ALMOST SURE LIMIT SETS OF RANDOM SERIES

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I would like to tell you about an easily stated problem which lies at the border of geometry, probability and classical analysis. In the summer of 2005 two undergraduates at McGill University worked in this general area, and were able to make some progress. When posing the project to them I made a certain conjecture, and (as expected) they did not get far enough in their eight week project to be able to seriously work on it. Allow me to pass the conjecture along to you in the hope that you might be able to do something with it!

Remarks for the close reader: this document is a set of lecture notes for a talk I gave in the UGA VIGRE (Graduate Student) Seminar on September 30, 2008. These notes give a slightly richer picture than I was able to present in the talk. If you are interested in a more minimalistic presentation, I also have available a separate "digression free" set of notes that I presented out for myself when giving the talk – these are available on request.

Because not attributing results has become a pet peeve of mine, I have included references for all of the important theorems discussed here, if possible to the original paper and also to a more modern (and anglophone) treatment. If a result is stated but no proof or attribution is given, it means that the proof is left **to you**, i.e., it is possible and not too difficult to give a proof without having any specialized background (taking for granted the previous results). I especially recommend that all but the most casual readers prove Theorem 8, since doing so will make clearer the analogy between this theorem and Theorem 3.

1. Rearrangements

Suppose $\sum_n a_n$ is a convergent real series. Recall the following question from analysis: when is it permissible to rearrange the terms of the series? Here, by a **rearrangement** of the series we understand a new series of the form $\sum_n a_{\sigma(n)}$, where $\sigma : \mathbb{N} \to \mathbb{N}$ is a permutation of the natural numbers, and by "permissible" we mean that every rearrangement should converge to a common sum. Let us call a series with this property **unconditionally convergent**, whereas a series which is convergent but not unconditionally convergent will be called **conditionally convergent**.

A sufficient condition for unconditional convergence was provided by K. Weierstrass. We say that a real series is **absolutely convergent** if $\sum_n |a_n| < \infty$. Then:

Theorem 1. *Absolute convergence implies unconditional convergence.*

The proof is immediate from the triangle inequality and the Cauchy criterion for convergence. Accordingly, it holds not just for real series, but for series with values

Thanks to Douglas Zare for pointing out an inaccuracy in the statement of Theorem 8.

in any Banach space (or even in any complete, normed abelian group).

The converse of this result for real series was established by Riemann.

Proposition 2. *(Weak Riemann Theorem) A real series which is unconditionally convergent is necessarily absolutely convergent. More precisely, if* $\sum_n a_n$ *is a convergent real series with* $\sum_n |a_n| = \infty$*, then there exists a divergent rearrangement.*

But in fact, in 1852 Riemann proved a much more precise and striking result [Rie]:

Theorem 3. *(Riemann Rearrangement Theorem) Let* $\sum_n a_n$ *be a real series which is convergent, but not absolutely convergent. For any extended real number* $L \in [-\infty, \infty]$*, there exists a permutation* σ *such that* $\sum_n a_{\sigma(n)} \to L$*.*

As an aside, we mention that more is true: we may rearrange any conditionally convergent series so as to be divergent, but with prescribed limiting behavior.

Theorem 4. *Let* $\sum_n a_n$ *be a conditionally convergent real series. Then, for any two extended real numbers* $A \leq B \in [-\infty, \infty]$*, there exists a permutation* σ *of* $\mathbb{Z}^+$ *such that the limit set of the series* $\sum_n a_{\sigma(n)}$ *– i.e., the set of limit points of the sequence of partial sums — is equal to* $[A, B]$*.*

For the proof, see e.g. [Rud].[1] Conversely:

Proposition 5. *If* $a_n \to 0$*, the limit set of* $\sum_n a_n$ *is connected.*

Another mild strengthening comes from the observation that the hypothesis that the original series $\sum_n a_n$ be conditionally convergent is not really necessary. It suffices for the series to be (i) nonabsolutely convergent and (ii) **rearrangeably convergent**, i.e., for it to converge after some permutation. It is easy to check that a real series $\sum_n a_n$ is rearrangeably convergent iff $a_n \to 0$ and either it is absolutely convergent or both the series of positive and negative parts are divergent.

