
On the Prime Number Theorems

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ON THE PRIME NUMBER THEOREM.*

By AUREL WINTNER.

A few years ago, Watson succeeded in proving an asymptotic congruence property formulated by Ramanujan, namely, the following theorem: If m and k are fixed positive integers, there are between $n = 1$ and $n = N$ only $o(N)$ integers n for which the sum of the $(2m - 1)$ -th powers of all divisors of n is not a multiple of k . This implies, in the particular case $2m - 1 = 11$, that Ramanujan's $\tau(n)$ is divisible by 691 except for $o(N)$ values of n .

Actually, Watson has improved on $o(N)$ by logarithmical factors (for arbitrary m, k), and has even proved asymptotic formulae for certain summatory functions on which the o -estimates depend¹; asymptotic formulae established by the classical methods of the theory of primes (that is, by contour integrations, based on estimates of $\zeta(s)$ in a certain domain containing the line $\sigma = 1$ in its interior). The singularities of the generating Dirichlet series and, correspondingly, the proof of the asymptotic formulae are similar to those occurring in Landau's result,² according to which the number of the positive integers which are less than x and are representable as a sum of two squares is asymptotically proportional to $x/(\log x)^{\frac{1}{2}}$. The following considerations, which deal with the general analytical background of asymptotic laws of this type, seem to be warranted in view of Hardy's recent presentation³ of the results just mentioned.

The asymptotic formula, given by Hardy in his discussion of Landau's problem, is false, the passage from the order of the singularity (at $s = 1$) to the order of the summatory function being erroneous. This can be seen, without any Tauberian argument, from Abelian reasons alone. (Among all the possible orders, Landau's exponent, $\frac{1}{2}$, represents the only case in which Hardy's calculation becomes correct.) However, my quarrel is not with this *lapsus calami* (which does not occur in Watson's paper), but with the method of approach. In fact, Hardy refers (as Watson did) to the parallel problem of

* Received November 10, 1941.

¹ G. N. Watson, "Ueber Ramanujansche Kongruenzeigenschaften der Zerfallungszahlen. (I)," *Mathematische Zeitschrift*, vol. 39 (1935), pp. 712-731.

² E. Landau, "Ueber die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammenetzung erforderlichen Quadrate," *Archiv der Mathematik und Physik*, ser. 3, vol. 13 (1908), pp. 305-312.

³ G. H. Hardy, *Ramanujan*, Cambridge University Press, 1940, pp. 61-63 and 167-169.

two squares; a problem presented along the lines of Landau's approach of 1908, requiring, as the prime number theorem did at that time, a definite amount of information on a domain contained in the half-plane $\sigma < 1$. And Hardy commences the presentation of this approach by a remark concerning its analytical interest, which is observed to be due to the fact that the pole of the prime number theorem becomes replaced by an algebraic singularity; the order of infinity, say μ , at $s = 1$ now being $\frac{1}{2}$ instead of 1. But all of this must be felt as rather disturbing, the more so as, after Wiener's work,⁴ the prime number theorem itself has nothing to do with what happens beyond the line $\sigma = 1$.

In the sequel, this methodical anomaly will be disposed of by showing that, on the one hand, there is no point in passing through the line $\sigma = 1$ in order to prove the theorem of either Landau or Watson, since a *unified*, and much shorter, approach to all these problems (including the prime number theorem itself) can be obtained by a straight-forward extension of Ikehara's theorem concerning the case $\mu = 1$ of a simple pole; and that, on the other hand, all these problems do not involve the algebraic character of the underlying singularities, since μ could be irrational, and therefore the singularity at $s = 1$ logarithmic, as far as the unified Tauberian approach is concerned. Even the fact that, in the problems of Landau and Watson, μ not only is rational but less than 1 as well, turns out to be without any significance, since nothing happens when μ passes through Ikehara's case, $\mu = 1$.

