pathological" but related to the issues of nonliftability and topological p-torsion discussed above. We end our survey by mentioning a relatively recent (1987) result of Deligne-Illusie which gives a spectacular vindication of algebraic geometry in characteristic p.

One final remark about the Witt vectors: the alert reader will have noticed that the proposed field of our p-adic cohomology theory was not $\mathbb{Q}_p$ but the algebraically closed field $\overline{W(k)}$ – unless we choose k to have absolute transcendence degree greater than the cardinality of the continuum, this field is abstractly isomorphic to $\mathbb{C}$. The "bar" is just to make good on our promise only to consider varieties over algebraically closed fields: DeRham cohomology (and crystalline cohomology) is defined relative to any field k of characteristic p and the coefficients live in $W(k)$. Thus, what can be done arithmetically over say $\mathbb{F}_p$ can be done in crystalline cohomology over $W(\mathbb{F}_p) = \mathbb{Z}_p$. This functoriality in the field of definition is key, because in fact there is no $\mathbb{Q}_p$-Weil cohomology theory in characteristic p:

Proposition 7 *(Serre) There is no Weil cohomology theory for varieties in characteristic p with coefficients in any of the following fields: $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{R}$.*

[4] Don't worry; there will be no derived categories.

Proof: Let H be a K-valued Weil cohomology theory, and consider $H^1(E)$, where E/k is a supersingular elliptic curve. A characterization of supersingular elliptic curves is that their endomorphism algebra $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a quaternion algebra B over $\mathbb{Q}$, ramified at precisely the places p and ∞ (in other words, $B \otimes \mathbb{Q}_p$ and $B \otimes \mathbb{R}$ are still division algebras, but for any finite l different from p, $B \otimes \mathbb{Q}_l \cong M_2(\mathbb{Q}_l)$.) Since every elliptic curve can be lifted to characteristic zero – e.g., just lift the j-invariant – and in characteristic zero the first Betti number of an elliptic curve is 2, our Weil cohomology theory must have $H^1(E)$ a 2-dimensional K-vector space, and by functoriality we get a unital, hence faithful as B is simple, representation $B \otimes K \hookrightarrow \mathrm{End}(H^1(K)) \cong M_2(K)$. This embedding of four-dimensional K-algebras must be an isomorphism – in other words, K is a splitting field for B. As discussed, this rules out $K = \mathbb{Q},\ \mathbb{Q}_p,\ \mathbb{R}$.

Chapter 2

Some geometry of sheaves

2.1 The exponential sequence on a $\mathbb{C}$-manifold

Let X be a complex manifold. An amazing amount of geometry of X is encoded in the long exact cohomology sequence of the **exponential sequence** of sheaves on X:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \stackrel{\exp}{\to} \mathcal{O}_X^\times \to 0,$$

where exp takes a holomorphic function f on an open subset U to the invertible holomorphic function $exp(f) := e^{(2\pi i)f}$ on U; notice that the kernel is the constant sheaf on $\mathbb{Z}$, and that the exponential map is surjective as a morphism of sheaves because every holomorphic function on a polydisk has a logarithm. Taking sheaf cohomology we get

$$0 \to \mathbb{Z} \to H(X) \stackrel{\exp}{\to} H(X)^\times \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^\times) \to H^2(X,\mathbb{Z}),$$

where we have written $H(X)$ for the ring of global holomorphic functions on X. Now let us reap the benefits:

I. Because of the exactness at $H(X)^\times$, we see that any nowhere vanishing holomorphic function on any simply connected $\mathbb{C}$-manifold has a logarithm – even in the complex plane, this is a nontrivial result.

From now on, assume that X is compact – in particular it homeomorphic to a finite CW complex, so its Betti numbers $b_i(X) = \dim_{\mathbb{Q}} H^i(X,\mathbb{Q})$ are finite. This also implies [Cartan-Serre] that $h^i(X,F) = \dim_{\mathbb{C}} H^i(X,F)$ is finite for all **coherent analytic sheaves** on X, i.e. locally on X F fits into an exact sequence $\mathcal{O}_U^m \to \mathcal{O}_U^n \to F \to 0$. Especially the **Hodge numbers** $h^{p,q} = h^p(X,\Omega_X^q)$ are finite.

II. $H^1(X,\mathcal{O}^\times)$ is the **Picard group** of holomorphic line bundles on X. The map $c : H^1(X,\mathcal{O}_X^\times) \to H^2(X,\mathbb{Z})$ is the **Chern class map**; the image of c modulo torsion is the **Néron-Severi group** $NS(X) \cong \mathbb{Z}^\rho$, which classifies line bundles up to **algebraic equivalence**; one says that ρ is the **Picard number** and that $b_2 - \rho$ is the **number of transcendental cycles**. Line bundles in the kernel of c are said to be **algebraically equivalent to zero** and this subgroup of $\mathrm{Pic}(X)$ is denoted by $\mathrm{Pic}^0(X)$. From the exact sequence we see $\mathrm{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$; In case we have $2h^{0,1} = b_1$ (which will occur if X is Kahler) we are modding out a $\mathbb{C}$-vector space by a full sublattice – that is, the Picard group is a **complex torus**. Notice that if X is simply connected (and Kahler) then $\mathrm{Pic}^0(X) = 0$, and the group of line bundles is just the finitely generated discrete group $H^2(X, \mathbb{Z})$; this occurs e.g. for $X = \mathbb{P}^n$. Finally, if X is Kahler then we will see in the next chapter that we have a Hodge decomposition

$$H^2(X, \mathbb{C}) = H^2(X, \mathcal{O}_X) + H^0(X, \Omega_X^2) + H^1(X, \Omega_X).$$

The N'eron-Severi group canonically is a $\mathbb{Z}$-lattice in $H^2(X, \mathbb{C})$, and it is contained in the $(1,1)$-subspace [Griffiths-Harris ??]; thus the Picard inequality can be refined to $\rho \leq h^{1,1}$.[1] Moreover one has (still in the Kahler case) that $H^1(X, \Omega_X) \cap (\mathrm{Im}(H^2(X, \mathbb{Z})) = NS(X)$, i.e., every integral $(1,1)$-form comes from the Picard group, the **Lefschetz (1,1) Theorem**. More generally, to an i-cycle Z on X – i.e., a $\mathbb{Z}$-linear combination of closed analytic subsets of dimension i – we can associate, via a triangulation and Poincaré duality, a cohomology class $c(Z) \in H^{2d-2i}(X, \mathbb{Z})$, and one finds that the image of $c(Z) \in H^{2d-2i}(X, \mathbb{C})$ lands in the $(d-i, d-i)$-subspace. Suppose finally that X is projective. It is not quite true that every integral $(d-i, d-i)$-class needs to be represented by an algebraic cycle, but this is supposed to be true "with denominators": i.e., every element of $H^{(d-i),(d-i)} \cap H^{2d-2i}(X, \mathbb{Q})$ should be a $\mathbb{Q}$-linear combination of algebraic cycles; this is the **Hodge conjecture**.[2]

2.2 Fiber bundles, locally constant sheaves, monodromy

2.2.1 Fiber bundles as an example of descent

We review the notion of an (F, G)-bundle on a topological space, the classification via sheaf cohomology, and the special place that locally constant sheaves have among fiber bundles.

Let F be another topological space. A map $\pi : E \to X$ is said to be an F-fiber bundle over X if there is an open covering $\{U_i\}$ of X such that π_{U_i} is isomorphic, over X, to the product $F \times U_i$; such an isomorphism is called a **local trivialization** $\varphi_i : E_{U_i} \to F \times U_i$. The "data" for a fiber bundle are

[1] One still need not have equality; e.g. for any d-dimensional complex abelian variety, one has $h^{1,1} = b_2 - 2h^{1,0} = \binom{2d}{2} - 2d$, whereas the rank of the Néron-Severi group is characterized in terms of the endomorphism algebra (the Rosati-invariant subalgebra), so is generically just 1 but can be as large as $\frac{d(d+1)}{2}$, attained when A is the dth power of a CM elliptic curve.

[2] Apparently the Hodge conjecture was first formulated in its integral version but was proven false with embarrassing swiftness. I do not know the full story nor even the counterexample.

its **transition functions**: namely on the overlaps $U_i \cap U_j$ we may consider the composite $\rho_{ij} := \varphi_j \circ \varphi_i^{-1} : U_i \cap U_j \to \mathrm{Aut}(F)$. The compatibility among triple intersections is equivalent to ρ being a **one-cocycle** in $Z^1(X, \mathrm{Aut}(F)_c)$; this is a (nonabelian) Cech cohomology group, and if G is any topological group, by G_c we mean the sheaf of **continuous** G-valued functions on X. This has been formulated in the topological category, but is easily modified: if X is a real manifold and G a real Lie group, we can work with G_∞, the sheaf of smooth functions $X \to G$; if X is a complex manifold and G a complex Lie group, we can work with G_h, the sheaf of holomorphic functions $X \to G$. (Unless we are considering more than one of these categories at once, we may abusively write just G, trusting that the context will make clear whether we are working with continuous, smooth or holomorphic functions.)

On the other hand, we probably do not want the transition functions to be *arbitrary* automorphisms of F – for instance if $F = \mathbb{R}^n$ its automorphism group is an enormous (infinite-dimensional) space. This leads to (F, G)-bundles: we prescribing a **structure group** $G \leq \mathrm{Aut}(F)$ and requiring the transition functions to lie in G. It is not news, but this simple idea is miraculous in its range of applicability. For instance if $F = \mathbb{R}^n$ and we want to get real vector bundles, we take $G = \mathrm{GL}_n(\mathbb{R})$; similarly if $F = \mathbb{C}^n$ we get complex vector bundles; if $G = \mathrm{GL}_n^+(\mathbb{R})$ we get oriented vector bundles; if $G = SO_n(\mathbb{R})$ we get oriented vector bundles endowed with a Riemannian metric, and so on. The basic result is as follows:

Proposition 8 *The set of (F, G)-bundles on X is naturally in bijection with $H^1(X, G_c)$; under the correspondence the trivial F-bundle corresponds to the identity cocycle.*

Given an acquaintance with Cech cohomology (we are passing to the direct limit over refinements of covers, of course), the proof is almost immediate: we have associated a Cech class to a fiber bundle; conversely, given a cocycle $\rho_{ij} \in Z^1(\{U_i\}, G_c)$, we form the space $\coprod_i U_i \times F$ and mod out by $(u, f) \sim (u, g_{ij}(u)f)$ whenever $u \in U_i \cap U_j$. One striking aspect of the correspondence is that the fiber F appears on one side but not on the other! One take on this is that it is enough to consider **principal** bundles, i.e., where the fiber $F = G$ acting (left or right; one must choose) regularly on itself.

Another viewpoint is that we have an instance of what (following Serre in the case of Galois cohomology) I call the **first principle of descent**: we start with an "object" F_0 on a "space" X (here we have a topological space; for algebraic purposes probably the best example is the flat site of a scheme, e.g. $\mathrm{Spec}\, k$!), and a covering $\{U_i\}$ of X. Let $Y = \coprod U_i$; there is a natural surjective local homeomorphism[3] $\pi : Y \to X$. An object F on X such that $\pi^* F \cong \pi^* F_0$ is called a **twisted form** of F_0; denote by $\mathcal{T}_{Y/X}(F_0)$ the space of all twisted forms

[3] which is not necessarily a covering map – it need not be "flat," i.e., the fibers may have different cardinalities

which are trivialized over Y. In our case every (F,G)-bundle E over X admits a covering such that the pullback to Y is equal to the pullback to Y of the trivial bundle. Then:

Proposition 9 *(Descent principle) The pointed set $\mathcal{T}_{Y/X}(F_0)$ of Y/X-twisted forms is naturally in bijection with $H^1(Y/X, \mathrm{Aut}(\pi^* F_0))$.*

Corollary 10 *Any two objects F_0 and G_0 on X – however dissimilar! – such that $\mathrm{Aut}(\pi^* F_0) \cong \mathrm{Aut}(\pi^* G_0)$ will have bijectively corresponding sets of Y/X-twisted forms: $\mathcal{T}_{Y/X}(F_0) \cong \mathcal{T}_{Y/X}(F_0)$.*

Here is an application of this:

Let X be a (real or complex) manifold, and consider the set of finite rank projective $\mathcal{O}_X$-modules. One can interpret "projective" purely algebraically: for all open subsets U, $M(U)$ is a finitely generated module over the ring $\mathcal{O}_X(U)$. Also as a matter of pure algebra, every finite rank projective module over a commutative ring R becomes free over a Zariski-open subset of every point of R – this is more than enough to ensure the existence of an open cover $\{U_i\}$ such that the pullback of M to $\coprod U_i$ is a free $\mathcal{O}_X$-module. So the set of rank n projective $\mathcal{O}_X$-modules is classified by $H^1(X, \underline{\mathrm{Aut}}(\mathcal{O}_X^n))$, where $\underline{\mathrm{Aut}}(M) = \underline{\mathrm{End}}_{\mathcal{O}_X-Mod}(M,M)^\times$ is the sheaf of automorphisms of the $\mathcal{O}_X$-module M, i.e., over any open subset U we take the $\mathcal{O}_U$-module automorphisms of $M|U$. We have that $\underline{\mathrm{Aut}}(\mathcal{O}_X^n) = \mathrm{GL}_n(\mathcal{O}_X) = (\mathrm{GL}_n)_h$. Because this is the same automorphism group for a rank n holomorphic vector bundle, we conclude:

Proposition 11 *On any (real or complex) manifold X, there is a canonical bijection of pointed sets between rank n projective $\mathcal{O}_X$-modules and (real or holomorphic) rank n vector bundles on X.*

Of course, in such a situation, one would like to have an explicit bijection. The descent principle does not tell us how to write down such a bijection (but assures us that we will find one, which gives us the motivation to look). In this case it is easy to go from a vector bundle to a locally free sheaf: we just take the sheaf of local sections. The inverse is not as transparent – see [Hartshorne, pp. 128-129].