Thus we have solved the problem of determining the "rearrangement limit set" of any series which is either absolutely convergent or rearrangeably conditionally convergent. What about other series? The cases in which $a_n \to 0$ but exactly one of the positive and negative series diverges are trivial. Remarkably, from the perspective of limit sets the case in which a_n does not tend to 0 is the most interesting For instance the rearrangement limit set of $\sum_n (-1)^n$ has a connection with random walks and probability. More on this later!

2. The Lévy-Steinitz Theorem

Suppose now we consider series $\sum_n a_n$ with values in $\mathbb{R}^N$. Now what can be said about absolute convergence, unconditional convergence and rearrangements?

We claim that again absolute convergence and unconditional convergence are equivalent. Indeed, the forward implication is provided by Weierstrass' theorem. Conversely, if for a vector valued series we have $\sum_n ||a_n|| = \infty$, then there must exist at least one i, $1 \leq i \leq N$, such that the series of ith coordinates is not absolutely convergent: $\sum_n |(a_n)_i| = \infty$. One might worry about this at first glance, because it looks like we are comparing L^2 and L^1 convergence. But on the finite dimensional

[1] Or work it out yourself – it is not harder than Theorem 3.

vector space $\mathbb{R}^N$ any two norms differ by a multiplicative constant, which means we shouldn't be worrying (and also that all of these problems are completely independent of the choice of norm on $\mathbb{R}^N$). So if the series is not absolutely convergent, then neither is at least one of the component series, and choosing some permutation σ that makes the component series diverge certainly makes the entire vector-valued series diverge.

As another aside, we remark that the finite-dimensionality of $\mathbb{R}^N$ is crucial here:

Theorem 6. *(Dvoretzky-Rogers, 1950* [DR]*) Let $(E, ||\ ||)$ be any infinite dimensional Banach space. Then there exists a E-valued series $\sum_n a_n$ which converges **unconditionally** but **nonabsolutely**: i.e., for all permutations σ, $\sum_n a_{\sigma(n)}$ converges and has sum independent of σ, while nevertheless $\sum_n ||a_n|| = \infty$.*

But not so fast! We established the weak Riemann theorem in $\mathbb{R}^N$. But we are left with the problem of the analogue of Riemann's rearrangement theorem: what are the possible limits of rearrangements of a given conditionally convergent series?

Half a second's thought reveals that the most naive generalization – i.e., the rearrangement set of any nonabsolutely convergent series will be $\mathbb{R}^N$ – is ridiculous. For instance, take your favorite conditionally convergent real series – say $\sum_n \frac{(-1)^n}{n^p}$ for some $p \in (0, 1]$ – and view it as a series in $\mathbb{R}^N$ whose final $(N-1)$-coordinates are all zero. Obviously since the partial sums of any rearrangement lie in the closed subspace $\mathbb{R} \times 0^{N-1}$, the same must be true of any limit. Clearly the rearrangement set of a series must lie in the linear subspace spanned by the coefficients!

Let $0 \neq v \in \mathbb{R}^N$ be any nonzero vector. Using the natural inner product structure on $\mathbb{R}^N$ we get an orthogonal projection of $\mathbb{R}^N$ onto $\mathbb{R}v$. In particular, for any $a_n \in \mathbb{R}^N$, write a_n^v for this projection. Then the projected series $\sum_n a_n^v$ is a real series, and is the projection of the original series onto the line $\mathbb{R}v$. So we know that if $\sum_n |a_n^v| < \infty$, this projected series is absolutely convergent, and no rearrangement will affect the sum of the projection of the series onto $\mathbb{R}v$. Consider now the set

$$V_1 = \{v \in \mathbb{R}^N \mid \sum_n |a_n^v| < \infty\}.$$

Then we easily see:

(i) V_1 is a linear subspace of $\mathbb{R}^N$.
(ii) The orthogonal projection of the series onto V_1 is absolutely convergent, and thus the projection onto V_1 of each rearrangement is convergent with the same sum.
(iii) V_1 is the largest subset of $\mathbb{R}^N$ for which (ii) holds.

