A SITE-THEORETIC CHARACTERIZATION OF POINTS IN A TOPOLOGICAL SPACE

ANDREW ARCHIBALD, PETE L. CLARK, AND DAVID SAVITT

Abstract. We study points in the context of Grothendieck topologies (sites), especially the relationship between "honest" points of a topological space X and site-theoretic points of X_{top}. We offer an elementary and self-contained proof of a theorem of Grothendieck-Hakim which gives necessary and sufficient conditions for these two notions to coincide. We also explore analogues of this result for other sites encountered in geometry by giving axioms on a site sufficient to ensure the correspondence between site-theoretic points and the points of a certain topological space. These axioms are satisfied for the étale, fppf, fpqc and crystalline sites. They are not satisfied by the large étale site of Spec $\mathbb{C}$, in which (as we show) the site-theoretic points form a proper class.

1. Introduction

Recall that a *site*, or *Grothendieck topology* $(\mathcal{C}, \mathcal{T})$, consists of (i) a category $\mathcal{C}$ possessing a final object and closed under fiber products, together with (ii) for each object U in $\mathcal{C}$, a distinguished collection of families of morphisms $\{U_i \to U\}_{i \in I}$, called the *covering families* of U. We require the following properties:

(GT1) If $U' \to U$ is an isomorphism, then $\{U' \to U\}$ is a covering family.
(GT2) If $\{U_i \to U\}_{i \in I}$ is a covering family and $V \to U$ is a morphism, then $\{U_i \times_U V \to V\}_{i \in I}$ is a covering family of V.
(GT3) If $\{U_i \to U\}_{i \in I}$ is a covering family and for each $i \in I$, $\{V_{i,j} \to U_i\}_{j \in J}$ is a covering family of U_i, then the composite family $\{V_{i,j} \to U_i \to U\}_{(i,j) \in I \times J}$ is a covering family of U.

For a topological space T, we define the topological site T_{top} whose underlying category has as objects the open subsets of T and as morphisms the inclusions between open sets; here the final object is X itself and $U_1 \times_V U_2 = U_1 \cap U_2$. As expected, the covering families are collections $\{U_i \to U\}_{i \in I}$ such that $\bigcup_{i \in I} U_i = U$.

A morphism of sites $F : (\mathcal{C}_1, \mathcal{T}_1) \to (\mathcal{C}_2, \mathcal{T}_2)$ is a functor $F^{-1} : \mathcal{C}_2 \to \mathcal{C}_1$ which preserves the final object and fibred products, and which preserves covering families. "Choose" a one-point space $\{\star\}$ and write 0 for $\emptyset$, 1 for $\{\star\}$, and $\{0, 1\}$ for the topological site of (the unique topological space with underlying set) $\{\star\}$.[1]

Key words and phrases. site, Grothendieck topology, étale site, fppf site, fpqc site, crystalline site, pointless topology, filter, morphism of sites.

[1]This notation is a nod at the connections with Boolean algebra, but we will leave to the interested reader the task of systematically rephrasing our results in this language.

In this note we are concerned with the following question: is it reasonable to define a "point" in a site $(\mathcal{C}, \mathcal{T})$ as a morphism of topologies from $\{0,1\}$ to $(\mathcal{C}, \mathcal{T})$?[2]

We will refer to a morphism of topologies

$$F : \{0,1\} \to (\mathcal{C}, \mathcal{T})$$

as a *site-theoretic point* of $(\mathcal{C}, \mathcal{T})$, and we denote by $(\mathcal{C}, \mathcal{T})(\star)$ the collection of all site-theoretic points of $(\mathcal{C}, \mathcal{T})$. If F is a site-theoretic point of $(\mathcal{C}, \mathcal{T})$ and V is an object of $\mathcal{C}$, we say that V pulls back to 0 or 1 if $F^{-1}(V) = 0$ or 1, respectively.

As motivation, observe that if the site $(\mathcal{C}, \mathcal{T})$ is the site X_{top} of a topological space X, we can associate to any point $P \in X$ a morphism of sites $F_P : \{0,1\} \to X_{\mathrm{top}}$ by decreeing that an open subset $V \subset X$ pulls back to 1 iff $P \in V$. So when $(\mathcal{C}, \mathcal{T}) = X_{\mathrm{top}}$ we have defined a map

$$\Phi : X \to X_{\mathrm{top}}(\star), \ P \mapsto F_P$$

from the points of X to the site-theoretic points of X_{top}. When is Φ a bijection?

Here are two examples in which it is *not*:

First: take a set X with cardinality greater than one, and endow it with the trivial topology $\mathcal{T}_X = \{\emptyset, X\}$. Then $X_{\mathrm{top}} \cong \{0,1\}$ as sites, and one checks immediately that the identity is the only morphism of sites from $\{0,1\}$ to itself. So all points P of X induce the same site-theoretic point F_P, and Φ is not injective.