We mention in passing one more example: let X be a scheme and $G = \mathrm{PGL}_{nX}$, considered as a representable sheaf on the étale site of X. Since PGL_n is the common automorphism group of both $\mathbb{P}^{n-1}_X$ and $M_n(X)_X$ (matrix algebra bundle), we find a canonical correspondence between **projective bundles** on X and bundles of central simple algebras on X, i.e. **Azumaya algebras**. This leads to the interpretation of the **Brauer group of** X as classifying both geometric and algebraic objects on X. For more details, see either [Grothendieck I,II,III] or (for a much-abbreviated version) [Clark].

A look at complex line bundles: suppose we want to study complex line bundles on a real manifold X. If we (temporarily) write $\mathcal{O}_X$ for the sheaf of $\mathbb{C}$-valued

C^∞-functions on X, then we still have the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$: even a smooth function nonzero at a point is, locally about that point, the exponential of another smooth function. But since $\mathcal{O}_X$ is **fine**, it is acyclic for sheaf cohomology, and the cohomology of the exponential sequence gives an isomorphism

$$c_1 : H^1(X, \mathcal{O}_X^\times) \xrightarrow{\sim} H^2(X, \mathbb{Z}).$$

That is, in the smooth category, a complex line bundle is determined by its Chern class.

When X is a complex manifold, this need not be the case, as we saw in the previous section: the kernel of c_1 is the complex torus $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$, whose dimension is the first Betti number $b_1(X)$. Thus we see that $c_1(L)$ determines L if and only if $b_1(X) = 0$, in particular if X is simply connected.

2.2.2 Locally constant sheaves

The aim of this section is to define locally constant sheaves, understand their relation to fiber bundles, and give their monodromy classification.

Definition: A **locally constant sheaf** of abelian groups F on X is a sheaf for which there admits a cover $\{U_i\}$ of X such that $F|_{U_i}$ is isomorphic to a constant sheaf. If X is connected, the stalks of a locally constant sheaf are mutually isomorphic to a common abelian group Λ.

Relation with fiber bundles: we can use the principle of descent to associate a fiber bundle to a locally constant sheaf: A locally constant sheaf with group Λ is a twisted form of the constant sheaf with group Λ, whose sheaf of automorphisms is just $\mathrm{Aut}(\Lambda)_c$. However, since Λ is merely an "abstract" abelian group, $\mathrm{Aut}(\Lambda)$ is given the **discrete** topology, and $\mathrm{Aut}(\Lambda)_c$ means the sheaf of **locally constant** functions from X to the group $\mathrm{Aut}(\Lambda)$. We can similarly speak of locally constant sheaves with G-structure, where $G \leq \mathrm{Aut}(\Lambda)$ − in particular taking, $\Lambda = \mathbb{R}^n$ or $\mathbb{C}^n$ and $G = \mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$, we have a notion of locally constant sheaves of vector spaces.

Write $\underline{G}$ for the constant sheaf with group G. Since $\underline{\mathrm{GL}_n} \hookrightarrow (\mathrm{GL}_n)_c$ (or $(\mathrm{GL}_n)_\infty$ or $(\mathrm{GL}_n)_h$), the mapping

$$H^1(X, \underline{\mathrm{GL}_n}) \to H^1(X, (\mathrm{GL}_{n\ c/\infty/h})$$

shows that any locally constant sheaf of vector spaces can be viewed as a continuous, smooth or holomorphic vector bundle on X, albeit one of a very special form.

Example: Consider the sheaf of differentials Ω^1 on $\mathbb{P}^1/\mathbb{C}$ as in [Hartshorne], [C-K]. In terms of transition functions, it is given by the standard covering

$U_1 = \mathbb{P}^1 - \infty = \mathbb{A}^1[x]$, $U_2 = \mathbb{P}^1 - 0 = \mathbb{A}^1[y]$ and with transition function $\rho_{12} : U_1 \cap U_2 \to \mathcal{O}^\times_{U_1 \cap U_2}$ given by $-1/x^2$. This is not a locally constant function! Moreover, since its divisor has degree -2, the line bundle is nontrivial. We will soon see that there are no nontrivial locally constant sheaves on $\mathbb{P}^1(\mathbb{C}) = S^2$, so that the line bundle Ω^1 cannot be given by locally constant transition functions.

We should also give an example of a locally constant sheaf that is not constant! Let $X = S^1$, and let $E \to X$ be the Mobius band, which *a priori* is a real line bundle on X in the broader sense of the previous section, famously nontrivial. But the structure group can be reduced from $\mathbb{R}^\times$ to $\mathbb{Z}/2\mathbb{Z}$, i.e., it is a locally constant sheaf. This is also well known and easy to check: indeed when we make the Mobius band out of two strips U_1 and U_2 with two components of intersection $U_1 \cap U_2 = V_{12}^1 \coprod V_{12}^2$, at one end we glue U_1 to U_2 identically, and at the other hand we glue by a uniform half twist – i.e., the first transition function is 1 and the second is -1.

Of course we could untwist the Mobius band by pulling back via $z^2 : S^1 \to S^1$, and this leads us to suspect that the reason there are no nonconstant locally constant sheaves on S^2 is that it is simply connected. This is true and leads us directly to the considerations of the next section.

2.2.3 Monodromy

In this section, we will need covering space theory to be applicable to X, so we suppose that X is connected, locally path-connected and semi-locally simply connected – in particular, it has a universal cover $\tilde{X} \to X$.

Let $f : X \to Y$ be a continuous map of topological spaces and F a locally constant sheaf on Y. Then f^*F is a locally constant sheaf on X.

Let F be a locally constant sheaf on X with fibers isomorphic to Λ. Fix a basepoint $x \in X$, and let $\gamma : [0, 1] \to X$ be a loop based at x. Then by the remark, γ^*F is a locally constant sheaf on $[0, 1]$. But we claim that any locally constant sheaf on the unit interval is constant. By an immediate compactness argument, it comes down to showing: if we have a sheaf F on $I = I_1 \cup I_2$ a union of overlapping intervals such that $F|_{I_1}$ and $F|_{I_2}$ are both isomorphic to constant sheaves, then so is F itself. But this is itself a kind of descent argument, involving the familiar ([Hartshorne], [Alon])

Lemma 12 *(Glueing lemma) If $\{U_i\}$ is an open cover of X and we have sheaves F_i on each U_i and the data of an isomorphism $\varphi_{ij} : F_i|_{U_i \cap U_j} \to F_j|_{U_i \cap U_j}$ satisfying the conditions $\varphi_{ii} = 1$, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$, then there is a unique sheaf F on X together with isomorphisms $\psi_i : F|_{U_i} \xrightarrow{\sim} F_i$ such that $\psi_j \circ \psi_i^{-1} = \varphi_{ij}$.*

We leave it to the reader to check that the glueing lemma implies the following generalization of our claim: let $X = U_1 \cup U_2$ such that $U_1 \cap U_2$ is connected. Then a sheaf F on X which restricts to a constant sheaf on U_1 and on U_2 is already constant on X.

Back to the case of our loop $\gamma : [0,1] \to X$. We now know that there is an isomorphism $\Psi : \gamma^* F \cong \underline{\Lambda}_{[0,1]}$: in particular we have $\Psi(0) : (\Gamma^* F)_0 \cong \Lambda$ and $\Psi(1) : (\Gamma^* F)_1 \cong \Lambda$. On the other hand, the stalks at 0 and 1 are identified with the stalks at $\gamma(0) = x = \gamma(1)$. It follows that the trivialization Ψ gives rise to an automorphism $\Psi(1) \circ \Psi(0)^{-1}$ of the stalk of F at x.

Exercise: Suppose $\gamma_1 \sim \gamma_2$ are homotopic paths. Show that the induced automorphisms are the same. (Hint: View the homotopy as giving a morphism $[0,1] \times [0,1] \to X$, divide the square into sufficiently small nicely overlapping squares on which the pulled back sheaf is constant, and argue as in the previous exercise.)

It follows that we have defined a homomorphism $\pi_1(X, x) \to \mathrm{Aut}(F_x)$, called the **monodromy representation**.

Theorem 13 *The monodromy representation gives a categorical equivalence between G-structured Λ-locally constant sheaves on X and G-compatible $\pi_1(X)$-module structures on Λ.*

Proof: We shall construct the inverse functor. Our hypotheses are such as to ensure that there is a universal cover $\tilde{X} \to X$, and since the theorem, if true, implies that the pullback of any locally constant sheaf to $\tilde{X}$ will be constant, this suggests our strategy: given the data of $\rho : \pi_1(X, x) \to \mathrm{Aut}\,\Lambda$, we will construct a locally constant sheaf F_ρ on X by descent from a constant sheaf on $\tilde{X}$. Indeed, let $\tilde{E} = \tilde{X} \times \Lambda$ (Λ is viewed as a discrete space), and consider the quotient space $E := \tilde{E}/\sim$, where $(x, f) \sim (gx, gf)$ for all $g \in \pi_1(X, x)$. Clearly projection onto the first factor gives a map $\pi : E \to X$. Since $\pi_1(X, x)$ acts discretely on $\tilde{X}$, every point of $\tilde{X}$ has a neighborhood $\tilde{U}$ such that $\tilde{U} \times F$ is mapped homeomorphically onto its image – i.e., $\pi : E \to X$ is a fiber bundle over F. We leave it as an exercise to show that the associated sheaf of local sections to π is a locally constant sheaf, and that this construction is indeed inverse to our association of a representation to a locally constant sheaf.

Remark: It is useful to recall that there were two steps to the proof of the monodromy theorem, the first being an argument that every locally constant fiber bundle became trivial when pulled back to the universal cover, and the second being an interpretation of such bundles as being equivalent to $(G\text{-})\pi_1(X)$-module structures on Λ. Note also that it is certainly not always the case that a fiber bundle on X must trivialize on its universal cover (again recall Ω_1 on $\mathbb{P}^1(\mathbb{C}) = S^2$), but this happens often enough that it is worth abstracting the second part of the argument as follows:

Proposition 14 *Let X be a topological space with universal cover $\pi : \tilde{X} \to X$ and fundamental group $\mathfrak{g}$. Then the pointed set of (topological, smooth or* *holomorphic) (F,G)-bundles which trivialize over $\tilde{X}$ is isomorphic to the group cohomology set $H^1(\mathfrak{g}, \pi^*(G))$.*

Compare with [Mumford, pp. 22-23] for an analogoue valid for all sheaves $\mathcal{F}$ on X (but with a somewhat weaker conclusion). The point is that the automorphism group (sheaf) G of a locally constant sheaf is a trivial $\mathfrak{g}$-module, but this is not necessarily true for more general fiber bundles.

Example: In the topological category, every fiber bundle over a contractible paracompact base is trivial, a consequence of the following basic result.

Theorem 15 *(Covering Homotopy Theorem) Let $\pi : E \to Y$ be an (F,G)-bundle over a paracompact base. Let $g_1,\ g_2 : X \to Y$ be homotopic maps. Then the pullbacks $g_1^*\pi$ and $g_2^*\pi$ are isomorphic. In particular, every (F,G)-bundle over a contractible base is trivial.*

For the proof see e.g. [Milnor-Stasheff]. It follows that (F,G)-bundles over a $K(\pi,1)$-space[4] are classified by $\mathrm{Hom}(\pi, G)$. There is, up to homotopy equivalence, a unique $K(\pi,1)$-space for each group π, but we are rather lucky if it is finite-dimensional – for instance, there is an isomorphism $H^\bullet(\pi,\Lambda) \cong H^\bullet(K(\pi,1),\Lambda)$ from the group cohomology of π (with coefficients in the trivial π-module Λ) to the singular cohomology of the Eilenberg-MacLane space [Brown], so for instance, for any $n > 1$, $H^k(K(\mathbb{Z}/n\mathbb{Z},1),\mathbb{Z}/n\mathbb{Z}) \neq 0$ for every even k. (One knows that in fact $K(\mathbb{Z}/2\mathbb{Z},1) = \mathbb{R}P^\infty = \varinjlim_n \mathbb{R}P^n$, and the other $K(\mathbb{Z}/n\mathbb{Z},1)$'s are infinite-dimensional "lens spaces.") But for all n, $T^n := S^1 \times \ldots \times S^1$ is $K(\mathbb{Z}^n,1)$. It follows that the (F,G)-bundles on an n-dimensional real torus are classified by $\mathrm{Hom}(Z^n, G) = G^n$.