So any "rearrangeability" in the sum must come from the directions other than V_1. The issue is: to what extent is the converse true? For instance, a representative special case (to which the general case can in fact be reduced) is when $V_1 = \{0\}$, i.e., when the series is conditionally convergent "in every direction." In this case we can rearrange so as to make any one-dimensional projection converge to whatever we want, but can we find a single rearrangement which simultaneously effects the desired rearrangement on, say, each coordinate projection? In fact, yes:

Theorem 7. *(Lévy-Steinitz) Let $\{a_n\}$ be a convergent sequence in $\mathbb{R}^N$. Put*

$$V_1 = \{v \in \mathbb{R}^N \mid \sum_n |a_n^v| < \infty\}$$

and

$$L = \sum_n a_n^V.$$

Then the set of all limits of convergent rearrangements of $\sum_n a_n$ is $L + V_1^{\perp}$, where $V_1^{\perp}$ is the orthogonal complement of V_1 in $\mathbb{R}^N$.

Remark: Again, this has an immediate generalization to rearrangeably convergent $\mathbb{R}^N$-valued series. I leave it to you to write down the condition for a vector-valued series to be rearrangeably convergent.

This result first appears in a 1905 paper of Paul Lévy [Lév]. In 1913 Ernst Steinitz pointed out that Lévy's argument was incomplete, and gave the first complete proof [Ste]. More recent treatments may be found in a 1986 paper of I. Halperin [Hal] and a 1987 Monthly article of P. Rosenthal [Ros]. None of these proofs are easy!

Note that one could ask for a "Rudinesque" refinement of this result: i.e., what are the possible limit sets upon rearrangement? So far as I know this is an open problem, but it is not the one I really want to talk about, so let's press on.

3. Series with random signs

Once we realize that what makes a nonabsolutely convergent real series convergent is that (i) both the positive and negative parts diverge, and (ii) the positive and negative contributions are ordered in such a way so that the net contribution is "nearly balanced", it is only a matter of time until we think of another "thing to do" to a conditionally convergent series: instead of swapping the order of the terms, why not change some of the signs?

In other words, given a conditionally convergent real series $S = \sum_n a_n$, let $\epsilon = (\epsilon_n)_{n=1}^{\infty} \in \mathcal{B} = \{\pm 1\}^{\infty}$ be a **sign sequence**, and consider instead the series

$$\epsilon(S) := \sum_n \epsilon_n a_n.$$

Certainly if S is absolutely convergent, then so will be $\epsilon(S)$ for any sign sequence ϵ. It is equally clear that sign-changing may affect the sum – indeed, changing the sign of exactly one nonzero term will certainly change the sum. With a little bit of thought one can prove the following "Riemannesque theorem":

Theorem 8. *Let $\{a_n\}$ be a sequence of real numbers with $a_n \to 0$ and $\sum_{n=1}^{\infty} |a_n| = \infty$. For any extended real number L, there exists a sign sequence ϵ such that $\sum_n \epsilon_n a_n \to L$.*

We could go to $\mathbb{R}^N$, but again, not so fast! There is a fascinating way to thicken the plot. Suppose we start with a sequence $\{a_n\}$ of real numbers with $a_n \to 0$. By the above theorem there exists at least one sign sequence ϵ such that $\sum_n \epsilon_n a_n$ converges. If $\sum_n |a_n| < \infty$, then every possible insertion of signs makes the series converge. That's boring, so assume that $\sum_n |a_n| = \infty$. Then there is also at least one sign sequence ϵ such that $\sum_n \epsilon_n a_n$ diverges. But **which is more likely?**

We can answer this because the space $\mathcal{B}$ of sign sequences is endowed with a natural probability measure μ. This can be seen in many ways. The two ways I like best are:

(i) There is a natural probability measure on each factor space $\{\pm 1\}$, and one can endow any Cartesian product of probability spaces with a natural measure;

(ii) Observe that $\mathcal{B}$ is, with respect to the product topology, a compact abelian group, so has a unique **Haar measure**, i.e., a translation-invariant measure.