Second: let X be an infinite set endowed with the cofinite topology, in which a subset is open if and only if its complement is finite. Using the fact that any two nonempty open subsets of X have nonempty intersection, one sees that by mapping *every* nonempty open set to 1 and the empty set to 0 we get a site-theoretic point G. But for every $P \in X$, there is an open set — namely $X \setminus P$ — which does not contain P, so G is not equal to F_P for any point P of X, and Φ is not surjective.

While the first example does not worry us — the trivial topological space is indeed rather trivial — the second is more alarming, since it is precisely the Zariski topology on the set of k-points of an irreducible algebraic curve C over an algebraically closed field k. However, when we consider not the k-points but all points on the associated *scheme* of C, we get an extra point — the generic point — which lies in every nonempty open set, so G does come from a point on this larger space.

The passage from the Zariski topology on $C(k)$ to the topological space of the associated scheme can be generalized: to any topological space X one can associate a new space $t(X)$ whose points are the irreducible closed subspaces of X, endowed with the canonical (quotient) topology. We call this process t-completion and call the spaces in its image t-complete: they are precisely the topological spaces for which every irreducible closed subspace is the closure of a unique point. This includes in particular the class of Hausdorff spaces. T-completion is precisely what one needs to do to a space in order to make the map Φ a bijection:

Theorem 1 (Main Theorem). *Let X be a topological space.*

(1) *The map $\Phi : X \to X_{\mathrm{top}}(\star)$ is a bijection if and only if X is t-complete. More precisely Φ is injective if and only if X is T_0, and Φ is surjective*

[2]This question arose in E. Goren's 2003 seminar on cohomology theories.

if and only if every irreducible closed subset is the closure of at least one point.

(2) *The map from X to its t-completion $t(X)$ induces an isomorphism of sites $X_{\mathrm{top}} \xrightarrow{\sim} t(X)_{\mathrm{top}}$. It follows that for any space X, there is a canonical bijection between the set of site-theoretic points of X and the usual points of $t(X)$.*

The data of a site-theoretic point of X is strongly reminiscent of that of a filter on X, but with two differences: it is a Boolean-valued function not on all the subsets of X but only on the open subsets, so it is what we will call a "filter of open sets." More significantly, that (not necessarily finite) covers be preserved corresponds to an extra, and quite strong, condition on the filter, our so-called Axiom A. We call a filter of open sets satisfying Axiom A an A-filter, and our main theorem is really a classification of A-filters on t-complete spaces.

We stress that Theorem 1 and Theorem 11, which compare points and continuous functions in topological spaces in the conventional sense to points and continuous functions on topological sites, are far from new. Rather Theorem 11 is stated in [SGA4, Exposé IV, 4.2.3, p. 338] and proved in the thesis of [Hak]. Moreover this result, which allows for a complete study of a t-complete topological space X in terms of the lattice of open subset of X, has become the jumping-off point for an entire school of mathematics ("pointless topology"). In pointless topology, Theorem 11 is expressed as a canonical equivalence from the category of t-complete spaces to the category of *locales*, or as canonical duality between t-complete spaces and *spatial frames* (a category of lattices of a certain type). This duality generalizes Stone's duality between totally disconnected compact Hausdorff spaces ("Stone spaces," or "Boolean spaces") and Boolean algebras, and indeed the general form also goes under the name *Stone duality* [Joh].

Challenged by Professor Goren's question, we found our own proof of these theorems. Upon searching the literature we found no shortage of proofs: however, all of the existing literature on the subject seems to be couched in the language of topos theory (as in Hakim's proof) or of lattice theory (as in e.g. [Joh]), or both. Many 21st century mathematicians who wish to use Grothendieck topologies in their work – e.g. most students and/or researchers in algebraic or arithmetic geometry (and in particular the authors of this paper) – lack fluency in either of these languages, so we feel that there may be some interest in our more elementary and directly "topological" approach. We believe that the material of §6 *is* new, and will be of interest to those learning about the étale, flat and crystalline sites.

The note is organized as follows: in Section 2 we recall basic material on filters and convergence. Moreover we introduce two invariants of a filter, its base locus and convergence set. We then consider how each of the separation axioms T_0, T_1 and T_2 (Hausdorff) can be formulated in terms of convergence of filters; this is summarized in Scholium 2. In Section 3 we explain why a site-theoretic point is equivalent to an A-filter, show that any A-filter on a topological space converges to at least one point, and recall basic notions about the t-completion functor. In Section 4 we prove the Main Theorem; in Section 5 we give conditions under which morphisms of (topological) sites arise from continuous functions; and in Section 6 we discuss generalizations to other sites encountered in algebraic geometry.

2. Filters on a set

A filter $\mathcal{F}$ on a set X is a family of subsets of X satisfying:

$$X \in \mathcal{F},\ \emptyset \notin \mathcal{F} \tag{1}$$

$$U,\ V \in \mathcal{F} \implies U \cap V \in \mathcal{F} \tag{2}$$

$$U \in \mathcal{F},\ U \subset V \implies V \in \mathcal{F} \tag{3}$$

The collection of filters on a set is partially ordered by inclusion, and an *ultrafilter* is a maximal element. That every filter is contained in an ultrafilter is an immediate consequence of Zorn's Lemma; moreover, ultrafilters are characterized among filters by the property that if $U \subset X$ is any subset, then precisely one of U and $X \setminus U$ is in $\mathcal{F}$. It follows that ultrafilters have the following fineness property (which we will later recognize as a weaker analogue of Axiom A): $\mathcal{F}$ is an ultrafilter on X and $U = \bigcup_{i=1}^{n} U_i$ are subsets of X such that $U \in \mathcal{F}$, then some $U_i \in \mathcal{F}$.