Example: A coherent analytic sheaf on a **Stein manifold** is acyclic for sheaf cohomology; this is a theorem due to Serre which is the analytic analogue (but proved first!) of the acyclity of coherent sheaves on affine varieties. In particular, a nonsingular affine analytic space is a Stein manifold – so $\mathbb{C}^n$ is a Stein manifold. If we further assume that $b_1(X) = h_1(X,\mathbb{Z}) = 0$, then the exponential sequence gives $H^1(X, \mathcal{O}_X^\times) = 0$, whence:

Proposition 16 *Let X be a complex manifold with fundamental group $\mathfrak{g}$ whose universal cover $\tilde{X}$ is Stein. Then holomorphic line bundles on X are classified by the group cohomology group $H^1(\mathfrak{g}, \mathcal{O}_{\tilde{X}}^\times)$.*

In particular this result applies to **complex tori**, and we get the fact that we can represent any line bundle on $\mathbb{C}^n/\Lambda$ as a collection of functions $\Lambda \to \mathcal{O}_{\mathbb{C}^n}^\times$ satisfying the cocycle condition, i.e., by **theta functions**.

Consider now the place of locally constant line bundles on an abelian variety among all line bundles: these are given by homomorphisms $\Lambda \to \mathbb{C}^\times$. The

[4]A space with a contractible universal cover and fundamental group π. These are also called **Eilenberg-MacLane** spaces.

group of all such is just $(\mathbb{C}^\times)^{2n}$, which is not quite what we expect to see. The problem is that the map which associates to a locally constant line bundle its associated holomorphic line bundle is neither surjective nor injective. To repair matters, one considers the composite "change of structure groups"

$$S^1 \to \mathbb{C}^\times \to H(\mathbb{C}^n)^\times,$$

and it turns out that the image in the Picard group of $\mathbb{C}^n/\Lambda$ of the locally constant sheaves with structure group S^1 coincides with the image of the locally constant sheaves with structure group $\mathbb{C}^\times$, and moreover the map $T^{2n} = \mathrm{H}^1(\Lambda, S^1) \to H^1(\Lambda, \mathcal{O}_{\mathbb{C}^n}^\times)$ is *injective*; indeed the image of the composite is precisely the Picard variety of line bundles algebraically equivalent to zero. These statements are not immediate; rather, they are much of the content of the **Appell-Humbert theorem** classifying line bundles on a complex torus. We will however be able to see later that every locally constant line bundle on a complex manifold has vanishing Chern class, by showing that it admits a flat connection. This brings us to the next section.

2.3 Flat connections, especially Gauss-Manin

Let $E \to X$ be a (say complex) vector bundle on a real manifold X. In this section, we plunge to the core of differential geometry (but of course for our own nefarious, ultimately algebraic, purposes) by defining a **connection** on E: it is a $\mathbb{C}$-linear morphism of sheaves $E \to \Omega^1(E) := \Omega^1 \otimes E$ satisfying the Leibniz rule

$$D(fg) = f \cdot dg + gD(f),$$

where d denotes the usual exterior derivative. Note that D is of course **not** an $\mathcal{O}_X$-module map: the special case to keep in mind is the trivial line bundle $L_0 \cong \mathcal{O}_X$ on X; then d itself gives a connection on L_0, and differentiation is by its nature $\mathbb{C}$-linear but not $\mathcal{O}_X$-linear.

The matrix of one-forms: it is quite easy to write down connections locally. Namely, over any trivializing open subset U for E, choose a local frame $e = (e_1, \ldots, e_d)$ – i.e., sections $e_i \in \Gamma(U, E)$ such that $e_1 \wedge \ldots \wedge e_d$ is a nowhere vanishing section of the line bundle $\Gamma(U, \Lambda^d E)$. Because of the Leibniz rule, D is determined by its action on e, and can in these local coordinates be given simply by a $d \times d$ matrix with entries in Ω^1, via

$$De_i = \sum_{j=1}^{d} \theta_{ij} e_i.$$

Viewing $E|_U = \mathcal{O}_{XU}^d$ via e, and writing a section $s \in \Gamma(U, E)$ in vector form as $s = \sum_{i=1}^d s_i e_i = s_e \cdot e$, a short calculation gives the matrix equation

$$Ds_e = (d + \theta)s_e.$$

There is no condition to be imposed on the matrix θ, so indeed every connection can be given in local coordinates as $d + M$ where $M \in M_d(\Omega^1)$. In particular, the difference of any two connections is an $\mathcal{O}_X$-linear map.

Globally, every vector bundle over a paracompact base admits a connection: in local coordinates we can take $D = d$, and smooth via a partition of unity.

The curvature matrix: It is defined in local coordinates e on U as

$$\Theta = \Theta_{D,e} = d\theta + \theta \wedge \theta,$$

i.e., it is an $n \times n$-matrix of two-forms on U. We have the equation

$$(d + \theta_e)(d + \theta_e)s_e = \Theta s_e.$$

In other words, the curvature matrix gives an $\mathcal{O}_X$-linear map $E \to \Omega^2(E)$ which looks for all the world like $D \circ D$. Indeed it is as soon as we extend the connection to a map $D : \Omega^i(E) \to \Omega^{i+1}(E)$, via

$$D(\eta_e) = d(\eta_e) + \theta_e \wedge \eta_e,$$

or globally by continuing to enforce the Leibniz rule:

$$D(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge D(\eta).$$

We say the connection D is **flat** if $\Theta = 0$.[5]

There is a clear "formal" reason to be interested in flat connections: it says precisely that the sequence

$$0 \to E \xrightarrow{D} \Omega^1(E) \xrightarrow{D} \Omega^2(E) \xrightarrow{D} \dots \tag{2.1}$$

is a **complex** of sheaves on X, a kind of generalized DeRham complex giving a resolution of the vector bundle E. We will see in Chaper 2 that the Hodge theorem can be generalized to a theorem about such a complex of sheaves.

But there are more immediate, geometric reasons to be interested in flat connections: suppose that E is a line bundle, so that Θ is just a 2-form on X. Notice that it is closed: $\theta \wedge \theta = -\theta \wedge \theta = 0$, so $d\Theta = d(d\theta + \theta \wedge \theta) = 0$. Therefore, via the DeRham theorem, $\Theta \in H^2(X, \mathbb{R})$.

Proposition 17 *Suppose that $E \to X$ is a complex line bundle on a real manifold X. The curvature two-form Θ of a line bundle E is the image of the Chern class $c_1(E)$ under the natural map $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$.*

[5] One also says that a connection D with $D^2 = 0$ is **integrable**, which could be preferred on the grounds that it uses a term not already ubiquitous in algebraic geometry (beware: every vector bundle is a flat module!) On the other hand, I think the differential geometers have us beat on this point: calling something which has zero curvature flat makes more sense than calling something for which tensoring with that thing is exact flat.

For a proof of this, see [Griffiths-Harris] or [Wells]. In particular, a line bundle is algebraically equivalent to zero if and only if it admits a flat connection.

More generally, (almost) the entire theory of characteristic (Chern) classes of complex vector bundles on a real manifold in terms of the curvature matrix Θ. The key observation is that, if (E, D) has fiber dimension > 1, it need not be that $d\Theta = 0$ but if $P : M_n(\mathbb{C}) \to \mathbb{C}$ is any polynomial function in the matrix entries with the invariance property $P(YXY^{-1}) = P(X)$ for all matrices X and Y, then $dP(\Theta) = 0$, so that $P(\Theta) \in H^{\bullet}_{DR}(X)$. Taking $P = \sigma_k$, the kth elementary symmetric function of the eigenvalues, gives the kth Chern class $c_k(E) \in H^{2k}_{DR}(X)$, up to a scaling. The reason for the "almost" in the first sentence of this paragraph, is: since $H^{2k}_{DR}(X) \cong H^{2k}(X, \mathbb{R})$, there is a slight loss of information over the topologically defined Chern classes $c_k \in H^{2k}(X, \mathbb{Z})$: a vector bundle which admits a flat connection is such that all of its topological Chern classes are *torsion*, but not in general identically zero. For all this, see Appendix C of [Milnor-Stasheff].

2.3.1 Flat connections versus locally constant sheaves

Let us look once again at the relationship between locally constant sheaves and fiber bundles. On the geometric side – or in terms of transition functions – we saw that a locally constant sheaf of complex vector spaces *is* a vector bundle with an impressively small structure group. On the sheaf side, this is not quite true: by definition, the stalks of a $\Lambda = \mathbb{C}^n$-locally constant sheaf are all isomorphic to $\mathbb{C}^n$, whereas the corresponding locally free sheaf has stalk at P isomorphic to the much larger group $\mathcal{O}^n_{X,P}$. But this is easily remedied: to go from a $\mathbb{C}^n$-locally constant sheaf F on X to the locally free sheaf corresponding to the corresponding vector bundle, we just take

$$F \mapsto F \otimes_{\mathbb{C}} \mathcal{O}_X.$$

(Depending upon what we mean by $\mathcal{O}_X$, this makes sense and is correct in the topological, smooth, and holomorphic categories.)

But now we have another instance of descent: because it came from F, $\mathcal{F} = F \otimes \mathcal{O}_X$ can be canonically endowed with a connection: namely, in local coordinates, we take $\sigma = \sum s_i e_i \in \mathcal{F}$, and define

$$D(\sigma) := \sum ds_i \otimes e_i \in \mathcal{F} \otimes \Omega_X.$$

The point being: this expression is independent of the coordinates, because any other trivialization is obtained from the first by a transition matrix with constant coefficients; such changes of variables are d-linear.

Moreover this connection D is flat, since in local coordinates it is just d, and indeed $d^2 = 0$. We now get the result that we mentioned earlier during our discussion of line bundles on abelian varieties:

Corollary 18 *A locally constant line bundle on a complex manifold is algebraically equivalent to zero.*

Conversely, if $(E \to X,\ D)$ is a vector bundle endowed with a flat connection, we define a sheaf F_D by taking for $F_D(U)$ the **horizontal** sections over U, namely the kernel of $D|_U$. One can show (using Frobenius' integrability criterion for distributions; see [Voisin]) that F_D is a $\mathbb{C}^n$-locally constant sheaf. These two constructions are mutually inverse to each other, i.e., one has:

Proposition 19 *There is a bijective correspondence (in either the smooth or the holomorphic category) between vector bundles on a (real/complex) manifold X endowed with a flat connection and $\mathbb{C}^n$-locally constant sheaves on X.*

Note that one important consequence of this is that a *smooth* complex vector bundle which can be endowed with a flat connection necessarily admits a canonical structure of a *holomorphic* vector bundle.

2.3.2 The Gauss-Manin connection

Let $\pi : X \to B$ be a proper submersion of (smooth or complex) manifolds. The implicit function theorem guarantees that the fibers $\pi^{-1}(b)$ are themselves manifolds, allowing us to think of π as giving a **family** of manifolds over the base B. In fact, our hypotheses ensure that, in the smooth category, we have a complete understanding of the local behavior of such a family:

Theorem 20 *(Ehresmann Lemma) Let $\pi : X \to B$ be a proper smooth submersion of real manifolds over a contractible pointed base $(B, 0)$, and write $X_0 := \pi^{-1}(0)$. Then there exists a diffeomorphism over the base $T : X \xrightarrow{\sim} X_0 \times B$.*

Actually we need this result only locally, where it is a special case of the existence of tubular neighborhoods. For a proof of the global case (involving some differential topology), see [Demailly].

In other words, in the smooth category all such families are locally constant (aha!). Imagine now a holomorphic family satisfying the same hypotheses; of course it need not be holomorphically locally trivial (there are moduli spaces, after all), but the fact that it is smoothly locally trivial allows us to view the family as a **deformation** of the complex structure on a fixed fiber.

Let $\underline{A}$ be the constant sheaf on X for some group A (think of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\ldots$). Let $H^k_A := R^k\pi_*(\underline{A})$, the kth derived functor of the pushforward. One knows that $R^k\pi_*(F)$ is the sheaf associated to $U \mapsto H^k(\pi^{-1}, F|_{\pi^{-1}(U)})$. Since B is locally contractible, the Ehresmann Lemma implies that $H^k(X_0 \times U, \underline{A}) \cong H^k(X_0, \underline{A})$ for a fundamental system of neighborhoods $X_0 \times U$ of B at 0. That is, $H^k_A = R^k\pi_*(A)$ is a locally constant sheaf, isomorphic in a neighborhood of 0 to $H^k(X_0, A)$.

Definition: The corresponding flat connection $\Delta : H^k \to \Omega^1(H^k)$ is called the

Gauss-Manin connection.

When $A = \mathbb{C}$, using the remark at the end of the previous section we may view H^k as a holomorphic vector bundle on B. We will use this structure at the end of the next chapter to give a meaning to the holomorphy of the Hodge filtration.

Chapter 3

Hodge theory and DeRham cohomology: the analytic case

3.1 Introduction

Let $X/\mathbb{C}$ be a (smooth, proper, irreducible) algebraic variety of dimension d. Classically, the algebraic geometry of X was developed alongside the algebraic topology of the associated $\mathbb{C}$-manifold $X(\mathbb{C})$ – in particular the intersection theory of algebraic cycles ($\mathbb{Z}$-linear combinations of irreducible subvarieties) was understood to take place in the cohomology ring $H^\bullet(X(\mathbb{C}), \mathbb{Z})$ via a **cycle class map** $c : Z_i(X) \to H^{2d-2i}(X(\mathbb{C}), \mathbb{Z})$. It is critically important that the singular cohomology groups are nonvanishing up to dimension $2d$. Since the Zariski topology on (the associated scheme of) X is a d-dimensional Noetherian space, by Grothendieck's vanishing theorem [C-K], we have that for any sheaf F on X $H^i(X, F) = 0$ for all $i > d$, and it seems like the Zariski cohomology groups are hopelessly incapable of capturing the topological data of the Betti cohomology groups.

