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Ext(Q,Z) is the additive group of real numbers

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Standard homological methods and a theorem of Harrison on cotorsion groups are used to prove the result mentioned.

In this note Z denotes an infinite cyclic group, Q the additive group of rational numbers, Z_{p^∞} a p-quasicyclic group, and I_p the group of p-adic integers.

Pascual Llorente proves in [3] that $\mathrm{Ext}(Q,Z)$ is an uncountable group, and gives explicitly a countably infinite subset. Very little extra effort produces the result embodied in the title, as follows.

Llorente applies the functor $\mathrm{Hom}(\ ,Z)$ to the exact sequence

(1) $$0 \to Z \to Q \to Q/Z \to 0 ,$$

and deduces almost immediately an exact sequence

(2) $$0 \to Z \to \mathrm{Ext}(Q/Z,Z) \to \mathrm{Ext}(Q,Z) \to 0 .$$

Since Q/Z is the restricted direct sum $\sum Z_{p^\infty}$, the sum taken over all primes, it follows [1, p.238] that $\mathrm{Ext}(Q/Z,Z)$ is the complete direct sum $\sum^* \mathrm{Ext}(Z_{p^\infty},Z)$.

To find the structure of $\mathrm{Ext}(Z_{p^\infty},Z)$, we apply the functor $\mathrm{Hom}(Z_{p^\infty},\ )$ to (1), giving a long exact sequence:

$$0 \to \mathrm{Hom}(Z_{p^\infty},Z) \to \mathrm{Hom}(Z_{p^\infty},Q) \to \mathrm{Hom}(Z_{p^\infty},Q/Z) \to \mathrm{Ext}(Z_{p^\infty},Z) \to \mathrm{Ext}(Z_{p^\infty},Q) \to \mathrm{Ext}(Z_{p^\infty},Q/Z) \to 0 .$$

All but the two central terms vanish, and we are left with an exact sequence

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$$(3) \qquad 0 \to \mathrm{Hom}(Z_{p^\infty}, Q/Z) \to \mathrm{Ext}(Z_{p^\infty}, Z) \to 0 .$$

However, it is clear that $\mathrm{Hom}(Z_{p^\infty}, Q/Z) \cong \mathrm{Hom}(Z_{p^\infty}, Z_{p^\infty}) \cong I_p$, so that (3) expresses an isomorphism

$$\mathrm{Ext}(Z_{p^\infty}, Z) \cong I_p .$$

It follows that $\mathrm{Ext}(Q/Z, Z)$ is a cotorsion group in the sense of Harrison [2]; the main feature which concerns us is that $\mathrm{Hom}(Q, \mathrm{Ext}(Q/Z, Z))$ and $\mathrm{Ext}(Q, \mathrm{Ext}(Q/Z, Z))$ are both zero [2, Proposition 2.1]. As a result, application of $\mathrm{Hom}(Q, \ )$ to (2) yields an exact sequence

$$0 \to \mathrm{Hom}(Q, \mathrm{Ext}(Q, Z)) \to \mathrm{Ext}(Q, Z) \to 0 ,$$

so that $\mathrm{Ext}(Q, Z) \cong \mathrm{Hom}(Q, \mathrm{Ext}(Q, Z))$. Since Q is divisible, this isomorphism gives [1, p. 207] that $\mathrm{Ext}(Q, Z)$ is torsion-free. Further, since Q is torsion-free, $\mathrm{Ext}(Q, Z)$ is divisible [1, p. 245]. Finally, (2) and the fact that $\mathrm{Ext}(Q/Z, Z)$ is isomorphic with $\sum^* I_p$ proves that $\mathrm{Ext}(Q, Z)$ has the cardinal of the continuum. This is enough to yield our claim that $\mathrm{Ext}(Q, Z)$ *is isomorphic with the additive group of real numbers*.

One slightly surprising feature is that (2) expresses $\mathrm{Ext}(Q, Z)$, a large torsion-free divisible group, as a factor-group of $\sum^* I_p$ by a *cyclic* subgroup. I would not have imagined that killing a cyclic subgroup could have so profound an effect.

References

[1] L. Fuchs, *Abelian groups* (Publishing House of the Hungarian Academy of Sciences, Budapest, 1958).

[2] D.K. Harrison, "Infinite abelian groups and homological methods", *Ann. of Math.* 69 (1959), 366-391.

[3] Pascual Llorente, "Construccion de grupos-estensiones", *Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat.* 4 (1966), 119-145.

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