Filter bases: Define a *prefilter* (or *filter base*) on X to be a collection F of nonempty subsets of X satisfying the axiom that for all $A,\ B \in F$, there exists $C \in F$ such that $C \subset A \cap B$. The collection of all filters containing a prefilter is nonempty and has a unique minimal element, the filter $\mathcal{F}$ generated by F: namely, $\mathcal{F}$ consists of all subsets V of X which contain at least one $U \in F$.

If $S \subset X$ is any nonempty subset, then $\{S\}$ is a prefilter on X; we denote by $\mathcal{F}_S$ the filter it generates; it is just the collection of all subsets of X containing S. Clearly $\mathcal{F}_S \subset \mathcal{F}_T$ if and only if $T \subset S$. Thus $\mathcal{F}_S$ can only be an ultrafilter if S is a one-point subset, and conversely for any $P \in X$, one sees easily that $\mathcal{F}_P := \mathcal{F}_{\{P\}}$ is an ultrafilter, called *principal.*

Define the *base locus* of a filter $\mathcal{F}$ to be $\bigcap_{U \in \mathcal{F}} U$. If the base locus of $\mathcal{F}$ is a nonempty subset S, then $\mathcal{F} \subset \mathcal{F}_S$; in particular, any filter whose base locus consists of more than a single point cannot be an ultrafilter, and the only ultrafilter whose base locus is P is $\mathcal{F}_P$, so the ultrafilters with nonempty base locus are precisely the principal ultrafilters. Consider now the class of all filters with empty base locus. Note first that if X is a finite set, the intersection in the definition of the base locus is a finite intersection, so every filter on a finite set has nonempty base locus. On the other hand, assume that X is infinite. The family of filters with empty base locus has a unique *minimal* element, namely the so-called *Frechet filter* comprised of all cofinite subsets. So every infinite set admits at least one nonprincipal ultrafilter.

2.1. Convergence.

Now let X be a topological space. If P is a point of X, define $\mathcal{N}_P$ to be the collection of neighborhoods of P — i.e. subsets Y of X for which P is in the interior of Y. $\mathcal{N}_P$ is a filter on X whose base locus contains P. We say that $\mathcal{F}$ *converges* if for some P, $\mathcal{N}_P \subset \mathcal{F}$.

We come now to the purpose of this section, which is to record the implications of each of the separation axioms T_0, T_1, T_2 on convergence. First, that X satisfies the T_1-separation axiom is equivalent to: for all $P \in X$, the base locus of $\mathcal{N}_P$ is P.

More generally, the relation $\mathcal{N}_P \subset \mathcal{N}_Q$ is equivalent to: P lies in the closure of Q, or in more geometric language, Q *specializes* to P. It follows that the base locus of $\mathcal{N}_P$ is the set of all points Q which specialize to P, or the set of all *generizations* of P. In these terms, T_1 is equivalent to the specialization relation being trivial: no point specializes to any distinct point. That X satisfies the T_0-separation axiom is equivalent to specialization being a partial ordering on X (i.e., if P specializes to some $Q \neq P$, then Q does not specialize to P); it follows that T_0 is also equivalent to the mapping $P \mapsto \mathcal{N}_P$ being injective.

Scholium 2.
(1) X is T_0 if and only if for all points $P \neq Q$ of X, there is a filter on X which converges to exactly one of P and Q.
(2) X is T_1 if and only if for every point P of X there is a filter converging to P and to no other point of X.
(3) X is T_2 (Hausdorff) if and only if every convergent filter converges to a unique point.

For example, in a trivial topological space every point specializes to every other point; equivalently $\mathcal{N}_P = \mathcal{N}_Q$ for all $P,\ Q$ in X.

Another invariant of a filter $\mathcal{F}$ is its *convergence set* $c(\mathcal{F})$, i.e., the set of all points of X to which $\mathcal{F}$ converges.