But we are giving up on the sheaf cohomology groups too easily: although no single sheaf F can play the role of a constant sheaf on X^{an}, we may still be able to read the data of the singular cohomology groups off of the cohomology of **a family** of sheaves on X. Indeed consider the family of sheaves $\Omega^i_{X/\mathbb{C}}$ of "regular i-forms," defined for all $i \in \mathbb{N}$. These are **coherent sheaves** of $\mathcal{O}_X$-modules on the scheme X: recall that $\Omega^0_{X/\mathbb{C}} = \mathcal{O}_X$ itself; $\Omega^1_{X/\mathbb{C}}$ is the globalization of the module of differentials. For any affine open subscheme given by a $\mathbb{C}$-algebra A, $\Omega^1_{A/\mathbb{C}}$ is the A-module generated by symbols da for $a \in A$ and subject to the relations $d(a+b) = d(a) + d(b), d(ab) = adb + da(b)$, $dc = 0$ for $c \in \mathbb{C}$.[1] This process is compatible with localization, so we can glue to get a coherent $\mathcal{O}_X$-module $\Omega^1_{X/\mathbb{C}}$. Indeed, $\Omega^1_{X/\mathbb{C}}$ is locally free of dimension d if and only if $X/\mathbb{C}$ is nonsingular, and is nothing but the **cotangent bundle**. For $i > 1$, we define $\Omega^i_{X/\mathbb{C}} := \Lambda^i \Omega^1_{X/\mathbb{C}}$, i.e., just the globalization of the exterior powers of modules. So if $X/\mathbb{C}$ is nonsingular, $\Omega^i_{X/\mathbb{C}}$ will be a locally free sheaf on X of rank $\binom{d}{i}$ – especially, $\Omega^d_{X/\mathbb{C}}$ is an invertible sheaf on X, the **canonical bundle**. Moreover, working purely at the level of exterior powers of modules, we have an exterior derivative $d : \Lambda^i M \to \Lambda^{i+1} M$ which, famously, satisfies $d^2 = 0$. Therefore we have

$$\mathcal{O}_X = \Omega^0_X \to \Omega^1_X \to \ldots \to \Omega^d_X \to 0,$$

the **DeRham complex** of $X/\mathbb{C}$.

Consider all possible cohomology groups $H^p(X, \Omega^q)$: they must vanish when $p > d$ or when $q > d$. Because we have assumed X is complete, the algebraic analogue of the Cartan-Serre finiteness theorem [Hartshorne, ???] tells us that the cohomology groups of any coherent sheaf on X are finite dimensional $\mathbb{C}$-vector spaces. We put $h^{p,q} = \dim_{\mathbb{C}} H^p(X, \Omega^q)$, and we are ready for the following celebrated theorem, implying in particular that the Betti numbers can be calculated from cohomology of coherent sheaves.

Theorem 21 *(Hodge Theorem) The Betti numbers of X^{an} are determined by the coherent cohomology of the sheaves Ω^i: for all n, we have*

$$\dim_{\mathbb{C}} H^n(X, \mathbb{C}) = \sum_{p+q=n} h^{p,q}.$$

Moreover, $h^{p,q} = h^{q,p}$.

In the next two sections we give the proof of this theorem, or rather the proof modulo some (not at all trivial) analytic and differential geometric facts. In fact, part of the point of giving the proof is to appreciate its essentially **non-algebraic** nature.

3.2 Summary of Hodge Theory on Riemannian manifolds

Let (M, g) be a compact oriented $\mathbb{R}^n$-manifold endowed with a Riemannian metric g. Every (paracompact!) real manifold can be so endowed – the easy way to do this is to take a locally finite covering of M by subsets homeomorphic to $\mathbb{R}^n$, endow each of these with the standard Euclidean metric, and add up all these individual metrics, smoothing with a partition of unity. Another way to prove this result is to realize M as a submanifold of $\mathbb{R}^N$ by Whitney embedding,

[1] A useful special case is that if $A = C[X_1, \ldots, X_n]/(f_j)$ is a finite-type $\mathbb{C}$-algebra, $\Omega_{A/\mathbb{C}}$ is the finitely generated A-module with generators dX_i for $1 \leq i \leq n$ and relations $d(f_j) = 0$.

take the Euclidean metric on $\mathbb{R}^N$ and restrict to N.[2] For now, we write $\Lambda_M^p = \Lambda_M^p(\mathbb{R})$ for the space of C^∞ p-forms on M with real coefficients. Let dV be the volume n-form on M associated to the metric g – in local coordinates $x_1, \ldots, x_n$, it is given by

$$dV = \sqrt{\det(g_{ij})} dx_1 \wedge \ldots \wedge dx_n.$$

We define, using the metric, the **Hodge star** operator

$$\star : \Lambda_M^p \to \Lambda_M^{n-p},$$

defined as follows: in a neighborhood U about every point M admits an orthonormal frame $e_1, \ldots, e_n$ of sections of the tangent bundle – i.e., $g(e_i(x), e_j(x)) = \delta_{ij}$ for all $x \in U$. Every i-form can be written as a sum of terms $\sum_I f_I(x) de_I$, where $I \subset \{1, \ldots, n\}$ is a subset of cardinality i and $de_I = \bigwedge_{i \in I} de_i$. Define $I^\star = \{1, \ldots, n\} \setminus I$, the complementary subset, and finally define

$$\star(\sum_I f_I(x) de_I) = \sum_I f_I(x) de_{I^\star}.$$

This allows us to endow Λ_M^i with an inner product, namely

$$\langle \alpha,\ \beta \rangle = \int \star(\alpha \wedge (\star\beta)) dV.$$

The completion of this real inner product space is denoted $L^2(\Lambda_M^p)$, the Hilbert space of square-integrable p-forms on M. It is built into our definition that the Hodge star operator is a Hilbert space isometry $L^2(\Lambda_M^p) \to L^2(\Lambda_M^{n-p})$. We put $L^2(\Lambda_M) := \bigoplus_p L^2(\Lambda_M^p)$ (Hilbert space direct sum, i.e., $\langle\ ,\rangle = \sum_p \langle\ ,\rangle_p$.)

Because of this, it makes sense to speak of the adjoint operator to the exterior derivative d on $L^2(\Lambda_M)$, denoted $d^\star$. One can check that it exists and is given by $(-1)^{n+1} \star \circ\, d \circ \star$.

Finally, we define the Laplace-Beltrami operator on $\Lambda_M^\bullet$ as

$$\Delta = d \circ d^* + d^* \circ d.$$

We remark that if M is the (noncompact; in this case we should take the completion of the space of compactly supported smooth forms) manifold $\mathbb{R}^n$ endowed with the Euclidean metric $ds^2 = \sum dx_i^2$ then, up to a sign, the Laplacian of a zero form is the familiar $\sum_i \frac{\partial^2 f}{\partial x_i^2}$.)

In general, we define $\mathcal{H}^p(M) = \ker(\Delta)$, the **harmonic** p-forms.

[2] The latter approach raises the issue of whether every compact Riemannian manifold arises as a submanifold of Euclidean space. The answer is yes; this is the celebrated Nash Embedding Theorem [Nasar].

Lemma 22 *For any* $s \in \Lambda^p_M$*, we have*

$$\langle \Delta s,\ s\rangle = ||ds||^2 + ||d^* s||^2$$

.

Moreover, $s \in \Lambda^p_M$ *is harmonic iff* $ds = d^* s = 0$.

Proof: The formula is immediate: $\langle \Delta s,\ s\rangle = \langle d(d^\star s) + d^\star(ds), s\rangle = \langle d(d^\star s), s\rangle + \langle d^\star(ds), s\rangle = \langle d^\star s, d^\star s\rangle + \langle ds, ds\rangle = ||ds||^2 + ||d^\star s||^2$. It clearly follows that a harmonic form is both d-closed and $d^\star$-closed. Conversely, if $ds = d^\star s = 0$, then

$$\langle \Delta s, \Delta s\rangle = \langle dd^\star s + d^\star ds, dd^\star s + d^\star ds\rangle = 0.$$

Theorem 23 *(Hodge theorem for Riemannian manifolds)*
a) For all p, there is an orthogonal decomposition $\Lambda^p_M = \mathcal{H}^p(M) \oplus \operatorname{Im} d \oplus \operatorname{Im} d^\star$.
b) Since d *and* d^* *are adjoint,* $\operatorname{Ker} d = (\operatorname{Im} d^*)^\perp$*, and we conclude that* $Z^p_{DR}(M) := \operatorname{Ker}(d : \Lambda^p_M \to \Lambda^{p+1}_M)$ *is naturally isomorphic to* $\mathcal{H}^p(M) \oplus \operatorname{Im} d$*. That is, each DeRham cohomology class contains a unique harmonic representative.*

"Proof": It is easy to see that $\mathcal{H}^p(M)$, $\operatorname{Im} d$ and $\operatorname{Im} d^*$ are mutually orthogonal subspaces of Λ^p_M: indeed $\langle ds,\ d^* t\rangle = \langle d^2 s, t\rangle = 0$. Moreover (using that harmonic forms are d-closed and $d^\star$-closed), since $\operatorname{Im} d^* = (\operatorname{Ker} d)^\perp$, no harmonic form is in the image of d^*; similarly, no harmonic form is in the image of d. To show that this subspace is all of Λ^p_M is another matter entirely. For this we need to know that Δ is an **elliptic operator** on M.

For completeness, we indicate briefly the definition of an elliptic differential operator: a differential operator of order at most m between vector bundles E and F on M with Riemannian metrics is a thing which can in local coordinates be written as a matrix $\sum_{I\ :\ |I|\le m} a^I_{ij}(x) D^I$, where e.g. $D^{(1,2)} = \frac{\partial}{\partial x_1}\frac{\partial^2}{\partial x_2^2}$. The associated **symbol** is obtained by dropping all the lower order terms and formally replacing the D^I's with $(\zeta)^I = (\zeta_1, \ldots, \zeta_n)^I = \zeta_1^{i_1} \cdots \zeta_n^{i_n}$, so

$$\sigma(D)(x,\zeta) = \sum_{I:|I|=m} a^I_{ij}(x) \zeta_1^{i_1} \cdots \zeta_n^{i_n}.$$

The operator is elliptic if for all $x \in M$ and all $\zeta \in \mathbb{R}^n \setminus O$, the symbol $\sigma(D)(x,\zeta)$ is an invertible matrix. For instance, since the homogeneous form $x_1^2 + \ldots + x_n^2$ has no nontrivial real zeros, the classical Laplacian on $\mathbb{R}^n$ is elliptic. It is not so hard to see that the general Laplace-Beltrami operator is elliptic; see e.g. [Demailly]. The hard part is the following result, which is an entirely serious theorem in the realm of PDEs, using Sobolev spaces, Garding's inequality, and so on.

Theorem 24 *(Finiteness theorem for elliptic operators) Let* P *be an elliptic operator on the sections of a vector bundle* $E \to M$*, whose fibres are equipped with an inner product. Then the* $\Gamma(M,E) = \operatorname{Im}(P) \oplus \operatorname{Ker} P^*$*, where the first summand is a closed subspace of finite codimension.*

Theorem 4 (proved by Hodge, of course, for the Laplace-Beltrami operator; later the general theory of elliptic operators developed around his proof) finishes the proof for us, since $\Delta = \Delta^*$ is self-adjoint and

$$\operatorname{Im} \Delta = \operatorname{Im}(d \circ d^* + d^* \circ d) \subset \operatorname{Im} d + \operatorname{Im} d^*.$$

An application: Let $\rho : \tilde{M} \to M$ be a degree N unramified cover of a compact smooth manifold M. One knows that $\chi(\tilde{M}) = N\chi(M)$ for truly topological reasons (pull back a sufficiently fine triangulation of M), but it is not as clear that we have inequalities $b_i(\tilde{M}) \geq b_i(M)$. But we claim that indeed $H^i(\rho)$ is an injection for all i, and harmonic cohomology gives an easy proof of this: indeed it is certainly true that pullback map is injective on the level of differential forms (as follows immediately from the chain-rule and that ρ is a submersion). Now choose any Riemannian metric on M and pull it back to $\tilde{M}$; since ρ is unramified, $\tilde{M}$ is locally isometric to M; since the Laplace-Beltrami operator is local by construction, it follows that the Laplacian **commutes** with pullback of differential forms. We conclude that the harmonic forms on M map monomorphically into the harmonic forms on $\tilde{M}$, whence the claim.[3]

3.3 A quick proof of the DeRham Theorem

For comparison, we recall the **DeRham Theorem**, which gives a canonical isomorphism between the DeRham cohomology $H^\bullet_D R(M)$ of a real manifold M and the singular cohomology with $\mathbb{R}$-coefficients. It is instructive to note that in constrast to the hard analysis of the Hodge theorem, the DeRham theorem can be proved using **only** the machinery of sheaf cohomology.

On the one hand we have the **DeRham resolution** of the constant sheaf $\underline{\mathbb{R}}$ on X:

$$0 \to \underline{\mathbb{R}} \xrightarrow{\iota} \Lambda^0_M \xrightarrow{d} \Lambda^1_M \ldots \to \Lambda^n_M \to 0.$$

Certainly $d^2 = 0$ – even at the level of presheaves. Moreover, upon restriction to any star-shaped domain, closed forms are exact (Poincar' Lemma), so as a sequence of sheaves it is exact, i.e., it gives a resolution of $\underline{\mathbb{R}}$. But the sheaf of sections of any vector bundle on a manifold is soft (indeed it is fine: we have partitions of unity), hence acyclic for sheaf cohomology (as discussed in [C-K]). This shows that the DeRham cohomology naturally isomorphic to $H^\bullet(X, \underline{\mathbb{R}})$.