The necessary modifications of the proof of Ikehara's theorem⁵ are so much on the surface that certain details will be given only because the possibility of these modifications appears to be overlooked in the literature. The only complication arising from the passage to the necessary generalization of Ikehara's case is the appearance of an additional factor. In fact, the Fourier transformation belonging to an arbitrary $\mu > 0$ corresponds to the Γ -integral

$$\int_0^{\infty} e^{-wx} x^{\mu-1} dx;$$

so that Ikehara's case, $\mu - 1 = 0$, of a simple pole might have disguised the

⁴ N. Wiener, *The Fourier Integral and Certain of its Applications*, Cambridge University Press, 1933, pp. 112-137. Actually, the first proof of the prime number theorem avoiding a crossing of the line $\sigma = 1$ (and based on the characteristic application of the Riemann-Lebesgue lemma for Fourier integrals) is due to Hardy and Littlewood, *Quarterly Journal of Mathematics*, vol. 46 (1915), pp. 215-219. However, their proof presupposes at any rate a rough (exponential) estimate of $\zeta(1 + it)$ for large t . But any such estimate depends on the same machinery as the crossing of the line $\sigma = 1$.

⁵ Cf. *loc. cit.*, pp. 127-130.

availability of the method in the general situation. The latter can be formulated as follows:

If a Dirichlet series

$$(1) \quad f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad (0 = \lambda_1 < \lambda_2 < \dots),$$

where

$$(2) \quad a_n \geq 0,$$

is convergent for $\sigma > 1$ and such that there exist two constants, $C \neq 0$ and $\mu > 0$, for which the regular function

$$(3) \quad f(s) - C/(s-1)^\mu, \text{ where } \sigma > 1, \quad (\arg(\sigma-1)^\mu = 0),$$

possesses a continuous boundary function as $\sigma \rightarrow 1 + 0$, then

$$(4) \quad \sum_{\lambda_n \leq x} a_n \sim \frac{C}{\Gamma(\mu)} e^{ax} x^{\mu-1} \quad \text{as } x \rightarrow \infty.$$

It is understood that by the existence of a continuous boundary function is meant this: For every fixed $T > 0$, the function (3) of $s = \sigma + it$ tends, as $\sigma \rightarrow 1 + 0$, to a function of t uniformly for $-T \leq t \leq T$. Actually, it is more than sufficient to assume the existence of two functions, say

$$(5) \quad g_0(t) \text{ and } g^*(t) \geq 0, \quad (-\infty < t < \infty),$$

such that the function (3) of $s = \sigma + it$ tends, as $\sigma \rightarrow 1 + 0$, to $g_0(t)$ if t is arbitrarily fixed on the complement of a t -set of measure 0, and $g^*(t)$ is L -integrable on every interval $-T \leq t \leq T$ and exceeds the absolute value of the difference (3) whenever $1 < \sigma < 2$ (say).

Even without this generalization of the italicized Tauberian theorem, the prime number theorem and Landau's result follow by choosing

$$\mu = 1, \quad f(s) = -\frac{\xi'(s)}{\xi(s)} \left(\text{or, equivalently, } f(s) = \sum_p \frac{\log p}{p^s} \right)^6$$

⁶ The summatory functions (4) belonging to these two ordinary Dirichlet series (the second of which defines a function meromorphic in the half-plane $\sigma > 0$, with poles clustering at every point of the line $\sigma = 0$) are $\psi(\exp x)$ and $\theta(\exp x)$, if the least common multiple of all integers between 1 and x and of all primes between 1 and x are respectively denoted by $\exp \psi(x)$ and $\exp \theta(x)$. It is interesting that the prime number theorem, $\psi(x) \sim x$, is equivalent not only to $\theta(x) \sim x$ but also to an asymptotic formula, $\psi(x) - \theta(x) \sim x^{1/2}$, for the excess represented by multiple prime factors (counted in ψ but not in θ); in fact,

$$\psi(x) = \sum_{n=1}^{\infty} \theta(x^{1/n}), \text{ where } \sum_{n=3}^{\infty} \theta(x^{1/n}) = x^{1/3} o(x^\epsilon) = o(x^{1/2}).$$

and

$$\mu = \frac{1}{2}, \quad \{f(s)\}^2 = \zeta(s) \frac{L(s)}{1 - 2^{-s}} \prod_{p \equiv 3 \pmod{4}} \left(1 - \frac{1}{p^{2s}}\right)^{-1}$$

respectively, where the L -series is that belonging to the non-principal character (mod 4). Correspondingly, Watson's theorem results by placing

$$\mu = 1 - \frac{1}{\phi(k)}, \quad f(s) = \zeta(s) \prod_{p \equiv -1 \pmod{k}} \left(1 + \frac{1}{p^s}\right)^{-1},$$

where ϕ is Euler's function and $k > 2$. Needless to say, the various extensions of these results (for instance, the asymptotic formula for the number of integers represented by a binary quadratic form,⁷ where, as in Landau's particular case, $\mu = \frac{1}{2}$) follow in the same way.