Proposition 3. *Every ultrafilter on a quasi-compact space is convergent.*

Proof. Since the intersection of any finite number of elements of $\mathcal{F}$ is nonempty, certainly the same is true of the intersection of finitely many closures of elements of $\mathcal{F}$. So we have a family of closed sets satisfying the finite intersection condition in a quasi-compact space, so $\bigcap_{U\in\mathcal{F}} \overline{U} \neq \emptyset$. Take x in the intersection. Then the collection of subsets obtained by intersecting elements of $\mathcal{F}$ with neighborhoods of x generates a filter at least as fine as $\mathcal{F}$, but since $\mathcal{F}$ is an ultrafilter, they must coincide, so that $\mathcal{F}$ is finer than the neighborhood filter of x. □

3. Filters of open sets and A-filters

3.1. **Filters of open sets.** Suppose now that X is a topological space and we are given a nonempty family F of nonempty *open* subsets of F which is closed under intersection and if $U \in F$ then any larger *open* subset of X is in F. We say F is a *filter of open sets*. These share many of the properties of filters (which we will now call, for clarity, filters of sets). Observe that any filter of open sets is a prefilter of sets, so if we like we can take the filter it generates. (In particular, our notation F for a filter of open sets is consistent with the notation F for a prefilter of sets; recall that we then wrote $\mathcal{F}$ for the filter it generates.) As an example, we have the filter N_P of *open* neighborhoods of X, and the filter it generates is just $\mathcal{N}_P$.

We define the base locus of a filter of open sets exactly as for filters of sets and define the convergence set of a filter of open sets in almost the same way: P is in $c(F)$ if and only if $N_P \subset F$. Equivalently, define the base locus and convergence sets of a filter of open sets in terms of the filter of sets $\mathcal{F}$ generated by F.

For filters of open sets the base locus and convergence sets have a certain complementarity: the base locus of F is closed under generization, while the convergence set of F is closed under specialization. Also, since $P \in b(F)$ means $F \subset N_P$ and $P \in c(F)$ means $N_P \subset F$, in a T_0-space $b(F) \cap c(F)$ contains at most one element.

We will be interested in when a filter F of open sets is equal to the neighborhood filter N_P of some point of X. Since F converges to P means that $N_P \subset F$, we shall say that F *equiconverges* to P if $N_P = F$ and that F *overconverges* to P if N_P is properly contained in F.

3.2. **Site-theoretic points are A-filters.** We come back to back to the notion of a site-theoretic point

$$F : \{0, 1\} \to X_{\text{top}}.$$

of a topological space X. Suppose that P is a point of X, and recall the associated morphism of sites $F_P = \Phi(P)$ from Section 1. Its data, as for any site-theoretic point, is just the collection of open subsets V of X that pull back to 1 under F, and by definition of F_P this is none other than N_P, the filter of open neighborhoods of P.[3]

As promised in Section 1, we now explain how a site-theoretic point of X is equivalent to a filter of open sets that satisfies an extra axiom.

I. Every site-theoretic point F on X_{top} determines a filter of open subsets of X, namely the family of open sets pulling back to 1.

Proof. Preservation of final objects gives $F^{-1}(X) = 1$, so the family is nonempty. Preservation of fibre products amounts to the familiar identity $F^{-1}(U \cap V) = F^{-1}(U) \cap F^{-1}(V)$; thus $F^{-1}(U) = F^{-1}(V) = 1$ implies $F^{-1}(U \cap V) = 1$. Finally, the fact that covers pull back to covers implies the stability under passage to larger open subsets: if $U \subset V$ are two open subsets and $F^{-1}(U) = 1$, then since $\{U, V \to V\}$ is a cover of V, it pulls back to the covering $\{F^{-1}(U),\ F^{-1}(V) \to F^{-1}(V)\} = \{1, F^{-1}(V) \to F^{-1}(V)\}$. Hence $F^{-1}(V)$ is nonempty, and so must equal 1. □

II. A filter of open sets F on X determines a site-theoretic point if and only if it satisfies the additional *Axiom A*: if $\{U_\alpha \to U\}$ is a covering of U by open subsets, then $U \in F$ if and only if some $U_\alpha \in F$. Indeed, whereas at the end of I. we used a special case of the covering axiom, this *is* the covering axiom. We call a filter of open sets satisfying Axiom A an *A-filter*.

Thus site-theoretic points of X_{top} correspond to A-filters of open subsets of X.

3.3. **Convergence of A-filters.** A-filters have the following nice property which is in analogy to Proposition 3 for ultrafilters of sets:

Proposition 4. *Any A-filter on a topological space X converges.*

[3]At this point we have three different objects $\mathcal{N}_P$, N_P and F_P which are all, in essence, keeping track of the neighborhoods of P. Such are the uneconomies of precision.

Proof. To say that a filter F does not converge to any point is to say that for all $P \in X$ there exists an open neighborhood U_P of P which is not an element of F. But then $\{U_P : P \in X\}$ gives an open cover of X by subsets not in F, contradicting Axiom A. □

We can bring out the analogy to ultrafilters by giving a different proof of Proposition 3 along these lines: as we remarked in Section 2, an ultrafilter of sets satisfies Axiom A for finite covers, and in a quasi-compact space we can begin the proof as above and then extract a finite subcover.