What about the singular cohomology? Let X be a locally contractible topological space and G an abelian group. We define a sheaf $\mathcal{S}^p(G)$ as follows: for any open subset U, we put $S^p(G)(U) := \operatorname{Hom}_{\mathbb{Z}}(S_p(U, \mathbb{Z}), G)$, where $S_p(U, \mathbb{Z})$ is the usual group of U-valued singular p-chains. We have coboundary maps $\delta : S^p(G)(U) \to S^{p+1}(U)$. Here's the punchline: of course this is not an exact

[3] In fact there is a proof using only DeRham cohomology: we must show that if $\rho^*(\omega)$ is exact, then so was ω. Writing $\rho^*(\omega) = d\theta$, it need not be the case that θ "descends" to M, but its "norm" (in the Galois-theoretic sense!) does; we leave the details to the reader.

sequence at the level of presheaves – indeed, taking U-sections, the cohomology is precisely the singular cohomology $H^\bullet_{sing}(U,G)$. But by the assumed local contractibility, on stalks we get an exact sequence. Therefore, letting $\mathcal{S}^p(G)$ be the sheafification of $U \mapsto S^p(G)(U)$, we have a long exact sequence of sheaves. Moreover, the kernel – which does not need to be sheafified – of $S^0(G)(U) \to S^1(G)(U)$ is canonically identified with the constant sheaf $\underline{G}$, so we find that $G \to \mathcal{S}^\bullet(G)$ is a resolution of $\underline{G}$. Moreover we claim it is a soft resolution: $S^0(G)(U) = \mathrm{Hom}_{\mathbb{Z}}(S_0(U,\mathbb{Z}),G) = \mathrm{Hom}_{\mathbb{Z}}(\bigoplus_{u\in U}\mathbb{Z}[u],G) = \bigoplus_{u\in U}\mathrm{Hom}(\mathbb{Z}[u],G) = \bigoplus_{u\in U} G = C^0(G)$, the canonical flasque sheaf (of discontinuous sections) associated to the constant sheaf $\underline{G}$. So $S^0(G)$ – which is already a sheaf – is flasque, and flasque sheaves are soft. Moreover, taking now $G=\mathbb{R}$, the $\mathcal{S}^i(\mathbb{R})$'s are modules over $\mathcal{S}^0(\mathbb{R})$, via the cup-product. But in general, a sheaf of modules F over a soft sheaf of rings R is soft. Indeed, take a section s of F over a closed subset K of M. By definition of the sections of a sheaf over closed subsets, s extends to some open neighborhood U of K. Since $K \cap (X \setminus U) = \emptyset$, we can define a section ρ of R over $K \cup (X \setminus U)$ by making it identically equal to the unity 1 on K and identically 0 on $X \setminus U$. Since R is assumed to be soft, ρ extends to all of X, and the product ρs gives an extension of s to all of X. Thus the singular resolution $\underline{\mathbb{R}} \to \mathcal{S}^\bullet(\mathbb{R})$ is also an acyclic resolution and can be used to compute the cohomology of X.

In fact, we can make the isomorphism between DeRham cohomology and singular cohomology **explicit**, as follows: first, we may as well work with differentiable p-chains (the above argument goes through verbatim). Then we have a commutative diagram

$$\underline{\mathbb{R}} \to \Lambda^\bullet_X$$

$$\underline{\mathbb{R}} \to \mathcal{S}^\bullet(X,\mathbb{R})$$

given by integration of p-forms against p-chains. Since both complexes are acyclic and the left-hand map is an isomorphism, the general theory of acyclic resolutions shows that the induced map on cohomology must be an isomorphism. This is the usual form of DeRham's theorem.

3.4 Hodge theory for complex & Kahler manifolds

Suppose X is now a $\mathbb{C}^n$-manifold, endowed with a Hermitian metric h. Note well that a Hermitian metric is still a C^∞-object – it has nothing to do with the $\mathbb{C}$-structure on X and indeed (unsing partitions of unity, as above) any $\mathbb{C}^n$-bundle on a real manifold can be endowed with a Hermitian metric. Viewing X as an $\mathbb{R}^{2n}$-manifold via local coordinates $z_1,\overline{z_1},\dots,z_n,\overline{z_n}$, we consider $\Lambda^\bullet_X = \Lambda^\bullet_X(\mathbb{C})$ the sheaves of $\mathbb{C}$-valued C^∞-differential forms on X, which are local expressions of the form $f_I(z)dz_I \wedge d\overline{z}_J$ – note well that $f_I(z)$ is a $\mathbb{C}$-valued merely C^∞ function. By definition of a complex manifold, transitions between coordinate systems preserve the decomposition into z-coordinates and $\overline{z}$-coordinates:

this allows us to decompose the exterior derivative as $d = \partial + \overline{\partial}$, where e.g. on zero forms $\partial f = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i$ and $\overline{\partial} f = \sum_{i=1}^n \frac{\partial f}{\partial \overline{z_i}} d\overline{z_i}$. We thus visibly get a decomposition of Λ_X^r into $\bigoplus_{p+q=r} \Lambda_X^{p,q}$, the sheaf of $\mathbb{C}$-valued $\mathbb{C}^\infty$ "(p,q)"-forms.

We also have a Hermitian Hodge-star operator defined by

$$u \wedge (\overline{\star v}) = \langle u,\ v \rangle dV,$$

where the volume form is associated to the "underlying" Riemannian metric – the real part of a Hermitian metric gives a Riemannian metric. The Hodge star operator gives a $\mathbb{C}$-linear isometry $\Lambda^{p,q} \to \Lambda^{n-q,n-p}$, and in this way we have not one but three Laplacians. The first is Δ, which is just obtained tensoring from $\mathbb{R}$ to $\mathbb{C}$ the Laplacian on the underlying real manifold. We also have Δ_1 and Δ_2 (slightly nonstandard notation, but the standard notation, $\square$ and $\overline{\square}$, seems rather silly), obtained by using ∂ (respectively $\overline{\partial}$) in place of d:

$$\Delta_1 = \partial \circ \partial^* + \partial^* \circ \partial,$$

$$\Delta_2 = \overline{\partial} \circ \overline{\partial}^* + \overline{\partial}^* \circ \overline{\partial}.$$

Among several identities relating these operators, we single out

$$\partial^* = - \star \overline{\partial} \star,\ \overline{\partial}^* = - \star \partial \star .$$

We work with Δ and Δ_2, defining

$$\mathcal{H}^p(X) = \mathcal{H}^p(X, \mathbb{C}) = \ker(\Delta : \Lambda^n \to \Lambda^n)$$

and

$$\mathcal{H}^{p,q}(X) = \mathcal{H}_2^{p,q}(X) = \ker(\Delta_2 : \Lambda^{p,q} \to \Lambda^{p,q}).$$

We speak of the **harmonic n-forms** and **harmonic (p,q)-forms** respectively.

Now we have three different versions of (p,q)-cohomology: the harmonic cohomology $\mathcal{H}_2^{p,q}(X)$; the coherent analytic sheaf cohomology $H^q(X, \Omega^p)$, and finally the **Dolbeault cohomology**, i.e., the "$\overline{\partial}$-DeRham cohomology":

$$H_{\overline{\partial}}^{p,q}(X) := H^q((\Lambda_X^{p,\bullet}, \overline{\partial}).$$

There is also a $\overline{\partial}$-analogue of the DeRham theorem: namely we have the **Dolbeault resolution**

$$0 \to \Omega_X^p \to \Lambda_X^{p,0} \xrightarrow{\overline{\partial}} \Lambda_X^{p,1} \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \Lambda_X^{p,n} \to 0,$$

and since the sheaves $\Lambda_X^{p,q}$ are fine, we conclude

$$H^q(X, \Omega_X^p) = \frac{(\mathrm{Ker}(\Lambda_X^{p,q} \xrightarrow{\overline{\partial}} \Lambda_X^{p,q+1})}{\mathrm{Im}(\Lambda_X^{p,q-1} \xrightarrow{\overline{\partial}} \Lambda_X^{p,q})} = H_{\overline{\partial}}^{p,q}(X).$$

The analogue of Theorem 16 for Δ_2 is:

Theorem 25 *For all* (p,q)*, there is an orthogonal decomposition*

$$\Lambda_X^{p,q} = \mathcal{H}_X^{p,q} \oplus \operatorname{Im} \overline{\partial} \oplus \operatorname{Im}(\overline{\partial}^*).$$

Corollary 26 *On a complex manifold, Dolbeault, harmonic and coherent cohomology coincide:*

$$H^p(X, \Omega^q) = \mathcal{H}_X^{p,q} = H_{\overline{\partial}}^{p,q}(X).$$

But we still do not know how any of these cohomology groups compute $H^\bullet(X, \mathbb{C})$. Indeed, they need not, until we add an extra hypothesis.

Our Hermitian metric, $\sum_{ij} h_{ij} z_i \overline{z_j}$ can be written as $h = S + iA$, where S is symmetric and A is skew-symmetric; put $\Omega := 1/2A$, a real-valued $(1,1)$-form. One says that h is a **Kahler metric** if $d\Omega = 0$. (Notice that any Hermitian metric on a one-dimensional $\mathbb{C}$-manifold is automatically Kahler.) A $\mathbb{C}$-manifold is said to be **Kahler** if it admits a Kahler metric.

The property of a metric being Kahler is preserved upon passage to submanifolds, so any submanifold of a Kahler manifold is Kahler. Moreover, $\mathbb{CP}^n$ has a canonical Kahler metric, the **Fubini-Study metric**; we conclude that any compact complex manifold which is algebraic is a Kahler manifold.

Theorem 27 *(Kahler identities) Let* (X, h) *be a Kahler metric on a complex manifold. Then*

$$\Delta_1 = \Delta_2 = 1/2\Delta.$$

Again this theorem has too much content for us to review here (the standard proof requires some representation theory of $\mathfrak{sl}_2(\mathbb{C})$) but unlike the purely analytic Theorem 17, it is discussed in *every* reputable text on Hodge theory, e.g. [Wells], [Griffiths-Harris], [Voisin I], [Demailly]. But it is certainly what we need: it tells us that on a Kahler manifold we have a unique notion of a harmonic form, so that

$$\mathcal{H}_X^n = \bigoplus_{p+q=n} \mathcal{H}_X^{p,q}.$$

Actually more is true: since (even without the Kahler condition), $\Delta_1 = \overline{\Delta_2}$, on a Kahler manifold we get that $\overline{\Delta_2} = \Delta_1 = \Delta_2$, so if a (p,q)-form is harmonic, so is its complex conjugate (q,p)-form. Thus we have canonical isomorphisms $H^{p,q}(X, \mathbb{C}) \cong \overline{H^{q,p}(X, \mathbb{C})}$, and in particular $h^{p,q} = h^{q,p}$.

Finally, we should discuss the *invariance* of the Hodge decomposition: *a priori* the direct sum decomposition $H^n(X, \mathbb{C}) = \mathcal{H}^n = \bigoplus_{p+q=n} \mathcal{H}^{p,q}$ seems to depend upon the choice of Kahler metric, but one can show that this is not the case. Probably the best way to see this is to observe that $\mathcal{H}^{p,q}$ can be intrinsically defined in terms of the **Hodge filtration** on the DeRham complex as $F^p \cap \overline{F^q}$; we will explore this viewpoint in Chapter 3. For an elementary proof involving yet a fourth kind of (p,q)-cohomology, namely the **Bott-Chern** cohomology groups

$$H^{p,q}_{BC}(X,\mathbb{C}) = \frac{\mathrm{Ker}(d : \Lambda^{p,q}(X) \to \Lambda^{p+q+1}(X))}{\partial\overline{\partial}(\Lambda^{p-1,q-1}(X))},$$

(which are at least *a priori* independent of the Kahler metric), see [Demailly, pp. 40-42]. In summary, we have "proved":

Theorem 28 *(Hodge theorem for Kahler manifolds) Let $X/\mathbb{C}$ be a compact Kahler manifold. Then there is a canonical isomorphism*

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X,\mathbb{C})$$

satisfying $H^{q,p} = \overline{H^{p,q}}$.

Remark: Let (E,D) be a vector bundle on X endowed with a flat connection. We have a notion of DeRham cohomology with coefficients in E, namely in the exact sequence (1.1) of Section 1.3, take global sections and then cohomology; we denote this $H^{\bullet}_{DR}(X,E)$. Simply by replacing d everywhere by D, one can redo all the constructions of this section, getting especially $\Delta_1(E) = \Delta_2(E) = 1/2\Delta(E)$ and at last an isomorphism

$$H^n_{DR}(X,E) = \bigoplus_{p+q=n} H^{p,q}(X,E)$$

satisfying $H^{p,q}(X,E) = \overline{H^{q,p}(X,E)}$. This generalization is not so important for us here, but what we have done is the complex-analytic analogue of taking crystalline cohomology of **crystals** rather than cohomology of the structure sheaf.