In order to facilitate a direct comparison with the standard treatment⁸ of Ikehara's case, the weight function of Tauberian averaging will be chosen in the usual way, that is, such that the kernel of the Fourier transformation becomes

$$(6) \quad S(u) = (\sin u/u)^2.$$

Since $|u|^\mu S(u) \rightarrow 0$ as $|u| \rightarrow \infty$ then holds only if

$$(7) \quad \mu < 2, \quad (\mu > 0),$$

the choice (6) will necessitate the restriction (7). In order to include arbitrarily large values of μ , all that is necessary is a replacement of the weight function

$$\omega(x) = \text{Max}(0, 1 - |x|), \quad (-\infty < x < \infty),$$

which leads to (6), by another weight function, leading to a Fourier transform which tends to 0 stronger than $|u|^{-\mu}$ (and is, as (6), non-negative). Such weight functions can be obtained by the well-known process of successive convolutions, for instance.⁹ Incidentally, not only (7) but even $\mu \leq 1$ is satisfied in all the applications referred to above.

⁷ Cf. P. Bernays, *Ueber die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nicht-quadratischen Diskriminante*, Dissertation, Göttingen, 1912.

⁸ Cf. E. Landau, "Ueber Dirichletsche Reihen," *Göttinger Nachrichten*, 1932, pp. 525-527.

⁹ Cf., e. g., J. Karamata, "Weiterführung der N. Wiener'schen Methode," *Mathematische Zeitschrift*, vol. 38 (1934), pp. 701-708 where, for other reasons, the square of (6) is applied (and, therefore, the first auto-convolution of the above weight function is considered).

The weight function $\exp(-\frac{1}{2}x^2)$ would have the advantage that the device of successive convolutions becomes unnecessary. But this self-reciprocal function has its own disadvantages, its order at infinity being too low.

If, for $-\infty < x < \infty$,

$$(8) \quad \alpha(x) = e^{-x} \sum_{\lambda_n \leq x} a_n, \quad (a_n \geq 0; 0 = \lambda_1 < \lambda_2 < \dots),$$

then, by (2),

$$(9) \quad e^x \alpha(x) \leq e^y \alpha(y) \text{ whenever } x < y,$$

and

$$(10) \quad \alpha(x) = 0 \text{ for } -\infty < x < 0.$$

According to (8), the assertion (4) can be written in the form

$$(11) \quad \alpha(x) \sim x^{\mu-1}, \quad (x \rightarrow \infty),$$

if, without loss of generality, $C = \Gamma(\mu)$ in (3).

Instead of (11), it will be sufficient to prove that

$$(12) \quad \int_{-\infty}^{Tx} \alpha(x - v/T) S(v) dv \sim \pi x^{\mu-1} \text{ as } x \rightarrow \infty,$$

if $T > 0$ is arbitrarily fixed. In fact, since

$$\int_{-\infty}^{\infty} S(v) dv = \pi, \text{ by (6),}$$

the passage from (12) to (11) requires the elimination of the weight factor. But it is clear from (8), (9), (10) that this elimination is supplied precisely by the standard Tauberian argument,¹⁰ if (12) holds for an *arbitrary* value of $T > 0$.

According to (8), (9) and (1),

$$f(s) = \int_0^{\infty} e^{-sx} d\{e^x \alpha(x)\} \equiv s \int_0^{\infty} e^x \alpha(x) e^{-sx} dx, \quad (\sigma > 1);$$

so that, since

$$\Gamma(\mu)/w^\mu \equiv \int_0^{\infty} e^{-wx} x^{\mu-1} dx, \quad (\Re w > 0, \mu > 0),$$

the difference (3), where $C = \Gamma(\mu)$, can be written in the form

$$s \int_0^{\infty} e^{(1-s)x} \alpha(x) dx - \int_0^{\infty} e^{(1-s)x} x^{\mu-1} dx, \quad (\sigma > 1).$$