Hausdorff spaces are easy:

Proposition 5. *If X is a Hausdorff space, then every A-filter equiconverges to a unique point: the map $\Phi : X \to X_{\mathrm{top}}(\star)$ is a bijection.*

Proof. By Scholium 2 and Proposition 4, every A-filter F on a Hausdorff space converges to a unique point P, so we need only show that F does not overconverge to P. Suppose it does: then there is $V \in F$ such that P is not in V. For every $Q \neq P$ in X, let U_P and V_Q be disjoint open neighborhoods of P and Q respectively. Since $\{V_Q \cap V\}$ is an open cover of V, by Axiom A some $V_Q \cap V$ is in F; since $U_P \in F$ and $U_P \cap (V_Q \cap V) = \emptyset$, this is a contradiction. □

3.4. **The t-completion functor.** On page 78 of [Har], a functor is given on the category of topological spaces: to any topological space X, one associates $t(X)$ the set of irreducible closed subspaces of X, and endows it with the canonical topology, i.e., the closed subsets are of the form $t(Y)$ as Y ranges over all the closed subspaces of X. The map $t_X : X \to t(X)$ taking $P \mapsto \overline{P}$ is continuous and functorial in X; moreover, the map $V \mapsto t_X^{-1}(V)$ is a bijection between the open subsets of $t(X)$ and those of X. The functor t is idempotent, hence can be viewed as a completion of X. We call this functor *t-completion* and say of a space for which $t_X : X \cong t(X)$ that it is *t-complete.*[4]

Remark 6. What we call t-complete is called *espace pur* in [Hak] and *sober* in [SGA4]. The latter terminology seems to have become standard, but as it forces us to speak of the "sobrification" ("sobering up"?) of a topological space, we will stick with our relatively innocuous neologism.

Any Hausdorff space is t-complete (the irreducible closed subsets are the points).

A t-complete space X is T_0: indeed if $P \in \overline{Q}$ and $Q \in \overline{P}$ then $\overline{P} = \overline{Q}$, so $t_X(P) = t_X(Q)$, so $P = Q$. In fact, one checks that a space X is T_0 if and only if the map t_X is injective. So for our purposes, T_0 is the "real" separation axiom — it is equivalent to "t-separated" — whereas t-completion can take a T_1 space to a space that is merely T_0 (and will do so, when the T_1 space is irreducible).

Remark 7. The underlying topological space of any scheme is t-complete. One checks this first for affine schemes, which is easy, and then for arbitrary schemes as follows: cover X by open affine subsets U_i. If two points P and Q had $N_P = N_Q$

[4]In particular, if you enjoy such things, t-completion is the left-adjoint of the forgetful functor from t-complete spaces to arbitrary topological spaces: for any X and t-complete Y, we have a natural isomorphism $C(t(X), Y) = C(X, Y)$.

then in particular they both lie in some U_i, contradicting the T_0 axiom for affine schemes. If $Y \subset X$ is an irreducible closed subset, it now suffices to show that Y, endowed with any closed subscheme structure, has a generic point. But of course it does: since Y is irreducible, the generic point of any member of an open affine cover of Y will do.

Proposition 8. *For any topological space, $t_X : X \to t(X)$ induces an isomorphism of sites $t_{\text{top}} : X_{\text{top}} \to t(X)_{\text{top}}$. In particular, there is a canonical bijection t_* between filters of open sets on X and on $t(X)$ under which the notion of A-filter is preserved.*

Proof. If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f : X \to Y$ is a map of sets inducing a well-defined bijection $\mathcal{T}_Y \to \mathcal{T}_X$, then $f_{\text{top}} : X_{\text{top}} \to Y_{\text{top}}$ is an isomorphism of sites. We observed above that the t-functor has this property. □

4. Proof of the main theorem

Our Main Theorem (Theorem 1) can be deduced immediately from the following two results:

Theorem 9. *Every A-filter F on a topological space X equiconverges to a unique point P on the t-completion $t(X)$. This point P can be characterized in any of the following ways:*

1. *It is the unique maximally special point in the base locus of $t_*(F)$, i.e., the unique point P of $b(t_*(F))$ such that for all $Q \in b(t_*(F))$, $P \in \overline{Q}$.*
2. *It is the unique maximally generic point in the convergence set of $t_*(F)$, i.e., the unique point P of $c(t_*(F))$ such that for all $Q \in c(t_*(F))$, $Q \in \overline{P}$.*
3. *It is the unique element of $b(t_*(F)) \cap c(t_*(F))$.*

Proof. Let F be an A-filter on a topological space X. By Proposition 8 we may (and shall) assume that X is t-complete. Let S denote the convergence set of F.

Step 1: We claim that S is an irreducible closed subset whose complement S^c is not in F. (This step does not use the t-completeness of X.) To see that it is closed, for a point Q not in S, F_Q is not contained in F, so there is an open neighborhood U_Q of Q which is not in F. But being open, U_Q is a neighborhood of each of its points, showing that F cannot converge to any point of U_Q and $S \cap U_Q = \emptyset$. Moreover, as we just covered S^c with open sets which are not in F, by Axiom A S^c is itself not in F. To see that S is irreducible: assume not, so we may write $S = S_1 \cup S_2$ with each S_i closed and such that there exist $P_1 \in S \setminus S_2$ and $P_2 \in S \setminus S_1$. But F converges to P_1, so the open neighborhood $X \setminus S_2$ of P_1 is an element of F, and similarly so is $X \setminus S_1$. But then $(X \setminus S_1) \cap (X \setminus S_2) = S^c$ is in F, a contradiction.