Finally, if $X/\mathbb{C}$ is projective nonsingular variety, then as mentioned above the associated complex manifold $X(\mathbb{C})$ is compact Kahler. We must appeal to Serre's GAGA theorem: there is a natural analytification functor from coherent sheaves on $X/\mathbb{C}$ in the algebraic sense to coherent sheaves on $X(\mathbb{C})$ in the analytic sense, such that coherent cohomology computed algebraically is canonically isomorphic to coherent cohomology computed analytically. At last we get our algebraic Theorem 14!

3.5 Implications for the topology of compact Kahler manifolds

The Hodge Theorem is intriguing even at the level of algebraic topology: it places constraints on the Betti numbers of compact Kahler manifolds that need not be satisfied for more general compact complex manifolds (in particular, the Kahler hypothesis is essential in the Hodge theorem and not just an artifice of the proof).

For instance, $h^{1,0} = h^{0,1} = 1/2b_1$ for any compact Kahler manifold. Thus the group of line bundles algebraically equivalent to zero $\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$ really is a complex torus, as promised in Section 1.1. When X is projective $\mathrm{Pic}^0(X)$ admits a Riemann form, i.e., is an abelian variety, the Picard variety. So $b_1(X) = 0$ implies the triviality of the Picard variety of X. Interpreting $b_1(X)$ in the sense of étale cohomology, this statement makes sense purely algebraically, i.e., in all characteristics. In Chapter 4, we will gain a profound appreciation for the "nonobviousness" of this algebraic statement (i.e., it can be false in positive characteristic!) In fact the equality $h^{1,0} = 1/2b_1$ is already "nonobvious" for complex manifolds:

Example (Hopf surfaces): Consider $X = (\mathbb{C}^2 \setminus \{0\})/\Gamma$, where for some fixed $\lambda \in (0, 1)$, $\Gamma = \lambda^{\mathbb{Z}}$, viewed as a group of homotheties of $\mathbb{C}^2$. Each element of Γ is a $\mathbb{C}$-manifold automorphism of $(\mathbb{C}^2 \setminus \{0\}$, so the quotient X is a $\mathbb{C}$-manifold. Since $\mathbb{C}^2 \setminus \{0\}$ is diffeomorphic to $\mathbb{R}^{>0} \times S^1$, we see that X is diffeomorphic to $S^1 \times S^3$. Using the Kunneth formula, we compute the Betti numbers of X: $b_0 = 1, b_1 = 1, b_2 = 0, b_3 = 1, b_4 = 1$. X is definitely not a Kahler manifold! Actually, the $b_2 = 0$ is also enough to ensure that a complex manifold is non-Kahler: one can show that the top wedge power of the fundamental form Ω is a positive scalar multiple of the volume form – in particular $[\Omega^n] \neq 0 \in H^{2n}_{DR}(X, \mathbb{C})$, which implies that **every** wedge power of Ω must be cohomologically nontrivial, and so all the even Betti numbers of a Kahler manifold are **positive**.

Fundamental groups of compact Kahler manifolds: Of course, that $b_1(X)$ must be even is saying something about $\pi_1(X)$, namely that the free rank of its abelianization is even. What if we want to know about $\pi_1(X)$ itself? It is well-known that every finitely presented group arises as the fundamental group of a compact $\mathbb{R}^4$-manifold. Less well-known but still true is that every finitely presented group is the fundamental group of a compact $\mathbb{C}^3$-manifold, so the above restriction on $\pi_1(X)$ for Kahler manifolds is actually rather surprising. Say that a group is a **Kahler group** if it arises as the fundamental group of a compact Kahler manifold.

Question 29 *Which finitely presented groups are Kahler groups?*

This is analogous to the question of which finite groups are Galois groups over $\mathbb{Q}$ and to the question of which finitely presented groups are π_1 of a compact $\mathbb{R}^3$-manifold but, purely on its own terms, seems more interesting than both, since the conjectured answer to the first question is "all of them" and to the second is "very few." In contrast, the frontier between Kahler and non-Kahler groups is remarkably rugged. For instance, we will see in Chapter 4 that every finite group is the fundamental group of an algebraic variety (even in characteristic p – this is a theorem of Serre). Since the class of Kahler groups is clearly closed under products and certainly $\mathbb{Z}^{2g}$ is a Kahler group (the fundamental group of a genus g curve or equally well of its Jacobian), we can completely characterize the abelianizations Kahler groups. To see that this is not enough: the class of

Kahler groups is closed under passage to subgroups (by covering space theory; a cover of a Kahler manifold is Kahler), so the free group on two generators is not Kahler (even though its abelianization is), since it contains free subgroups on every *odd* number of generators.

One might ask why we study Kahler groups instead of fundamental groups of projective manifolds. The answer is that so far no one has ever found a Kahler group which is non-projective; moreover, it is conjectured that the complex structure on a Kahler manifold can be *deformed* (in the sense of Section 1.3.2) to a projective complex structure, which would imply that the two classes are the same. In practice, most constructions of Kahler groups can be done with algebraic manifolds, while non-existence arguments tend to work for the larger class of Kahler manifolds. For much more on this fascinating question, see [Amoros et. al.].

Chapter 4

Topological Invariants in Characteristic p

Here we want to use an example of Serre to show that – even when the geometry is "the same" in characteristic p as in characteristic zero, the Hodge and DeRham numbers can be different. This will motivate us to define carefully three different kinds of Betti numbers for a variety in characteristic p.

4.1 Serre's example

We begin with a further application of the Artin-Schreier isogeny considered already in Chapter 1. So let X/k be a smooth projective variety over an algebraically closed field of characteristic p, and consider the Artin-Schreier sequence of sheaves on $X_{\text{ét}}$:

$$0 \to \underline{\mathbb{Z}/p\mathbb{Z}}_X \to \mathbb{G}_{aX} \stackrel{F-1}{\to} \mathbb{G}_{aX} \to 0.$$

Since $X(k) = k$ and k is algebraically closed, $F - 1$ induces a surjection on $H^0(\mathcal{O}_X)$, so we deduce immediately that

$$H^1(X, \underline{\mathbb{Z}/p\mathbb{Z}}) \cong H^1(X, \mathcal{O}_X)^{F-1}.$$

Moreover, since $H^1(X, \underline{\mathbb{Z}/p\mathbb{Z}}) = \mathrm{Hom}(\pi_1(X), \mathbb{Z}/p\mathbb{Z})$, we find that if X admits a connected étale $\mathbb{Z}/p\mathbb{Z}$-covering $Y \to X$, then $H^1(X, \mathcal{O}_X)^{F-1} \neq 0$; *a fortiori* $h^{0,1} = \dim_k H^1(X, \mathcal{O}_X) > 0$.

Next we have the following "classical" fact, whose proof we omit.

Proposition 30 *(Serre) For $p \geq 5$ and $k = \overline{k}$ any algebraically closed field, there exists a hypersurface $Y \subset \mathbb{P}^3$ and a free action of $\mathbb{Z}/p\mathbb{Z}$ on Y, i.e., an unramified $\mathbb{Z}/p\mathbb{Z}$-cover $Y \to X$. Indeed, there exists $\mathcal{Y}/W(k)$ smooth and proper with such an action. Since $\pi_1(\mathcal{Y}_\eta) = 0 = \pi_1(\mathcal{Y}_0)$, we have that $\pi_1(X_\eta) \cong \mathbb{Z}/p\mathbb{Z} \cong \pi_1(X_0)$.*

In other words, equally well in characteristic zero and in characteristic p, we can construct a surface with fundamental group $\mathbb{Z}/p\mathbb{Z}$ and universal covering a hypersurface in $\mathbb{P}^3$. So in characteristic 0 the first Betti number of X is 0. By the Hodge theorem, this implies that $h^{1,0} = h^{0,1} = 0$. However, in characteristic p, we have an unramified $\mathbb{Z}/p\mathbb{Z}$-covering $Y \to X$ of our surface, and the previous proposition implies that $h^{0,1} > 0$. (Serre sketches the argument for $h^{0,1} = 1$.) On the other hand, standard considerations will show that $h^{1,0} = 0$ independent of the characteristic: indeed, by Serre duality, $h^{1,0}(Y) = h^{2,1}(Y) = \dim H^1(Y, \omega_Y)$, and since $\omega_Y = \mathcal{O}(n)$ is some multiple of the hyperplane section (precisely $n = d - 4$, where d is the degree of the hypersurface), it follows from [Hartshorne, Ex. 3.5.5] that $H^1(Y, \omega_Y) = 0$. Note that this was for Y, but now since $Y \to X$ is surjective, hte pullback on differentials is *injective*, so *a fortiori* $H^0(X, \Omega^1_X) = h^{1,0}(X) = 0$. (Indeed in our case since $Y \to X$ is étale, $\Omega_{Y/X} = 0$ and the aforementioned pullback is an isomorphism.)

The moral of the story is that the Hodge numbers are sensitive to the p-torsion in characteristic p.

4.2 An Embarrassment of Riches

We actually have three different kinds of Betti number in characteristic p:
i) The ℓ-adic Betti numbers $b_{i,\ell}$ (equal for all $\ell \neq p$)
ii) The Hodge Betti numbers $b_{i,H} = \sum_{p+q=i} h^{p,q}$.
iii) The DeRham Betti numbers $b_{i,DR}$, which it is the subject of the next chapter to define.

They are related as follows:

$$b_{i,H} \geq b_{i,DR} \geq b_{i,\ell}.$$

In general, both inequalities can be strict, for very different reasons. The strictness of the first inequality is equivalent to the failure of the Hodge to DeRham spectral sequence to degenerate at E_1. Although this may sound impressive, we will see in the next chapter that it is an immediate consequence of the spectral sequence setup. A much deeper explanation for this strictness is that it related to the **nonliftability** of X/k even to $W_2(k)$ by work of Deligne-Illusie. We will say (only) a little more about this later. In turn, the strictness of the second inequality, which Serre's example is a case of[1] will in all generality turn out to be equivalent to the existence of p-torsion in the "true p-adic cohomology," i.e., the crystalline cohomology, and this is our best motivation for studying crystalline cohomology.

[1] I believe the first example was due to Igusa

Chapter 5

Algebraic DeRham Cohomology

5.1 Souping up the Hodge Theorem: spectral sequences and hypercohomology

In the previous chapter we proved the Hodge theorem for smooth, projective complex varieties – but only by translating the statement into a statement about Kahler manifolds. We now want to recast a portion of the Hodge theorem in terms of a statement about degeneration of spectral sequences. The translated statement, namely, "The Hodge to de Rham spectral sequence for a smooth projective complex variety degenerates at the E_1 term" is itself purely algebraic, so it is at least meaningful to ask whether it is true in characteristic p.

5.1.1 Spectral sequence of a double complex

Let $(K^{p,q}, d' + d'')$ be a double complex with horizontal and vertical differentials d' and d''. We assume it is concentrated in the first quadrant, i.e. $K^{p,q} = 0$ unless $p,\ q \geq 0$. From the double complex we pass to the associated **total complex**, $K^n := \bigoplus_{p+q=n} K^{p,q}$, endowed with the the differential $d = d' + (-1)^q d''$. On the total complex one has a decreasing filtration

$$F^p K^n := \bigoplus_{p \leq j \leq n} K^{j,n-j}.$$

This induces a filtration on the cohomology groups $H^\bullet(K^\bullet)$ of the total complex, namely

$$F^p H^l(K^\bullet) := \mathrm{Im}(H^l(F^p K^\bullet) \to H^l(K^\bullet)).$$

There is a spectral sequence

$$E_1^{p,q} = H^q((K^{p,\bullet}, d'')) \implies H^{p+q}(K^\bullet).$$

Recall this means that for all $r \geq 1$ we have differentials $d_r : E_r^{p,q} \to E^{p+r,q-r+1}$, such that, inductively, $E_{r+1} = H^\bullet(E_r)$. Since the complex is concentrated in the first quadrant, for any given (p,q) eventually the head or the tail of every "arrow" lies outside of the first quadrant, so that the process stabilizes pointwise: $\lim_{r\to\infty} E_r^{p,q} = E_\infty^{p,q}$ exists. The convergence means that $E_\infty^{p,q} = G^p H^{p+q}(K^\bullet)$, the pth graded piece of the filtration.

Finally, we say the spectral sequence **degenerates** at the E_r-term if all the differentials d_{r+i}, for all $i \geq 0$, are zero. Then indeed $E_r^{p,q} = E_\infty^{p,q}$. One simply says the spectral sequence **degenerates** if it degenerates at the E_1-term (or at the first term under consideration; in a slightly different context, many spectral sequences start with the E_2-term).[1]

5.2 The Hodge to DeRham spectral sequence

Let $X/\mathbb{C}$ be a complex manifold – not yet assumed to be compact or Kahler. Dolbeault's theory provides us with a double complex, namely $K^{p,q} = \Lambda_X^{p,q}$. Our two differentials are just ∂ and $\overline{\partial}$ – or, to adhere strictly with the sign conventions of the previous section, take $d'' =(-1)^q\overline{\partial}$; we will not be so careful about this – with total differential d. Notice that the asasociated total complex is in degree n $\bigoplus_{p+q=n} \Lambda^{p,q}$ with differential d – i.e, the $\mathbb{C}$-valued DeRham complex $\Lambda_X^\bullet$. The associated spectral sequence is called the **Hodge to DeRham** spectral sequence: let's look at it. The E_1 terms are $E_1^{(p,q)} = H^q((\Lambda^{p,\bullet}, \overline{\partial})) = H^{p,q}(X,\mathbb{C}) = \mathcal{H}^{p,q}(X) = H^q(X,\Omega^p)$, the Hodge groups. So we can write the spectral sequence as

$$H^{p,q}(X,\mathbb{C}) \implies H_{DR}^{p+q}(X,\mathbb{C}).$$

We saw above that the convergence is phrased in terms of a canonical decreasing filtration on the limiting object. In our case, we get a filtration $F^k H_{DR}^\bullet(X,\mathbb{C})$, the **Hodge filtration**.