¹⁰ Cf., e. g., *loc. cit.*⁸, 1)-2) on pp. 526-527

Thus the assumptions specified after (5) mean that, if

$$(13) \quad s = \sigma + it, \quad \sigma = 1 + \epsilon > 1, \quad (-\infty < t < \infty),$$

and

$$(14) \quad F_\epsilon(t) \equiv \int_0^\infty \{\alpha(x) - x^{\mu-1}\} e^{(1-s)x} dx, \quad (\epsilon > 0),$$

then there exist for $-\infty < t < \infty$ two functions, say $F_0(t)$ and $F^*(t)$, such that

$$(15) \quad F_\epsilon(t) \rightarrow F_0(t) \text{ as } \epsilon \rightarrow 0 \text{ holds for almost all } t$$

and

$$(16) \quad |F_\epsilon(t)| < F^*(t) \text{ for arbitrary } \epsilon \text{ and } t,$$

where $F^*(t)$ is L -integrable on every finite interval $-T \leqq t \leqq T$.

Since the integral (14) is absolutely convergent for all values of s mentioned under (13), it follows by a legitimate interchange of the order of integrations, that

$$\int_{-2T}^{2T} e^{ixt} (1 - \frac{1}{2} |t|/T) F_\epsilon(t) dt = 2T \int_0^\infty \{\alpha(u) - u^{\mu-1}\} e^{-\epsilon u} S(Tx - Tu) du$$

is, by virtue of the definition (14) and (6), an identity in the three parameters ϵ, T, x . But (15) and (16) assure that, if $\epsilon \rightarrow 0$, the integral on the left tends to the (finite) limit

$$\int_{-2T}^{2T} e^{ixt} (1 - \frac{1}{2} |t|/T) F_0(t) dt.$$

And the integral analogue of the Riemann-Lebesgue lemma shows that the last integral tends to 0 as $x \rightarrow \infty$, if T is fixed. Consequently, there exists a (finite) limit

$$(17) \quad \lim_{\epsilon \rightarrow 0} T \int_0^\infty \{\alpha(u) - u^{\mu-1}\} e^{-\epsilon u} S(Tx - Tu) du,$$

where T and x are arbitrary. And (17) tends to 0 as $x \rightarrow \infty$, if T is arbitrarily fixed.

Accordingly, if u in (17) is replaced by the integration variable $v = Tx - Tu$,

$$(18) \quad \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{Tx} \{\alpha(x - v/T) - (x - v/T)^{\mu-1}\} e^{-\epsilon(x-vT)} S(v) dv \rightarrow 0 \text{ as } x \rightarrow \infty,$$

where $T (> 0)$ is arbitrary. Furthermore, from (6) and (7),

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{Tx} (x - v/T)^{\mu-1} e^{-\epsilon(x-v/T)} S(v) dv = \int_{-\infty}^{Tx} (x - v/T)^{\mu-1} S(v) dv,$$

and it is also clear from (6) and (7) that the ratio of the last integral to $x^{\mu-1}$ is

$$\equiv \int_{-\infty}^{Tx} \left(1 - \frac{v}{Tx}\right)^{\mu-1} \frac{\sin^2 v}{v^2} dv \rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 v}{v^2} dv = \pi, \quad x \rightarrow \infty,$$

where T is arbitrarily fixed. It follows therefore from (18) that

$$(19) \quad x^{1-\mu} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{Tx} \alpha(x - v/T) e^{-\epsilon(x-v/T)} S(v) dv \rightarrow \pi, \quad x \rightarrow \infty,$$

holds for every T .

In order to complete the proof of (12), it remains to ascertain that, if both x and T are fixed, the limit process $\epsilon \rightarrow 0$ can be carried out beneath the integral sign in (19):

$$(20) \quad \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{Tx} \alpha(x - v/T) e^{-\epsilon(x-v/T)} S(v) dv = \int_{-\infty}^{Tx} \alpha(x - v/T) S(v) dv.$$

But the existence of the (finite) limit on the left of (20) was proved above. Furthermore, both functions (6), (8) are real and non-negative. Hence, (20) can be reduced to the trivial fact that, for every positive $R (< \infty)$,

$$\lim_{\epsilon \rightarrow 0} \int_{-R}^{Tx} \alpha(x - v/T) e^{-\epsilon(x-v/T)} S(v) dv = \int_{-R}^{Tx} \alpha(x - v/T) S(v) dv.$$

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