Step 2: Using Step 1 and now the t-completeness of X, S has a unique generic point P. In other words, statement (2) of the Theorem holds for F and P: in the natural ordering of points on a T_0 space, P is the unique greatest point to which F converges: $N_P \subset F$.

Step 3: We finish by showing that part (2) of the theorem holds for P: it remains to see that P is in the base locus, i.e. $F \subset N_P$. Indeed if so $F = N_P$ and P is the unique element of the base locus and the convergence set. So suppose that U is an open subset of X which does not contain P. Then $U \cap S = \emptyset$. (Indeed, if $Q \in U \cap S$, then since Q is in the closure of P, every open set containing it meets P, a contradiction. In other words, if an open set contains any point of an irreducible closed subset, it contains the generic point.) So $U \subset S^c$, but as we observed above that S^c is not in F, neither can U be, a contradiction. Thus $P \in b(F)$, completing the proof of the theorem. □

Proposition 10. *If X is a topological space such that every A-filter on X equiconverges to a unique point, then X is t-complete.*

Proof. By Scholium 1, if X is not T_0 then there are distinct points P and Q with coincident neighborhood A-filters: N_P equiconverges to P but also to Q. So we may assume that X is T_0 hence that it injects into its t-completion. A point $t(S)$ of $t(X) \setminus X$ corresponds to an irreducible closed subset S of X which is not the closure of any point of X. By Proposition, the neighborhood A-filter $F_{t(S)}$ on $t(S)$ can be canonically viewed as an A-filter on X, which, since S lacks a generic point, is clearly not the neighborhood filter of any point on X. □

5. Morphisms of sites and continuous functions

As an application of Theorem 1, we give a site-theoretic characterization of continuous functions from an arbitrary space X to a t-complete space Y.

Theorem 11. *Let X and Y be topological spaces. Let $C(X,Y)$ denote the set of continuous functions from X to Y, and let $Mor(X_{\text{top}}, Y_{\text{top}})$ be the collection of morphisms from the topological site on X to the topological site on Y. The natural map $\Phi : C(X,Y) \to Mor(X_{\text{top}}, Y_{\text{top}})$ is injective if and only if Y is T_0, and is surjective if and only if every irreducible closed subset in Y has a generic point.*

Proof. Suppose to begin with that Y is t-complete, and let $F : X_{\text{top}} \to Y_{\text{top}}$ be a morphism of sites. We define a function $f : X \to Y$ in the following way: if $P \in X$, then we may compose $F_P : \{0,1\} \to X_{\text{top}}$ with the morphism of sites F to obtain a morphism of sites $F'_P : \{0,1\} \to Y_{\text{top}}$. By Theorem 1, F'_P defines a unique point $Q \in Y$, and we set $f(P) = Q$.

Now let V be an open set in Y which contains $f(P)$. Then $F_P^{-1}(F^{-1}(V)) = (F'_P)^{-1}(V) = 1$, so $F^{-1}(V)$ contains P. It follows that $F^{-1}(V)$ and $f^{-1}(V)$ are actually equal; in particular, $f^{-1}(V)$ is open and f is continuous.

Now suppose Y is arbitrary. We have a continuous function $t_Y : Y \to t(Y)$. Then

$$\text{Mor}(X_{\text{top}}, Y_{\text{top}}) = \text{Mor}(X_{\text{top}}, t(Y)_{\text{top}}) \overset{\Phi^{-1}}{=} C(X, t(Y))$$

by Proposition 8 and the above argument. So we are reduced to comparing $C(X,Y)$ and $C(X,t(Y))$, and specifically we would like to know: if $f : X \to t(Y)$ is any continuous function, when can we lift it to a continuous function g making the following diagram commute, and when are such lifts unique?

$$\begin{array}{ccc} & & Y \\ & {\scriptstyle g} \nearrow & \downarrow {\scriptstyle t_Y} \\ X & \xrightarrow{f} & t(Y) \end{array}$$

The fact that t_Y^{-1} induces a bijection between the open subsets of $t(Y)$ and those of Y implies that any set-theoretic mapping g making the diagram commute is necessarily continuous. Indeed, any open $V \subset Y$ is of the form $t_Y^{-1}(V')$ for an open $V' \subset t(Y)$ (namely, $V' = t(V^c)^c$) and then $g^{-1}(V) = g^{-1}(t_Y^{-1}(V')) = f^{-1}(V')$ is open. So we can answer the question in the category of sets, where it reduces to the usual conditions on uniqueness and existence of factorization of maps.

If t_Y is surjective, then such a function g always exists. Conversely, if t_Y is not surjective, so that there is $Q \in t(Y)$ which is not in the image of t_Y, then the function taking all of X to Q does not lift to a function $g : X \to Y$. Recall that t_Y is surjective iff every irreducible closed subset of Y has a generic point.