So for any complex manifold, the Hodge groups are related to the DeRham cohomology – somehow. The question is: does this spectral sequence degenerate (immediately)?

Suppose now that X is compact; then by the finiteness theorem of Serre we know that $H^{p,q}(X,\mathbb{C})$ are all finite dimensional $\mathbb{C}$-vector spaces; we may write $h^{p,q}$ for their dimensions and $b_n := \dim H_{DR}^n(X,\mathbb{C})$ for the Betti numbers. Now if the spectral sequence degenerates, we can sum along the line $x+y=n$ to get the nth Betti number: i.e., degeneration implies $\sum_{p+q=n} h^{p,q} = b_n$, the greater part of the Hodge theorem. But in fact the converse is true: notice that since a

[1]To be sure: in contrast to most instances in mathematics (and in life), degeneration of a spectral sequence is a joyous occasion: it means that two quantities which abstract nonsense says are related, albeit in a very complicated way, are actually related in the simplest possible way, aka the way in which you wanted them to be related.

spectral sequence involves repeated passage to subquotients, the dimensions of the $\mathbb{C}$-vector spaces $E_r^{p,q}$ are nonincreasing functions of r, and that a single differential is nonzero is precisely the condition for some subquotient to be proper. In other words if the spectral sequence does not degenerate we must have for some n that $\sum_{p+q=n} h^{p,q} > b_n$. In summary:

Proposition 31 *Let X be any compact complex manifold. The Hodge to DeRham spectral sequence degenerates at the E_1-term iff for all n we have $\sum_{p+q=n} h^{p,q} = b_n$.*

So the following is an immediate consequence of the Hodge theorem:

Theorem 32 *The Hodge to DeRham spectral sequence of a compact Kahler manifold degenerates at the E_1 term.*

Remark: The part of the Hodge theorem that says that a compact Kahler manifold has $H^{p,q}(X,\mathbb{C}) = \overline{H^{q,p}(X,\mathbb{C})}$ is therefore not guaranteed by the degeneration of this spectral sequence. Indeed, it turns out that if $X/\mathbb{C}$ is any compact complex surface, the spectral sequence degenerates. Moreover X will be Kahler iff b_1 is even; otherwise it turns out that $h^{1,0} = h^{0,1} + 1$ [BPV]. Notice that, together with Serre duality, this computes the Hodge diamond of the Hopf surfaces studied in Chapter 2.

5.3 Hypercohomology

Let us at long last return to the algebraic category: in particular, suppose X/k is a smooth projective variety over an algebraically closed field of positive characteristic p. We still have Hodge numbers, defined via coherent cohomology: $h^{p,q} := \dim_k H^q(X, \Omega^p)$. However we do not have anything like DeRham resolution of the constant sheaf $\underline{\mathbb{C}}$, because indeed constant sheaves on Noetherian spaces are flasque and do not need to be resolved. Nor do we have the Dolbeault double complex $\Lambda_X^{p,q}$. Nevertheless, we can still construct a **Hodge to DeRham** spectral sequence whose E_1 term is $H^q(X, \Omega^p)$ by using a construction of pure homological algebra: hypercohomology.

Namely, let $S^\bullet$ be a bounded below (cohomological) complex of sheaves on a topological space.[2] Choose $I^\bullet$ an **injective resolution** of the complex $S^\bullet$: by definition, this means a morphism of complexes $\varphi : S^\bullet \to I^\bullet$ to a complex of injective objects such that $H^\bullet(\varphi) : H^\bullet(S^\bullet) \to H^\bullet(I^\bullet)$ is an isomorphism (a so-called **quasi-isomorphism** of complexes). Note well that this generalizes the notion of an injective resolution of a single sheaf as soon as we identify the sheaf S with the complex $S \to 0 \to 0 \to \ldots$. We need two facts about resolutions of complexes of sheaves whose analogues in the case of a single sheaf are familiar from [CK]: first, that injective resolutions exist, and second that they

[2]It will be clear that we could work in more generality: in an arbitrary abelian category with enough injectives and with some left-exact functor R.

are unique up to homotopy; for the proofs of these facts (which require no new ideas), see e.g. [Iversen].

So, given $S^\bullet$ our complex of sheaves, we define its **hypercohomology** groups $\mathbb{H}^n(S^\bullet) := H^n(\Gamma(X, I^\bullet))$; observe that this too generalizes the definition of cohomology groups of a single sheaf, and are similarly independent of the choice of injective resolution.

In our algebraic setting we have the DeRham complex

$$\Omega_X^\bullet : \ \mathcal{O}_X = \Omega_X^0 \to \Omega_X^1 \to \ldots \to \Omega_X^d \to 0,$$

and we define the **algebraic DeRham cohomology of** X to be the hypercohomology of the DeRham complex:

$$H^n_{DR}(X/k) = \mathbb{H}^n(\Omega^\bullet_{X/k}).$$

In the remainder of this section we explain the following two important facts:

• Why the algebraic DeRham cohomology coincides with the analytic DeRham cohomology in the complex case.

• How to construct a purely algebraic Hodge to DeRham spectral sequence

$$H^q(X, \Omega_X^p) \implies H^{p+q}_{DR}(X). \tag{5.1}$$

When $k = \mathbb{C}$, the Poincaré Lemma holds for holomorphic differentials:

$$0 \to \underline{\mathbb{C}} \to \Omega_X^0 \to \Omega_X^1 \to \ldots \to \Omega_X^n \to 0$$

so that $\Omega_X^\bullet$ is a resolution – not acyclic! – of the constant sheaf $\underline{\mathbb{C}}$. But consider: to take the cohomology of $\underline{\mathbb{C}}$, we take any injective resolution of $\underline{\mathbb{C}}$. Since $\Omega_X^\bullet$ is itself a resolution of $\mathbb{C}$, taking an injective resolution $I^\bullet$ of the complex $\Omega_X^\bullet$, the fact that $\Omega^\bullet \to I^\bullet$ is a quasi-isomorphism precisely means that $\ker(I^0 \to I^1) \cong \ker(\Omega^0 \to \Omega^1) \cong \underline{\mathbb{C}}$ and that thereafter the complex $I^\bullet$ is exact, so that $I^\bullet$ is itself an injective resolution of $\underline{\mathbb{C}}$ and $H^n(X, \underline{\mathbb{C}}) = \mathbb{H}^n(\Omega_X^\bullet)$. So algebraic DeRham cohomology computes DeRham cohomology in the complex case.

Finally, any time we have a complex of sheaves $S^\bullet$ we will get a **hypercohomology spectral sequence**

$$H^q(X, S^p) \implies \mathbb{H}^{p+q}(X, S^\bullet) \tag{5.2}$$

Indeed we take for each S^p an injective resolution $S^p \to I^{p,\bullet}$: these successive injective resolutions form the columns of a double complex. Moreover, since we have a natural bijection in the homotopy category between $\mathrm{Hom}(S^p, S^{p+1})$ and $\mathrm{Hom}_{complexes}(I^{p,\bullet}, I^{p+1,\bullet})$ we can choose essentially unique horizontal maps from one injective resolution to the next. The associated total complex is a complex of injective sheaves quasi-isomorphic to $S^\bullet$ – draw a picture! – i.e., upon taking global sections and then cohomology we have computed the hypercohomology of $S^\bullet$. It follows that if we take global sections of the entire complex, we get a double complex with $E_1^{p,q}$ =Ker($\Gamma(X, I^{p,q}) \to \Gamma(X, I^{p,q+1}))/\operatorname{Im}(\Gamma(X, I^{p,q-1}) \to \Gamma(X, I^(p,q)) = H^q(X, S^p)$. This shows that in general, there is a hypercohomology spectral sequence as in (2) above. Applying it to $\Omega_X^\bullet$, we get a Hodge to DeRham spectral sequence, which, although purely algebraic in nature, coincides in the complex case with the Hodge to DeRham spectral sequence constructed using Dolbeault cohomology groups.

5.4 An example of nondegeneration of the Hodge to DeRham spectral sequence

I want to give some indication of an example of a variety for which the Hodge to DeRham spectral sequence fails to degenerate at E_1. The most "concrete" example I have seen was done by William Lang in his Harvard thesis. Our discussion of this example is nothing close to complete since, for one, Lang's computation of the DeRham Betti numbers uses crystalline cohomology. Nevertheless it is instructive on a rather confusing point in characteristic p geometry: a variety over a nonperfect field can be noninsgular without being smooth!

Let k be an algebraically closed field of characteristic $p > 0$. A **quasi-elliptic surface** is a nonsingular surface X/k which admits a dominant morphism $X \to C$ (C a smooth curve) whose generic fibre is a (geometrically integral) **nonsmooth** curve of genus one. (Compare with the definition of an elliptic surface, which is the same except that the generic fibre is smooth of genus one.)

The first thing to notice is that quasi-elliptic surfaces can only exist in positive characteristic. Indeed, in characteristic zero a morphism $f : X \to Y$ of nonsingular varieties is **generically smooth**, i.e., there exists an open subset U of X such that $f|U$ is smooth (this is equivalent to the generic fibre being smooth) – see [Hartshorne, 3.10.5 3.10.7], but indeed the result is immediate once one remembers that if $k(V)/k$ is a finitely generated field extension of dimension d, then $\dim_k \Omega_{k(V)/k} = d$ iff $k(V)/k$ has a separable transcendence basis of cardinality d – no worries about this in characteristic 0 In general, what can be said is that the generic fibre is a **nonsingular** curve over the imperfect field $k(C)$, and – alas – over an imperfect field nonsingularity (i.e., regularity of the local rings) and smoothness (the Jacobian condition) are distinct notions. We can see this explicitly [Hartshorne, Exercise III.10.1]: consider

$$y^2 = x^p - t,$$

which we may equally well view as giving (the affine model of) a hypersurface in $\mathbb{P}^3$ or as a curve X over $k(\mathbb{P}^1)$. This curve is geometrically singular: as soon as we pass to the field extension $k' = k(t^{1/p})$, the equation becomes

$$y^2 = (x - t^{(1/p)})^p,$$

and certainly $(t^{1/p}, 0)$ is a singular point. To see that over k it is not smooth, consider the differential condition: $\Omega^1_(X/k)$ is generated by dx and dy and subject only to the relation $0 = d(y^2 - x^p - t) = 2ydy$ – at the unique (scheme-theoretic, not k-valued!) point $P \in X$ with $y(P) = 0$, this means that the module of differentials is free of rank $2 > 1$. On the other hand, this strange point P is not a singular point (over k). Indeed, I claim that $m_P = (y)$, so that the local ring $\mathcal{O}_P$ is a DVR and is hence nonsingular. And indeed, $\mathcal{O}_X/(y) = k[x, y]/(y^2 - x^p + t, y) = k[x]/(x^p - t)$ which is a purely inseparable **field** extension of k.

Taking $p = 3$ in the preceding discussion, we do indeed get a quasi-elliptic surface $X/\mathbb{P}^1$ via $y^2 = x^3 - t$. (In fact, quasi-elliptic surfaces exist only in characteristics 2 and 3. The generic fibre, being nonsingular of arithmetic genus 1, must be analytically irreducible – in other words, the preceding considerations require us to have a cusp and not a regular double point. And it is intuitively clear that cuspidal curves can behave in weird ways only when $p = 2$ or 3.) One of the results in Lang's thesis computes the Hodge and DeRham Betti numbers of a quasi-elliptic surface $\pi : X \to C$ fibred over an elliptic curve C/k for which there exists a section $s : C \to X$ such that $s(C)$ is contained in the smooth locus of X. Then, if the line bundle $R^1\pi_*\mathcal{O}_X$ on C has degree 1, one has

$$h^{0,0} = h^{2,2} = 1,\ h^{0,1} = h^{1,0} = 2,\ h^{1,1} = 4,\ h^{0,2} = h^{2,0} = 1,\ h^{1,2} = h^{2,1} = 2.$$

$$b_{0,DR} = 1,\ b_{1,DR} = 3,\ b_{2,DR} = 4,\ b_{3,DR} = 3,\ b_{4,DR} = 1.$$

So we have strict inequality wherever possible: for $1 \leq i \leq 3$, $b_{i,H} > b_{i,DR}$.

5.5 The Deligne-Illusie Theorem

Quasi-hyperelliptic surfaces are "weird" because they can only exist in characteristic p (and indeed only in characterstics 2 and 3). Another example of nondegeneration is that of a **supersingular Enriques surface**, which has the same "weird" flavor: in characteristic zero (and, indeed, in odd positive characteristic) an Enriques surface is by definition a surface which admits an unramified double cover which is a K3 surface ($\pi_1(S) = 0$, ω_S is trivial), so in particular has fundamental group $\mathbb{Z}/2\mathbb{Z}$ (in fact Serre's construction applied to a suitable quartic hypersurface would yield such a "classical" Enriques surface in any characteristic). It turns out that in characteristic 2 the classification of surfaces with Kodaira dimension zero becomes wildly more difficult than in the classical case (it was done by two Fields Medalists, Bombieri and Mumford) and one finds an entirely new surface X with the properties $h^{0,1} = h^{0,2} = 1$ (classical Enriques surfaces have $h^{0,1} = 0$, by Hodge theory) and such that Frobenius acts as zero on $H^1(X, \mathcal{O}_X)$. One finds for these surfaces that $b_{1,H} = 2 > 1 = b_{1,DR}$, so we get another example of nondegeneration which this time only occurs in characteristic two.