Here the translation-invariance means that in our intuitive picture of the space as a countably infinite sequence of indepenent coin flips, suppose we line up infinitely many coin-flipping machines each of which really can flip a coin with equal probabilities heads and tails, and the machines come out of the factory with the coin resting in the heads position. Then, before the flipping takes place, if some interloper comes in and changes some of the coins to be lying tails up in the machine, this is not going to affect the probability of any event.

As a sample case, take $a_n = \frac{1}{n^p}$ for $p \in \mathbb{R}$. When $p > 1$, the sequence is ℓ^1 convergent, so the probability of convergence is certainly 1. If $p \geq 0$, then the general term does not tend to zero, so the convergence is impossible: probability zero. It is very plausible that the function "probability of convergence of the p-series with random signs" should be increasing as p ranges from 0 to 1. If we didn't know any better, maybe we would postulate some nice simple function interpolating between the endpoints: could it be that the probability of convergence is just p??

If you know some probability theory, you can see in advance that this guess is wrong. For this, consider a special type of subset ("event") of $\mathcal{B}$ called a **tail set**. Namely, a subset $S \subset \mathcal{B}$ is a tail set iff for all $x \in \mathcal{B}$ and any sign sequence ϵ with $\epsilon_n = 1$ for all sufficiently large n, then $x \in S$ iff $\epsilon \cdot x \in S$. In other words, if we take any element and mess with it in any finite number of coordinates, then in so doing we do not change whether the event is in S or not.

Aside: Using the group theory, a nice way to express this is via the subgroup $\mathfrak{b}$ of $\mathcal{B}$ consisting of all sign sequences with only finitely many negative sign sequences. This is none other than the infinite direct sum sitting inside the direct product. Then a subset S is a tail subset iff it is invariant under $\mathfrak{b}$ iff it is a union of cosets of $\mathfrak{b}$.[2]

Indeed, for any fixed sequence $\{a_n\}$, "$\sum_n \epsilon_n a_n$ converges" is a tail event: changing finitely many signs means changing finitely many terms which will, famously, not affect the convergence or divergence. So the following is applicable here:

Theorem 9. *(Kolmogorov 0-1 Law) Let $S \subset \mathcal{B}$ be any tail event. Then either $\mu(S) = 0$ or $\mu(S) = 1$.*

[2]Technical aside: from this description it is not hard to see that there are tail sets which are not Borel measurable. Thus we should require that an "event" be an element of the Borel σ-algebra. In practice, all the tail events that are of interest are easily seen to be Borel measurable.

In other words, the function which assigns to a p-series the probability of its convergence with random signs is $\{0, 1\}$-valued, so cannot be everywhere continuous! Intuitively one still hopes that it is monotone, so our next guess is that there is some special, transitional value $p_0 \in (0, 1)$ such that for $p < p_0$ the probability of convergence is 0 and for $p > p_0$ the probability of convergence is 1.

In fact, this turns out to be the case:

Theorem 10. *(Rademacher-Paley-Zygmund,* [Kac]*) Let $\{a_n\}$ be any real sequence. The following are equivalent:*
(i) The probability that $\sum_n \epsilon_n a_n$ converges is 1.
(ii) The probability that $\sum_n \epsilon_n a_n$ is bounded is 1.
(iii) $\sum_n a_n^2 < \infty$.

Applying this to the case of $a_n = \frac{1}{n^p}$, we see that when we put random signs into the p-series, it converges with probability 0 iff $p \leq \frac{1}{2}$ and with probability 1 iff $p > \frac{1}{2}$.

Note that we did not make the assumption that $a_n \rightarrow 0$. Indeed an interesting case is $a_n = 1$: we then get $\sum_n \epsilon_n$, the **one-dimensional random walk**. This sequence is not ℓ^2-convergent, so according to the theorem, with probability 1 the partial sums are unbounded. Because of the fundamental symmetry between $+$ and $-$, the probability that the sum is unbounded above is certainly the same as the probability that the sum is unbounded below. Each of these is a tail event, so by the $0-1$ Law their probabilities are either both 0 or both 1. But if they were both 0, then the probability that the series would be unbounded would be 0 – which we just saw was not the case – so rather with probability 1 the partial sums are unbounded both above and below. A moment's thought gives:

Corollary 11. *(Recurrence of the one-dimensional random walk) With probability 1, the limit set of the random walk is $\overline{\mathbb{Z}} = \pm\infty \cup \mathbb{Z}$.*

A similar argument shows the following:

Corollary 12. *Suppose $\{a_n\}$ is a real sequence with $a_n \rightarrow 0$ and $\sum_n a_n^2 = \infty$. Then, with probability 1, the limit set of $\sum_n \epsilon_n a_n$ is the entire real line.*

4. And on to n dimensions?

The RPZ theorem holds verbatim in $\mathbb{R}^N$[3] – and, in fact, for series with values in any Hilbert space – as long as we replace a_n^2 with $||a_n||^2$, i.e., the fundamental dichotomy is whether the sequence is ℓ^2 convergent or not (which, notice, is weaker than ℓ^1 convergence, a.k.a. absolute convergence). Especially, if our vector-valued sequence is not in ℓ^2, then with probability 1 it is unbounded.

But again, in $\mathbb{R}^1$, assuming $a_n \rightarrow 0$ we showed more: if the sequence is in ℓ^2, its is almost surely convergent, in other words its limit set is almost surely a single (finite) point. On the other hand if it is not in ℓ^2 its limit set is almost surely the entire real line. I find it remarkable that no intermediate behavior is possible: why shouldn't the probability 1 limit set be something like a closed interval of bounded nonzero length, or an infinite ray?

[3]In fact, it is easy to reduce to the case of $N = 1$ as we did in the Weak Riemann Theorem.

So finally, here is the question I am interested in: let $\{a_n\}$ be a sequence in $\mathbb{R}^N$ for which $a_n \rightarrow 0$. By RPZ, if the sequence is ℓ^2-convergent, the limit set of $\sum_n \epsilon_n a_n$ is almost surely a single finite point. But what if the sequence is ℓ^2-divergent? What is the almost sure behavior of the limit set?

Let me say a few words to make you believe that this question is sensible and rigorous. If we start with any given series and change finitely many signs, then this has the effect of translating the sequence of partial sums by a fixed vector. Any statement about the limit set which is translation-invariant is therefore susceptible to the $0-1$ Law and must occur with probability 0 or 1. An example of a translation-invariant statement about the limit set is that it is unbounded, and indeed RPZ assures that if $\sum_n ||a_n||^2 = \infty$ this will be the case with probability 1. Another translation invariant statement would be if the limit set is all of $\mathbb{R}^N$.[4]

Is this then what we are going to guess, that the limit set is almost surely all of $\mathbb{R}^N$? Of course not. As before, if all of the a_n's lie in some fixed linear subspace V, then any limit point of any $\sum_n \epsilon_n a_n$ will also have to lie in V.

So probably we should, as above, consider projections, right? As above, for any $0 \neq v \in \mathbb{R}^N$, it makes sense to speak of the ℓ^2 convergence of the projected series a_n^v, and it is easy to see that there is a unique maximal subspace V_2 of $\mathbb{R}^N$ such that $\sum_n ||a_n^{V_2}|| < \infty$, i.e., the projection onto V_2 is ℓ^2-convergent. By RPZ applied inside the subspace V_2, with probability 1 the limit set of this projection is a single finite point. The question is about the converse. But in view of the Lévy-Steinitz theorem I find the following irresistible:

Conjecture. *Let $\{a_n\}$ be a sequence in $\mathbb{R}^N$ with $a_n \rightarrow 0$. Put*

$$V_2 = \{v \in \mathbb{R}^N \mid \sum_n ||a_n^v||^2 < \infty\}.$$

Then, with probability 1, the limit set of $\sum_n \epsilon_n a_n$ is a translate of $V_2^\perp$.

As above, a special case which should be sufficient is: suppose that the series is ℓ^2-divergent in every direction. Then with probability 1 its limit set is all of $\mathbb{R}^N$.

Can you prove this? (Or disprove it?) Let me know!

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[4] A better perspective: for any subset $S \subset \mathbb{R}^N$, the statement "The limit set is of the form $v + S$ for some $v \in \mathbb{R}^N$" is tautologically translation invariant.

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