Similarly, if t_Y is injective, then there is at most one lift. If t_Y is not injective, then there exist distinct points Q_1 and Q_2 such that $t_Y(Q_1) = t_Y(Q_2)$. Then the continuous function taking all of X to $t_Y(Q_1)$ has at least two lifts to Y, the constant lifts to Q_1 and Q_2. Recall that t_Y is injective if and only if Y is T_0. □

Taking $X = \{\star\}$ in Theorem 11, we recover Theorem 1. We also have:

Corollary 12. *The natural map $\Phi : C(X,Y) \to Mor(X_{\text{top}}, Y_{\text{top}})$ is a bijection if and only if Y is t-complete.*

6. Extension to other sites

To return to the setting of the introduction, let $(\mathcal{C}, \mathcal{T})$ be an arbitrary site. Can we compute the site-theoretic points $(\mathcal{C}, \mathcal{T})(\star)$ as we did in the case of X_{top}? We will see that in general the answer is no, but for many of the sites encountered in algebraic geometry — sites which, roughly, refine the Zariski site of a scheme X — Theorem 1 can be used to show that $(\mathcal{C}, \mathcal{T})(\star)$ is still canonically in bijection with the points of the underlying topological space of X.

6.1. A correspondence between site-theoretic points and honest points.

Consider first the étale site $X_{\text{ét}}$ of a scheme X. Still a point P of X induces a site-theoretic point F_P on $X_{\text{ét}}$: an étale map $f : U \to X$ pulls back to 1 if and only if $P \in f(U)$. If $F \in X_{\text{ét}}(\star)$ is any site-theoretic point, then composing with the natural morphism of sites $X_{\text{ét}} \to X_{Zar}$ and applying Theorem 1, we can associate to F an honest point P on X. Since every étale map $f : U \to X$ factors through the Zariski open immersion $f(U) \hookrightarrow X$ and a collection $\{U_i \to X\}$ of étale maps is by definition an étale covering if and only if $\{f(U_i) \to X\}$ is a covering in the usual topological sense, it follows that $F = F_P$, i.e., the natural map $X \to X_{\text{ét}}(\star)$ is a bijection.

We give axioms on a site sufficient to carry through this argument, as follows.

Theorem 13. *Let $T = (\mathcal{C}, \mathcal{T})$ be a site with final object X. Suppose that there is a functor $\pi : \mathcal{C} \to$ **Top**, the category of topological spaces, and a section $\iota : (\pi X)_{\mathrm{top}} \to \mathcal{C}$ of π on the subcategory of* **Top** *consisting of inclusions among open subsets of πX. Suppose moreover that:*

- *if U is an object in $\mathcal{C}$ with structure morphism $f : U \to X$, then $\pi f(\pi U)$ is open in πX,*
- *$\{f_\alpha : U_\alpha \to U\}_\alpha$ is a covering in T if and only if $\pi U = \bigcup \pi f_\alpha(\pi U_\alpha)$, and*
- *if U is an object in $\mathcal{C}$, then the structure morphism $f : U \to X$ factors through $\iota(\pi f(\pi U)) \to X$, and*
- *ι preserves fibre products.*

Then the site-theoretic points of T are in bijection with the points of $t(\pi X)$.

Remark 14. The Zariski, étale, fppf, fpqc, and infinitesimal sites over a scheme all satisfy these conditions, as does any crystalline site $\mathrm{Cris}_{X/S}$. In each case, π takes a scheme to its underlying topological space, while ι takes an open subset U of πX to the open subscheme induced on U by X.

Proof. To begin, note that any morphism $\{0, 1\} \to T$ induces a morphism $\{0, 1\} \to (\pi X)_{\mathrm{top}}$ by restriction to the image of ι in $\mathcal{C}$. On the other hand, let F be a morphism $\{0, 1\} \to (\pi X)_{\mathrm{top}}$. Then define a morphism $G : \{0, 1\} \to T$ as follows: let U be an object of $\mathcal{C}$ with structure morphism $f : U \to X$. Then f factors through $\iota(\pi f(\pi U)) \to X$, and we set $G^{-1}(U) = F^{-1}(\pi f(\pi U))$.

Note that since the image of the composition $\pi f : \pi U \to \pi\iota(\pi f(\pi U)) = \pi f(\pi U) \subset \pi X$ is exactly $\pi f(\pi U)$, the map $\pi U \to \pi\iota(\pi f(\pi U))$ is surjective, and so $\{U \to \iota(\pi f(\pi U))\}$ is a cover in T. Therefore it is clear that if G is indeed a morphism, then G is the unique morphism $\{0, 1\} \to T$ restricting to F.