In fact this "weirdness" is characteristic of nondegeneration of the spectral sequence.

Theorem 33 *(Deligne-Illusie) Let X/k be a smooth projective variety over an algebraically closed field of characteristic $p > 0$ and of dimension $d < p$. If X lifts even to $W_2(k)$, then the Hodge to DeRham spectral sequence degenerates.*

Of course liftability to $W_2(k)$ certainly does not imply liftability all the way to $W(k)$, so their proof cannot possibly use the Hodge theorem for Kahler manifolds. So consider a variety X over a number field K: for all but finitely many places v of K, X does extend to a smooth scheme over $\mathcal{O}_K$, and by the previous theorem one knows that in characteristic p the spectral sequence degenerates. Deligne and Illusie exploit this degeneration to show that the spectral sequence of the generic fibre degenerates. In summary, they give a **purely algebraic** proof of the (degeneration of the spectral sequence part of the) Hodge theorem for algebraic $\mathbb{C}$-manifolds by reducing to characteristic p, where the degeneration of the spectral sequence is in general false! Those who prefer to keep their distance from the "pathologies" of algebraic geometry in positive characteristic would do well to remember this remarkable success story.

5.6 Relative Hodge theory of Kahler manifolds

In this section we will say a bit about the Hodge theory of a smooth family $\pi : X \to S$. This material belongs at the end of Chapter 2, but because we will use the language of spectral sequences, we have chosen to put it here instead. The source for most of the material in this section was Section 10 of [Demailly]; our remarks about smooth versus holomorphic families from Section 1.3 will be helpful here.

A clue to the fact that one should be able to consider a much more general Hodge theory can be found already in the fact that one has not merely sheaves of differentials for varieties but sheaves of *relative* differentials $\Omega_{X/S}$ associated to an arbitrary morphism of schemes $X \to S$. Moreover $X \to S$ is smooth of dimension d if and only if $\Omega_{X/S}$ is a vector bundle of rank d on X. We assume for the remainder of the section that we have a proper smooth family of complex manifolds over a connected base $\pi : X \to S$. The first basic result is the following

Theorem 34 *(Kodaira-Spencer Semicontinuity Theorem)[Demailly] Let $X \to S$ be a proper smooth $\mathbb{C}$-analytic map and $E \to X$ a locally free sheaf on X; put $h^q(t) := h^q(X_t, E_t)$. Then the $h^q(t)$ are upper-semicontinuous functions on S, and more precisely, so is*

$$h^q(t) - h^{q-1}(t) + \ldots + (-1)^q h^0(t),\ 0 \le q \le n = \dim X_t.$$

Corollary 35 *Let $X \to S$ be a smooth, proper morphism of $\mathbb{C}$-analytic spaces whose fibres X_t are Kahler manifolds. Then the Hodge numbers $h^{p,q}(X_t)$ of the fibres are constant. Moreover, in the Hodge decomposition*

$$H^k(X_t, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X_t, \mathbb{C}),$$

the mappings $t \mapsto H^{p,q}(X_t, \mathbb{C})$ give C^∞ (but in general not holomorphic) subbundles of the bundle $tH^k(X_t, \mathbb{C})$.

Proof: By the Ehresmann Lemma, the Betti numbers $b_k = H^k(X_t, \mathbb{C})$ are constant. Since $h^{p,q}(X_t, \mathbb{C}) = h^q(X_t, \Omega^p_{X_t})$ is by the theorem an upper semicontinuous function of t and

$$h^{p,q}(t) = b_k - \sum_{r+s=k,(r,s)\neq(p,q) h^{r,s}(X_t),}$$

they are clearly lower-semicontinuous as well. So they are continuous, and hence constant.

Theorem 36 *(Kodaira)[Voisin I] For our smooth, proper holomorphic family $\pi : X \to S$, the Kahler locus – i.e., the subset of $s \in S$ such that $\pi^{-1}(s)$ is Kahler – is open. Indeed, if ω_0 is a Kahler metric on the fiber $\pi^{-1}(s_0)$, then on a neighborhood of s_0 one can endow the fibers with Kahler metrics $\omega(s)$ such that $s \mapsto \omega(s)$ is C^∞.*

More precise and more general results are available, using Grauert's direct image theorems. Recall that if $f : X \to Y$ and E is a sheaf on X, the higher direct image sheaves $R^k f_* E$ on Y are given as the sheafification of $U \mapsto H^k(f^{-1}(U), E)$. We have a hyperanalogue: if $A^\bullet$ is a complex of sheaves, we have complexes $\mathbb{R}^q f_*(A^\bullet)$,

$$U \mapsto \mathbb{H}^k(f^{-1}(U), A^\bullet).$$

We have the following fundamental result:

Theorem 37 *(Direct image theorem) Let $\sigma : X \to S$ be a proper morphism of $\mathbb{C}$-analytic spaces and $A^\bullet$ a bounded complex of coherent sheaves of $\mathcal{O}_X$-modules. Then:*
a) $\mathbb{R}^k \sigma_* A^\bullet$ is a complex of ocherent sheaves on S.
b) Every point of S admits a neighborhood $U \subset S$ on which there exists a bounded complex $W^\bullet$ of $\mathcal{O}_S$-modules whose sheafified cohomology $\mathcal{H}^k(W^\bullet)$ are isomorphic to the complexes $\mathbb{R}^k \sigma_* A^\bullet$).
c) If σ has equidimensional fibers, the hypercohomology of the fiber X_t with values in $A^\bullet_t := A^{bullet} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_t}$ ($\mathcal{O}_{X_t} = \mathcal{O}_X / \sigma^* \mathfrak{m}_{S,t}$) is given by

$$\mathbb{H}^k(X_t, A^\bullet_t) = \mathbb{H}^k(W^\bullet_t),$$

where $W^\bullet_t$ is the finite-dimensional complex of sheaves $W^k_t := W^k \otimes_{\mathcal{O}_S,t} \mathcal{O}_{s,t} / \mathfrak{m}_{S,t}$.
d) Under the hypothesis of c), if the hypercohomology fibrations $\mathbb{H}^k(X_t, A^\bullet_t)$ have constant dimension, the sheaves $\mathbb{R}^k \sigma_* A^\bullet$ are locally free on S.

From part b) and (the proof of) the Kodaira-Spencer theorem, one deduces:

Theorem 38 *(Semicontinuity theorem) If $X \to S$ is a proper morphism of $\mathbb{C}$-analytic spaces with equidimensional fibers and E/X is a coherent sheaf, then, putting $h^q(t) := H^q(X_t, E_t)$, we find that*

$$h^q(t) - h^{q-1}(t) + \ldots + (-1)^q h^0(t)$$

are upper semicontinuous functions of t, (even) for the analytic ***Zariski*** *topology on S (i.e., where the closed sets are the zero sets of finitely many analytic functions).*

Now is the time to recall (Section 1.3) that the fiber cohomologies $H^k(X_t, \mathbb{C})$ are locally constant functions of t and are thus canonically endowed with a flat connection, the Gauss-Manin connection. As we noted at the time, this implies that $t\mathrm{H}^k(X_t, \mathbb{C})$ has the canonical structure of a holomorphic vector bundle. The total cohomology $\bigoplus_k H^k(X_t, \mathbb{C})$ is called the **Hodge bundle** of $X \to S$.

Consider now the relative DeRham complex $\Omega^\bullet_{X/S}, d_{X/S})$ of $X \to S$. This complex furnishes us with a resolution of $\sigma^{-1}(\mathcal{O}_S)$,

$$\mathbb{R}^k \sigma_* \Omega^\bullet_{X/S} = R^k \sigma_*(\sigma^{-1}(\mathcal{O}_S)) = R^k \sigma_*(\sigma^{-1}\mathcal{O}_S) = (R^k \sigma_* \mathbb{C}_X) \otimes_{\mathbb{C}} \mathcal{O}_S. \quad (5.3)$$

The last – important! – equality comes from the $\mathcal{O}_S(U)$-linearity for the cohomology caluclated on $\sigma^{-1}\mathcal{O}_S$. In other words, $\mathbb{R}^k \sigma_* \Omega^\bullet_{X/S}$ *is* the locally free $\mathcal{O}_S$-module associated to the locally constant sheaf $t \mapsto H^q(X_t, \mathbb{C})$.

We get a spectral sequence of hypercohomology

$$E_1^{p,q} = R^q \sigma_* \Omega^p_{X/S} \implies G^p \mathbb{R}^{p+q} \sigma_* \Omega^\bullet_{X/S} = G^p R^{p+q} \sigma_* \mathbb{C}_X.$$

(This spectral sequence is obtained from the prior (general) hypercohomology spectral sequence by sheafifying.) Since the cohomology of $\Omega^p_{X/S}$ along the fibres X_t is nothing but the constant rank guy $H^q(X_t, \Omega^p_{X_t})$, part d) of the direct image theorem shows that the $R^q \sigma_* \Omega^p_{X/S}$ are locally free. Finally, the filtration $F^p H^k(X_t, \mathbb{C}) \subset H^k(X_t, \mathbb{C})$ is obtained at the level of locally free $\mathcal{O}_S$-modules by taking the image of the $\mathcal{O}_S$-linear map

$$\mathbb{R}^k \sigma_* F^p \Omega^\bullet_{X/S} \to \mathbb{R}^k \sigma_* \Omega^\bullet_{X/S},$$

a coherent subsheaf (in fact locally free, because of the constancy of rank of the fibres). From this and equation (6) one gets:

Theorem 39 *(Holomorphy of the Hodge filtration) The Hodge filtration $F^p H^k(X_t, \mathbb{C}) \subset H^k(X_t, \mathbb{C})$ is a holomorphic subbundle, with respect to the holomorphic structure defined by the Gauss-Manin connection.*

In general, $H^{p,q}(X_t, \mathbb{C}) = F^p H^k(X_t, \mathbb{C}) \cap \overline{F^q H^k(X_t, \mathbb{C})}$ has no reason to be a holomorphic **subbundle** of $H^k(X_t, \mathbb{C})$, even though $H^{p,q}(X_t, \mathbb{C})$ has a natural structure of a holomorphic vector bundle, obtained either from the cohrent sheaf $R^q \sigma_* \Omega^p_{X/S}$ or as a quotient of $F^p H^k(X_t, \mathbb{C})$. Otherwise put, it is the Hodge decomposition which need not be holomorphic.

Chapter 6

What should crystalline cohomology do?

Let X/k be a proper smooth variety over a field of characteristic p (which we no longer assume to be algebraically closed). Next week we will see the definition of the **crystalline cohomology groups** $H^n_{crys}(X)$, but at least now we know what to expect: first of all, as in Chapter 1, it is unreasonable for them to be $\mathbb{Z}_p$-valued, unless $k = \mathbb{F}_p$. We could agree to work only with a fixed algebraically closed field k and get a Weil cohomology with coefficients in $W(k)$, but crystalline cohomology is actually more flexible than this. Namely, the functor $X \mapsto H^\bullet_{crys}(X)$ goes from varieties over k to graded $W(k)$-algebras which are finitely generated as $W(k)$-modules in a way which is also functorial in k. Here are three important results about this crystalline cohomology:

Theorem 40 *(Comparison Theorem) if X/k lifts to a smooth $\mathcal{X}/W(k)$, then $H^n_{crys}(X) \cong H^n_{DR}(\mathcal{X}/W(k))$, showing that the DeRham cohomology groups are independent of the choice of a lifting when a lifting exists.*

Theorem 41 *(Universal Coefficient Theorem) The DeRham Betti numbers $b^i_{DR}(X)$ may be computed in terms of the crystalline cohomology, as follows:*

$$b_{i,DR}(X) = b_{i,crys} + t_{i,crys} + t_{i+1,crys} = b_{i,\ell} + t_{i,crys} + t_{i+1,crys}$$

where $b_{i,crys}$ is the free rank of $H^i_{crys}(X/k)$ (crystalline Betti number) and $t_{i,} =$ $\dim_k H^n_{crys}(X) \otimes_{W(k)} k$ *(crystalline torsion coefficent).*

In particular, the difference between $b_{i,DR}$ and $b_{i,\ell}$ is $t_{i,crys} + t_{i+1,crys}$ so is "due to torsion".

Theorem 42 *(Frobenius functoriality) For all n, $H^n_{crys}(X)$ has a canonical semilinear action of F and V making it into a Dieudonné module (or **F-crystal**) in such a way that $H^1_{crys}(A) \cong DA$ for an abelian variety A/k.*

This last theorem has been used by Katz, Mazur and others to relate the Hodge filtration on the DeRham cohomology to the **slopes** of the Dieudonné module $H^n_{crys}(X)$. This work can be viewed as a p-adic analogue of the Riemann hypothesis: rather than studying the Archimedean valuations of the Frobenius eigenvalues, we study their p-adic valuations, leading to p-adic estimates on the number of rational points of a variety over a finite field.

In summary, crystalline cohomology was motivated by the search for a cohomology theory in characteristic p that would explain topological p-torsion. The final theory accomplishes this, but is much more interesting than one would expect from merely topological considerations: it is intimately related to both Hodge theory and DeRham cohomology.