We wish to verify that G is a morphism of sites. Note that $G^{-1}(X)$ is defined to be $F^{-1}(\pi f(\pi X))) = F^{-1}(\pi X)$, so $G^{-1}(X) = 1$. Next, if we have a covering $\{p_\alpha : U_\alpha \to U\}$, then $\pi U = \cup_\alpha \pi p_\alpha(\pi U_\alpha)$. Let f_α, f denote the structure morphisms of U_α, U respectively. Then $\pi f(\pi U) = \cup_\alpha \pi f_\alpha(\pi U_\alpha)$, so U pulls back to 1 under G if and only if some $\pi f_\alpha(\pi U_\alpha)$ pulls back to 1 under F, i.e., if and only if some U_α pulls back to 1 under G. Finally, if U and V (with structure morphisms f, g) pull back to 1 under G, we must show that $U \times_X V$ does as well. Observe that if $\{U \to U'\}$ and $\{V \to V'\}$ are covers in T, then so are $\{U \times_X V \to U' \times_X V\}$ and $\{U' \times_X V \to U' \times_X V'\}$, and hence so is $\{U \times_X V \to U' \times_X V'\}$. Applying this observation with $U' = \iota(\pi f(\pi U))$ and $V' = \iota(\pi g(\pi V))$ and recalling that ι is assumed to preserve fibre products, we find that $\{U \times_X V \to \iota(\pi f(\pi U) \cap \pi g(\pi V))\}$ is a cover. Therefore the image of $\pi(U \times_X V)$ in X is exactly $\pi f(\pi U) \cap \pi g(\pi V)$. Since the latter pulls back to 1 under F, so does $U \times_X V$ under G.

We have now shown that the site-theoretic points of T are in bijection with the morphisms $\{0, 1\} \to (\pi X)_{\mathrm{top}}$. But by Theorem 1, the points of $t(\pi X)$ are in bijection with the morphisms $\{0, 1\} \to (\pi X)_{\mathrm{top}}$, and the result follows. ∎

6.2. **A site with too many points.** Since a point $F : \{0,1\} \to (\mathcal{C}, \mathcal{T})$ is determined by a map from isomorphism classes of objects of $\mathcal{C}$ to $\{0,1\}$, whenever these isomorphism classes form a set of cardinality κ, we can be sure that $(\mathcal{C}, \mathcal{T})(\star)$ forms a set of cardinality at most 2^κ. In particular, it is clear *a priori* that $(\mathcal{C}, \mathcal{T})(\star)$ forms a set in the following cases: the topological site of any topological space, the étale site of a scheme, and the fppf site of a scheme. For the fpqc, infinitesimal and crystalline sites, the isomorphism classes of objects do not form a set; however, Theorem 12 implies that $(\mathcal{C}, \mathcal{T})(\star)$ nevertheless forms a set. The following result shows that this is *not* a general fact about sites.

Proposition 15. *Let* $\mathbb{C}_{\text{Ét}}$ *be the large étale site on* $\operatorname{Spec} \mathbb{C}$*. For any infinite cardinal* κ*, there are* κ*-many distinct members of* $\mathbb{C}_{\text{Ét}}(\star)$*. In particular, the site-theoretic points do not form a set.*

Proof. Recall that the underlying category of $\mathbb{C}_{\text{Ét}}$ is the category of all $\mathbb{C}$-schemes, but the covering relation is still that a family $\{U_i \to U\}$ is a covering of U if each $U_i \to U$ is an étale morphism and if the union of the images is all of U.

For any cardinal κ, fix an index set S_κ of cardinality κ, and let $U_\kappa = \operatorname{Spec} A_\kappa$, where

$$A_\kappa = \mathbb{C}[\{X_i\}_{i \in S_\kappa}]/(\{X_i\}_{i \in S_\kappa})^2.$$

Observe that A_κ is, as a $\mathbb{C}$-vector space, of dimension $\kappa + 1$, so if $\kappa_1 < \kappa_2$, U_{κ_1} and U_{κ_2} are nonisomorphic $\mathbb{C}$-schemes. Moreover A_κ is a strictly Henselian local ring with residue field $\mathbb{C}$, and the structure map $U_\kappa \to \operatorname{Spec} \mathbb{C}$ induces at the level of topological spaces the homeomorphism from a one-point space to itself. It follows from this that the étale site on U_κ is canonically identified with the étale site on $\operatorname{Spec} \mathbb{C}$, i.e., the only étale U_κ-schemes are finite disjoint unions of copies of U_κ.[5]

For any infinite cardinal κ, define a point $F_\kappa : \{0,1\} \to \mathbb{C}_{\text{ét}}$ as follows: a scheme pulls back to 1 iff it is isomorphic to a finite disjoint union of copies of $U_{\kappa'}$ for $\kappa' \leq \kappa$. In particular, the final object $U_0 = \operatorname{Spec} \mathbb{C}$ pulls back to 1, and since the class of objects pulling back to 1 is closed under fibre products and passage from source to target and target to source of an étale map, it does indeed give a point.

If $\aleph_0 \leq \kappa_1 < \kappa_2$ are distinct infinite cardinals, then $F_{\kappa_1} \neq F_{\kappa_2}$. Since the collection of infinite cardinals does not form a set, neither does $\mathbb{C}_{\text{Ét}}(\star)$. □

Similar considerations show that all the "large" versions of the standard sites are too large in this sense: the covering relation is too coarse.

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[5] U_κ is also an object of the fpqc and infinitesimal = crystalline sites over $\operatorname{Spec} \mathbb{C}$, which explains the above remark that the isomorphism classes of objects in these sites do not form a set.